Geometric Study of a Family of Integrable Systems

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ABSTRACT

The aim of this paper is to demonstrate the rich interaction between complex algebraic geometry, the theory of integrable systems and the geometry of its asymptotic solutions. We present a family of integrable hamiltonian systems. We study theses systems from a different angle, assemble different geometric methods and several views.

1. Introduction

In this paper, we shall be concerned with finite dimensional algebraic completely integrable systems. A dynamical system is algebraic completely integrable [2, 3, 30] if it can be linearized on an abelian variety; a complex algebraic torus $\mathbb{C}^n/lattice$. The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equals to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines. However, besides the fact that many hamiltonian completely integrable systems posses this structure, another motivation for its study is algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate. Indeed a theorem of Adler-Kostant-Symes [3] applied to Kac-Moody algebras provides such systems which, by a theorem of van Moerbeke-Mumford [44], are algebraic completely integrable. Therefore there are hidden symmetries which have a group theoretical foundation. In recent decades there has been much effort given for finding integrable hamiltonian systems, not only because they have been on the subject of powerful and beautiful theories of mathematics, but also because the concepts of integrability have been applied to an increasing number of physical systems, biological phenomena, population dynamics, chemical rate equations, to mention only a few. However, it seems still hopeless to describe or even to recognize with any facility, those hamiltonian systems which are integrable, though they are quite exceptional. In fact, the overwhelming majority of dynamical systems, hamiltonian or not, are non-integrable and possess regimes of chaotic behavior in phase space. The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results.

In the present paper, we discuss an interesting interaction between complex algebraic geometry and dynamics. The paper is organized into eight sections, the first of which is this introduction. In section 2, we discuss the integrability of a family of hamiltonian systems. In section 3, we study this problem using the Lax spectral curve technique. We show that the system (2.2) in this case is linearized in the jacobian variety of a genus two hyperelliptic Riemann surface Γ. In section 4, we solve the equations of motion in terms of elliptic functions with the help of finite-gap integration theory and this can be done only for some special relations between the parameters a_1, a_2 and the constants of motion. The main problem of sections 5, 6 and 7, is to show that the affine surface V_c (3.5) obtained by setting the two quartics invariants of the problem equal to generic

constants, is the affine part of an abelian surface $\widetilde{V}_c$. A naive guess would be to take the natural completion $\overline{V}_c$ of V_c in $\mathbb{CP}^4$. We show that $\overline{V}_c$ is not an abelian surface and $\overline{V}_c$ must have a singularity somewhere along the locus at infinity $I = \overline{V}_c \cap \{X_0 = 0\}$, the latter consists of two lines d_ε, $\varepsilon = \pm 1$. The trajectories of the vectors fields (5.2) hit every point of these lines. We prove that around the points of hitting, the system of differential equations (5.2) admit Laurent expansion solutions whose coefficients depend on three free parameters. These parameters can be used to blowing up the variety $\overline{V}_c$ along the lines d_ε ($\varepsilon = \pm 1$). The new complex surface obtained is an abelian surface $\widetilde{V}_c$ and the two lines d_ε turn into two copies C_ε ($\varepsilon = \pm 1$) of the same smooth genus 3 hyperelliptic curve C, intersecting in two distinct points at which they are tangent to each other. The curve C is a double ramified cover of an elliptic curve C_0 ramified at four points and a $2-1$ unramified cover of a genus 2 hyperelliptic curve $\mathcal{H}$. The 16 functions $1, f_1, ..., f_{15}$ defined in (7.4) provide the embedding of $\widetilde{V}_c$ in projective 15-space $\mathbb{CP}^{15}$. The divisor $D = C_{+1} + C_{-1}$ obtained in this way has genus 5 and thus $S = 2D$ has genus 17. In section 7, we prove that the flow (5.3) evolves on abelian surface $\widetilde{V}_c = \mathbb{T}^2_c \subset \mathbb{CP}^{15}$ with the polarization of type $(2,2)$ and it will be expressed in terms of abelian integrals, involving the differentials (7.5). In section 8, we present a brief discussion and concluding remarks about similar problems.

2. A family of integrable systems

We consider the hamiltonian

$$H = \frac{1}{2}\left(x_1^2 + x_2^2 + a_1 y_1^2 + a_2 y_2^2\right) + \frac{1}{4}y_1^4 + \frac{1}{4}a_3 y_2^4 + \frac{1}{2}a_4 y_1^2 y_2^2, \tag{2.1}$$

where a_1, a_2, a_3, a_4 are arbitrary constants. The corresponding system is given by

$$\ddot{y}_1 + \left(a_1 + y_1^2 + a_4 y_2^2\right) y_1 = 0, \qquad \ddot{y}_2 + \left(a_2 + a_3 y_2^2 + a_4 y_1^2\right) y_2 = 0. \tag{2.2}$$

The integrability of this hamiltonian system has been studied by several authors. It arises in connection with some problems in scalar field theory and in the semi-classical method in quantum field theory. It corresponds to the Garnier system and to the anisotropic harmonic oscillator in a radial quartic potential. Also, this hamiltonian can be connected to the coupled nonlinear Schrödinger equations. For more information on these connections, noticed long ago, we can see for example the following articles : [11, 12, 24, 38, 46, 48 and 50]. Furthermore, the hamiltonian (2.1) is part of the well known family

$$H = \frac{1}{2}\left(x_1^2 + x_2^2\right) + \frac{1}{2}\left(a_1 y_1^2 + a_2 y_2^2\right) + A y_1^4 + B y_1^2 y_2^2 + C y_2^4 + \frac{1}{2}\left(\frac{E}{y_1^2} + \frac{D}{y_2^2}\right),$$

which describe the motion of particles interacting with a quartic potential $A y_1^4 + B y_1^2 y_2^2 + C y_2^4$ and perturbed by an inverse squared potential. There are four nontrivial known cases of complete integrability for the nonperturbed quartic potential and were all studied in detail from different points of view: **1)** A:B:C=1:2:1, **2)** A:B:C=1:12:16, **3)** A:B:C=1:6:1, **4)** A:B:C=1:6:8. Cases 1), 2) and 3) are separable in ellipsoidal, parabolical and cartesian coordinates respectively, whilst case 4) is separable in the general sense [39]. The integrability of case (i) and separability in ellipsoidal coordinates was proved in [47] (see also [25, 41]). The case 2) appears as one of the entries in the polynomial hierarchy discussed in [14]. The cases 3) and 4) are proved to be canonically equivalent under the action of a Miura map restricted to the stationary coupled KdV systems associated with a fourth order Lax operator [6]. Moreover all the cases 1)-4) allow the deformation of the potential by linear combination of inverse squares and squares with certain limitations on the coefficients (see [11], [4]). There are also Lax representations known for all these cases which yield hyperelliptic algebraic curves in the cases 1) and 2) and a 4-gonal curve in the cases 3) and 4). Various results concerning cases 1)-4) can be found in [22, 23, 17, 38]. For a review of known results see also [13, 45] and the references therein. When $a_1 = a_2 = A = C = E = D = 0$, $B = \frac{1}{2}$, this hamiltonian can be connected to the Yang-Mills system for a field with gauge group $SU(2)$ (indeed, consider the Yang-Mills system : $D_i F_{ik} = \partial_i F_{ik} + [A_i, F_{ik}] = 0$, where $F_{ik}, A_i \in T_e SU(2)$, $1 \le i, k \le 4$ and $F_{ik} = \partial_i A_k - \partial_k A_i + [A_i, A_k]$. In the case of homogeneous two-component field, we have $\partial_i A_k = 0$, $i \ne 1$, $A_1 = A_2 = 0$, $A_3 = n_1 U_1$, $A_4 = n_2 U_2$, $n_{1,2} = \text{constant}$, $n_1 = [n_2, [n_1, n_2]]$, $n_2 = [n_1, [n_2, n_1]]$ and the system becomes $\partial^2 U_1 + U_1 U_2^2 = 0$, $\partial^2 U_2 + U_2 U_1^2 = 0$. By setting $U_1 = y_1$, $U_2 = y_2$, $\frac{\partial U_1}{\partial t} = x_1$, $\frac{\partial U_2}{\partial t} = x_2$, Yang-Mills equations are reduced to hamiltonian system with the hamiltonian $H = \frac{1}{2}\left(x_1^2 + x_2^2 + y_1^2 y_2^2\right)$. The self-dual Yang-Mills (SDYM) equations is an universal system for which some reductions include all classical

tops from Euler to Kowalewski (0+1-dimensions), K-dV, Nonlinear Schrödinger, Sine-Gordon, Toda lattice and N-waves equations (1+1-dimensions), KP and D-S equations (2+1-dimensions).

The hamiltonian (2.1) has the Painlevé property (i.e. the general solutions have no movable singularities other than poles) only if i) $a_1 = a_2, a_3 = a_4 = 1$, ii) $a_1 = a_2, a_3 = 1, a_4 = 3$. In the case i), the second integral has the form $H_2 = x_2 y_1 - x_1 y_2$, whereas in the case ii) the second integral is $H_2 = x_1 x_2 + y_1 y_2 \left(a_1 + y_1^2 + y_2^2 \right)$. Also, it was shown [24] that if iii) $a_1 = \frac{a_2}{4}, a_3 = 16, a_4 = 6$, the system (2.2) is integrable and the second integral is $H_2 = a_1 y_1^2 + y_1 y_2 \left(y_1^3 + 2 y_1 y_2^2 \right) + x_1 \left(x_2 y_1 - x_1 y_2 \right)$. This paper deals with the problem of integrability of the system (2.2) corresponding to the choice : a_1, a_2 arbitrary and $a_3 = a_4 = 1$. For $a_1 = a_2$, it is easy to show that the problem can be integrated in terms of elliptic functions. In the next sections, we suppose that $a_1 \neq a_2$.

3. Lax representation and integrability

Using the results given in [14, 15], we consider the Lax representation in the form

$$\dot{A}_h = [B_h, A_h] = B_h A_h - A_h B_h, \qquad \cdot \equiv \frac{\partial}{\partial t} \tag{3.1}$$

with the following ansatz for the Lax operator

$$A_h = \begin{pmatrix} u(h) & v(h) \\ w(h) & -u(h) \end{pmatrix}, \qquad B_h = \begin{pmatrix} 0 & 1 \\ r(h) & 0 \end{pmatrix},$$

where

$$\begin{aligned}
v(h) &= -(a_1 + h)(a_2 + h) \left(1 + \frac{1}{2} \left(\frac{y_1^2}{a_1 + h} + \frac{y_2^2}{a_2 + h} \right) \right), \\
u(h) &= \frac{1}{2}(a_1 + h)(a_2 + h) \left(\frac{x_1 y_1}{a_1 + h} + \frac{x_2 y_2}{a_2 + h} \right), \\
w(h) &= (a_1 + h)(a_2 + h) \left(\frac{1}{2} \left(\frac{x_1^2}{a_1 + h} + \frac{x_2^2}{a_2 + h} \right) - h + \frac{1}{2} \left(y_1^2 + y_2^2 \right) \right), \\
r(h) &= h - y_1^2 - y_2^2.
\end{aligned} \tag{3.2}$$

The system (2.2) where $a_3 = a_4 = 1$ admits a Lax representation given by (3.1). The proof is straightforward and based on direct computation : we have

$$[B_h, A_h] = \begin{pmatrix} w(h) - v(h)r(h) & -2u(h) \\ 2u(h)r(h) & v(h)r(h) - w(h) \end{pmatrix},$$

and it follows from (3.1), (3.2) and (2.2) with $a_3 = a_4 = 1$ that

$$\dot{u}(h) = w(h) - v(h)r(h), \qquad \dot{v}(h) = -2u(h), \qquad \dot{w}(h) = 2u(h)r(h).$$

Equation (3.1) means that for $h \in \mathbb{C}$ and under the time evolution of the system, $A_h(t)$ remain similar to $A_h(0)$. So the spectrum of A_h is conserved, i.e. it undergoes an isospectral deformation. The eigenvalues of A_h, viewed as functionals, represent the integrals (constants of the motion) of the system. To be precise, a hamiltonian flow of the type (3.1) preserves the spectrum of A_h and therefore its characteristic polynomial $\det(A_h - zI)$. We form the Riemann surface in (z, h) space : $\Gamma : \det(A_h - zI) = 0$, whose coefficients are functions of the phase space. Explicitly, this equation looks as follows

$$\begin{aligned}
\Gamma : z^2 &= u^2(h) + v(h)w(h) \equiv P_5(h), \tag{3.3} \\
&= (a_1 + h)(a_2 + h)(h^3 + (a_1 + a_2)h^2 + (a_1 a_2 - H_1)h - H_2),
\end{aligned}$$

where $H_1 = H$ is defined by (2.1) with a_1, a_2 arbitrary, $a_3 = a_4 = 1$ and a second quartic integral H_2 of the form :

$$\begin{aligned}
H_2 &= \frac{1}{4}(a_2 y_1^4 + a_1 y_2^4 + (a_1 + a_2) y_1^2 y_2^2 + (x_1 y_2 - x_2 y_1)^2) \tag{3.4} \\
&\quad + \frac{1}{2}(a_2 x_1^2 + a_1 x_2^2 + a_1 a_2 (y_1^2 + y_2^2)).
\end{aligned}$$

The curve Γ determined by the fifth-order equation (3.3) is smooth, hyperelliptic and its genus is two. Obviously, Γ is invariant under the involution $(h, z) \longmapsto (h, -z)$. The second hamiltonian vector field is written as

$$
\begin{aligned}
\dot{y}_1 &= \frac{1}{2}(x_1 y_2 - x_2 y_1)y_2 + a_2 x_1, \qquad \dot{y}_2 = -\frac{1}{2}(x_1 y_2 - x_2 y_1)y_1 + a_1 x_2, \\
\dot{x}_1 &= -a_2 y_1^3 - \frac{1}{2}(a_1 + a_2)y_1 y_2^2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)x_2 - a_1 a_2 y_1, \\
\dot{x}_2 &= -a_1 y_2^3 - \frac{1}{2}(a_1 + a_2)y_1^2 y_2 - \frac{1}{2}(x_1 y_2 - x_2 y_1)x_1 - a_1 a_2 y_2.
\end{aligned}
$$

These vector fields are in involution with respect to the associated Poisson bracket. For generic $c = (c_1, c_2) \in \mathbb{C}^2$,

$$
V_c = \{H_1 = c_1, H_2 = c_2\}, \tag{3.5}
$$

is a smooth affine surface. The linearized flow could be realized on the jacobian variety $Jac(\Gamma)$ of the curve (3.3). Indeed, as the structure of the matrices A_h and B_h (degree of the polynomials in h) is the same as for the well-known Neumann system, so both systems are linearized in the same way. We can construct an algebraic map from V_c to the Jacobi variety $Jac(\Gamma)$:

$$
V_c \longrightarrow Jac(\Gamma), \quad p \in V_c \longmapsto (s_1 + s_2) \in Jac(\Gamma),
$$

and the flows generated by the constants of the motion are straight lines on $Jac(\Gamma)$, i.e., the linearizing equations are given by

$$
\sum_{i=1}^{2} \int_{s_i(0)}^{s_i(t)} \omega_k = c_k t, \quad 0 \le k \le 2,
$$

where ω_1, ω_2 span the two-dimensional space of holomorphic differentials on the curve Γ and s_1, s_2 two appropriate variables, algebraically related to the originally given ones, for which the Hamilton-Jacobi equation could be solved by separation of variables. Consequently, we have

Theorem 3.1. *The system (2.2) is completely integrable for all $a_1, a_2, a_3 = a_4 = 1$ and admits a Lax representation given by (3.1). The invariants of A_h are integrals of motion in involution. The first integral is given by the hamiltonian $H_1 = H$ (2.1) whereas the second integral H_2 is also quartic and has the form (3.4). The flows generated by H_1 and H_2 are straight line motions on the jacobian variety $Jac(\Gamma)$ of a smooth genus two hyperelliptic curve Γ (3.3) associated to Lax equation (3.1).*

We introduce coordinates s_1 and s_2 on the surface V_c (3.5), such that $v(s_1) = v(s_2) = 0$, $a_1 \ne a_2$,

$$
y_1^2 = 2\frac{(a_1 + s_1)(a_1 + s_2)}{a_1 - a_2}, \qquad y_2^2 = 2\frac{(a_2 + s_1)(a_2 + s_2)}{a_2 - a_1},
$$

i.e.,

$$
s_1 + s_2 = \frac{1}{2}(y_1^2 + y_2^2) - a_1 - a_2, \qquad s_1 s_2 = -\frac{1}{2}(a_2 y_1^2 + a_1 y_2^2) + a_1 a_2.
$$

After some manipulations, we obtain the following equations for s_1 and s_2 :

$$
\dot{s}_1 = 2\frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \qquad \dot{s}_2 = 2\frac{\sqrt{P_5(s_2)}}{s_2 - s_1},
$$

where $P_5(s)$ is defined by (3.3). These equations can be integrated by the abelian mapping

$$
\Gamma \longrightarrow Jac(\Gamma) = \mathbb{C}^2/L, \quad p \longmapsto \left(\int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2 \right),
$$

where the genus two hyperelliptic curve Γ is given by the equation (3.3), L is the lattice generated by the vectors $n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2$, Ω is the matrix of period of Γ, (ω_1, ω_2) is a basis of holomorphic differentials on Γ, i.e.,

$$
\omega_1 = \frac{ds}{\sqrt{P_5(s)}}, \qquad \omega_2 = \frac{s \, ds}{\sqrt{P_5(s)}},
$$

and p_0 is a fixed point. Consequently, we have

Theorem 3.2. *The system of differential equations (2.2) with $a_3 = a_4 = 1$, can be integrated in terms of genus two hyperelliptic functions of time.*

4. The gap theory and linearization using elliptic functions

In this section, we show that at some special values of the parameters a_1 and a_2, the solutions for the integrable case proposed in this paper may be expressed in terms of elliptic functions. We solve the equations of motion with the help of finite-gap integration theory. The reader is referred to the book [10] for this theory and its applications to nonlinear integrable equations. For doing that, we need some notions about elliptic Weierstrass $\wp$-function which we summarize here after.

Let $\Lambda = \{2m\omega_1 + 2n\omega_2,\ (m,n) \in \mathbb{Z}^2\} \subset \mathbb{C}$ be a lattice. We recall that the elliptic Weierstrass $\wp$-function with respect to Λ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\neq(0,0) \\ (m,n)\in\mathbb{Z}^2}} \left\{ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\},\ z \in \mathbb{C}.$$

This series converges absolutely and uniformly on every compact set in $\mathbb{C}\backslash\Lambda$, thus giving a meromorphic function having a double pole at each lattice point. This function is doubly periodic $\wp(z + 2m\omega_1 + 2n\omega_2) = \wp(z)$, $(m, n \in \mathbb{Z})$ with periods $2\omega_1, 2\omega_2$ such that $Im\frac{\omega_2}{\omega_1} > 0$ and has a singularity at $z = 0$ such as

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \cdots,$$

$$g_2 = 60 \sum_{\substack{(m,n)\neq(0,0) \\ (m,n)\in\mathbb{Z}^2}} \frac{1}{(2m\omega_1 + 2n\omega_2)^2}, \qquad g_3 = 140 \sum_{\substack{(m,n)\neq(0,0) \\ (m,n)\in\mathbb{Z}^2}} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

The derivative $\wp'(z)$ is again an elliptic function. The $\wp$-function satisfies the addition theorem

$$det \begin{pmatrix} 1 & \wp(a) & \wp'(a) \\ 1 & \wp(b) & \wp'(b) \\ 1 & \wp(c) & \wp'(c) \end{pmatrix} = 0, \qquad a + b + c = 0,$$

as well as the differential equation

$$\begin{aligned} (\wp'(z))^2 &= 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \\ &= 4\wp^3(z) - g_2\wp(z) - g_3, \end{aligned}$$

where $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_1 + \omega_2)$ and $e_3 = \wp(\omega_2)$. The map

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{CP}^2, z \longmapsto [1, \wp(z), \wp'(z)], z \neq 0 \text{ and } 0 \longmapsto [0, 0, 1],$$

is an isomorphism between the complex torus $\mathbb{C}/\Lambda$ and the elliptic curve

$$y^2 = 4x^3 - g_2 x - g_3,$$

where $x = \wp(z)$ and $y = \wp'(z)$.

The elliptic solutions for which we look for are given by the spectral problem (the Schrödinger equation) :

$$\left(\frac{\partial^2}{\partial x^2} - \mathcal{U}(x) \right) \Psi = \lambda\Psi, \tag{4.1}$$

depending in the elliptic potential

$$\mathcal{U}(x) = 2\sum_{i=1}^{3} \wp(x - x_i) + C. \tag{4.2}$$

Here $\Psi = \Psi(x, \lambda)$ is an eigenfunction (elliptic Backer-Akhiezer function) of operator $\mathcal{L}$, $\wp$ is the elliptic Weierstrass function defined above, C is a constant and x_1, x_2, x_3 are on the locus

$$\Delta = \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 : \sum_{i\neq j} \wp'(x_i - x_j) = 0,\ x_i \neq x_j,\ j = 1, 2, 3 \right\}.$$

The locus Δ is non-empty and its geometry was studied by several authors [4, 32]. It is shown in [4] that if $x_i = x_i(t)$, $j = 1, 2, 3$ evolve according to the law

$$\dot{x}_i = -12 \sum_{i \neq j} \wp \left(x_i - x_j \right),$$

then the function (4.2) is an elliptic solution of the well-known Korteweg-de Vries equation (KdV) :

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

and is connected with the completely integrable Calogero-Moser system described by the hamiltonian :

$$H = \frac{1}{2} \sum_{i=1}^{3} y_i^2 - 2 \sum_{i \neq j} \wp \left(x_i - x_j \right),$$

with y_i, x_i, $i = 1, 2, 3$ being canonical variables. The Jacobi inversion problem associated with 2-gap solutions of the KdV equation is determined by

$$\int_{\infty}^{s_1} \frac{dz}{w} + \int_{\infty}^{s_2} \frac{dz}{w} = -8it + c_1, \qquad \int_{\infty}^{s_1} \frac{zdz}{w} + \int_{\infty}^{s_2} \frac{zdz}{w} = 2ix + c_2,$$

where c_1, c_2 are constants, $w^2 = \prod_{j=1}^{5}(z - z_j)$, $s_j = s_j(x, t)$, $j = 1, 2$ and z_j, $j = 1, ..., 5$, are expressed in terms of integrals of motion. After an appropriate change of variables and corresponding choices of z_j, the Jacobi inversion problem above coincides with that associated with Hénon-Heiles equations [16]. Let us consider the two-gap potential for equation (4.2) normalized by its expansion near $x = 0$ as

$$\mathcal{U}(x) = \frac{6}{x^2} + \alpha_1 x^2 + \alpha_2 x^4 + \alpha_3 x^6 + \alpha_4 x^8 + O(x^{10}), \tag{4.3}$$

where α_1, α_2, α_3, α_4 are functions of the moduli g_2, g_3 of the underlying elliptic curve. This potential satisfies the Novikov equation [37]: $\sum_{i=-1}^{2} a_i \frac{\delta S_i}{\delta u} = 0$, where δ is the variational operator of the calculus of variations, a_1, a_2, are constants and

$$S_{-1} = \int u dx, \qquad S_1 = \int \left(\frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + u^3 \right) dx,$$

$$S_0 = \int u^2 dx, \qquad S_2 = \int \left(\frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 - \frac{5}{2} u^2 \frac{\partial^2 u}{\partial x^2} + \frac{5}{2} u^4 \right) dx,$$

are the first integrals of the KdV equation. The algebraic curve associated with the potential (4.3) has the form [9]:

$$w^2 = z^5 - \frac{35}{2} \alpha_1 z^3 - \frac{63}{2} \alpha_2 z^2 + \frac{1}{4} \left(\frac{567}{2} \alpha_1^2 + 297 \alpha_3 \right) z + \frac{1377}{4} \alpha_1 \alpha_2 - \frac{1287}{2} \alpha_4. \tag{4.4}$$

As the potential is elliptic (2-gap Lamé potential) the curve (4.4) can be written explicitly as, $w^2 = (z^2 - 3g_2)(z - 3e_1)(z - 3e_2)(z - 3e_3)$. All this can be done by classical means following the work of Hermite [21] and Halphen [20]. In addition, it follows from the trace formulas [49] that

$$s_1 + s_2 = - \sum_{j=1}^{3} \wp \left(x - x_j \right) + \frac{1}{2} \sum_{j=1}^{5} z_j - \frac{C}{2},$$

$$s_1 s_2 = 3 \sum_{\substack{i,j=1 \\ i \langle j}}^{3} \wp \left(x - x_i \right) \wp \left(x - x_j \right) - \frac{3}{8} g_2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \langle j}}^{5} z_i z_j - \frac{3}{8} \left(\sum_{j=1}^{5} z_j \right)^2$$

$$+ \frac{3C}{2} \sum_{j=1}^{3} \wp \left(x - x_j \right) + \frac{3C^2}{8},$$

where $x_1, x_2, x_3 \in \Delta$ and $z_1, ..., z_5$ are the branching points of the genus 2 hyperelliptic curve (4.4). We want the curve Γ (3.3) to be associated with the genus 2 curve (4.4). Let z_α, z_β be two distinct branch points of the curve (4.4) which are the shifted values of the branch points a_1 a_2 of the curve Γ (3.3). From system (2.2), we obtain

$$\ddot{y}_1 = \left(a_1 - y_1^2 - y_2^2\right) y_1, \qquad \ddot{y}_2 = \left(a_2 - y_1^2 - y_2^2\right) y_2,$$

implying that

$$\ddot{y}_1 - \mathcal{U}(x) y_1 = \left(a_1 - 2\left(z_\alpha + y_\beta\right)\right) y_1, \quad \ddot{y}_2 - \mathcal{U}(x) y_2 = \left(a_2 - 2\left(z_\alpha + y_\beta\right)\right) y_2.$$

Consequently, we have

Theorem 4.1. *Suppose that $a_1 = 3z_\alpha + 2z_\beta$ and $a_2 = 2z_\alpha + 3z_\beta$ where z_α and z_β are two distinct branch points of the curve (4.4). Then, formula for elliptic solutions of (2.2) is given by*

$$y_1^2 = \frac{1}{z_{\alpha\beta}} \left\{ \Phi + 2z_\alpha^2 + (3C - 2z_\alpha) \sum_{i=1}^{3} \wp(x - x_i) + \left(\frac{3C}{4} - z_\alpha\right) C \right\},$$

$$y_2^2 = \frac{-1}{z_{\alpha\beta}} \left\{ \Phi + 2z_\beta^2 + (3C - 2z_\beta) \sum_{i=1}^{3} \wp(x - x_i) + \left(\frac{3C}{4} - z_\beta\right) C \right\},$$

where $z_{\alpha\beta} \equiv z_\alpha - z_\beta$, $\Phi \equiv 6 \sum_{1 \leq i \langle j \leq 3} \wp(x - x_i) \wp(x - x_j) - \frac{3g_2}{4} + \sum_{1 \leq i < j \leq 5} z_i z_j$ and $z_1, ..., z_5$ are the branching points of the curve (2.2).

5. Algebraic complete integrability and intersection of quartics in complex projective 4-space

First we recall some basics concepts. Consider hamiltonian problems of the form

$$\dot{x} = J \frac{\partial H}{\partial x} , \ x \in \mathbb{R}^m, \tag{5.1}$$

where H is the hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket $\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial x}, J \frac{\partial H_j}{\partial x} \right\rangle$, satisfies the Jacobi identity. The system (5.1) with polynomial right hand side is called algebraic complete integrable when :

$a)$ The system (5.1) possesses $n + k$ independent polynomial invariants $H_1, ..., H_{n+k}$ of which k lead to zero vector fields $J \frac{\partial H_{n+i}}{\partial x} = 0$, $1 \leq i \leq k$ (called Casimir functions) and the n remaining ones are in involution (i.e. $\{H_i, H_j\} = 0$) and $m = 2n + k$. For most values of $c_i \in \mathbb{R}$, the invariant manifolds $\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\}$, are compact and connected. Then, according to the Arnold-Liouville theorem [5], there exists a diffeomorphism

$$\Phi : \bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\} \longrightarrow \mathbb{R}^n / Lattice,$$

and the solutions of the system (5.1) are straight lines motions on these tori.

$b)$ The invariant manifolds, thought of as affine varieties in $\mathbb{C}^m$ (non-compact) can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k} \{x \in \mathbb{C}^m : H_i = c_i\} = \mathbb{T}_c^n \setminus \text{a divisor } D \text{ (i.e. codimension 1 subvarieties)},$$

where the tori

$$\mathbb{T}_c^n = \frac{\mathbb{C}^n}{period\ Lattice} = \text{complex algebraic torus (abelian variety)},$$

depend on the $c's$.

Recall that algebraic means that $\mathbb{T}_c^n$ can be defined as an intersection $\bigcap_{i=1}^{M} \{(X_0, ..., X_N) : P_i(X_0, ..., X_N) = 0\}$, involving a large number of homogeneous polynomials P_i. The functions x_i are required to be meromorphic on

$\mathbb{T}_c^n$, and in particular in the natural coordinates $(t_1, ..., t_n)$ of $\mathbb{T}_c^n$ coming from $\mathbb{C}^n$, the functions $x_i = x_i(t_1, ..., t_n)$ are meromorphic and (5.1) defines straight line motion on $\mathbb{T}_c^n$. Condition b) means, in particular, there is an algebraic map $(x_1(t), ..., x_m(t)) \mapsto (s_1(t), ..., s_n(t))$, making the following sums linear in t :

$$\sum_{i=1}^{n} \int_{s_i(0)}^{s_i(t)} \omega_j = \mu_j t , \ 1 \leq j \leq n, \ \mu_j \in \mathbb{C},$$

where $\omega_1, ..., \omega_n$ denote holomorphic differentials on some algebraic curves.

Adler and van Moerbeke [2] have developed and used the following algebraic complete integrability criterion : If the hamiltonian system (5.1) is algebraic completely integrable, then each x_i blows up for some value of $t \in \mathbb{C}$ and whenever it blows up, the solution $x(t)$ behaves as a Laurent series

$$x_i = t^{-k_i} \left(x_i^{(0)} + x_i^{(1)}t + x_i^{(2)}t^2 + \cdots \right), \ k_i \in \mathbb{Z}, \ \text{some } k_i > 0,$$

which admits $\dim(phase\ space) - 1 = m - 1$ free parameters. To explain the criterion, if the hamiltonian flow (5.1) is algebraic completely integrable, it means that the variables x_i are meromorphic on the torus $\mathbb{T}_c^n$ and by compactness they must blow up along a codimension one subvariety (a divisor) $D \subset \mathbb{T}_c^n$. By the algebraic complete integrability definition, the flow (5.1) is a straight line motion in $\mathbb{T}_c^n$ and thus it must hit the divisor D in at least one place. Moreover through every point of D, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equation must admit Laurent expansions which depend on the $n - 1$ parameters defining D and the $n + k$ constants c_i defining the torus $\mathbb{T}_c^n$, the total count is therefore $m - 1 = \dim(phase\ space) - 1$ parameters.

The system (2.2) can be written in the form (5.1) with $m = 4, n = 2, k = 0$

$$\dot{x} = J\frac{\partial H}{\partial x} \equiv f(x) \qquad x = (y_1, y_2, x_1, x_2)^{\intercal}, \tag{5.2}$$

$H = H_1$ (2) with $a_3 = a_4 = 1$, $\frac{\partial H}{\partial x} = (\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2})^{\intercal}$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ a skew-symmetric matrix where I is the 2×2 unit matrix and O the 2×2 zero matrix. The second flow commuting with the first is regulated by the system

$$\dot{x} = J\frac{\partial H_2}{\partial x} , \ x = (y_1, y_2, x_1, x_2)^{\intercal}, \tag{5.3}$$

with H_2 defined by (3.4). The main problem will be to complete the affine surface V_c (3.5) into an abelian surface. A naive guess would be to take the natural compactification $\overline{V}_c = \bigcap_{i=1}^{2} \left\{ X \in \mathbb{CP}^4 : H_i(X) = c_i X_0^2 \right\}$ of V_c by projectivizing the equations. Indeed, this can never work for a general reason: an abelian variety $\widetilde{V}_c$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{CP}^n$ by n-dim $\widetilde{V}_c$ global polynomial homogeneous equations. In other words, if V_c (4.1) is to be the affine part of an abelian surface, $\overline{V}_c$ must have a singularity somewhere along the locus at infinity $\overline{V}_c \cap \{X_0 = 0\}$. An explicit calculation using the adjunction formula [18, 31] shows that we have :

Theorem 5.1. *Let $\overline{V}_c$ be the projective variety on $\mathbb{CP}^4$ defined by*

$$\frac{1}{2} \left(X_1^2 + X_2^2 + a_1 Y_1^2 + a_2 Y_2^2 \right) X_0^2 + \frac{1}{4} \left(Y_1^2 + Y_2^2 \right)^2 = c_1 X_0^4,$$

$$\frac{1}{4} \left(a_2 Y_1^4 + a_1 Y_2^4 + (a_1 + a_2) Y_1^2 Y_2^2 \right) + \frac{1}{4} \left(X_1 Y_2 - X_2 Y_1 \right)^2$$
$$+ \frac{1}{2} \left(a_2 X_1^2 + a_1 X_2^2 \right) X_0^2 + \frac{1}{2} a_1 a_2 \left(Y_1^2 + Y_2^2 \right) X_0^2 = c_2 X_0^4.$$

Then $\overline{V}_c$ is not an abelian surface. The variety $\overline{V}_c$ intersects the hyperplane at infinity $X_0 = 0$ according to two lines

$$d_\varepsilon = (X_0, X_1, X_2, Y_1, Y_2) = (0, X_1, \varepsilon i X_1, Y_1, \varepsilon i Y_1), \ \varepsilon \equiv \pm 1, \tag{5.4}$$

intersecting into one point $(0, X_1, X_2, 0, 0)$ and $\overline{V}_c$ is singular all along d_ε.

The trajectories of the vector fields (5.2) hit every point of the singular locus at infinity and ignore the smooth locus at infinity. In fact, we shall show that the existence of meromorphic solutions to the differential equations (2.2) depending on 3 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor.

More precisely, we shall show that around the points of hitting, the system of differential equations (5.2) admit a Laurent expansion solution depending on 3 free parameters and in order to regularize the flow at infinity, we use these parameters to blowing up the variety $\overline{V}_c$ along the lines d_ε (5.4). The new complex surface obtained in this fashion is compact, smooth and has two commuting vector fields on it; it is therefore an abelian surface and the two lines d_ε turn into two copies of a smooth genus 3 hyperelliptic curves, intersecting doubly into two different points.

6. Laurent series solutions and algebraic curves

Consider points at infinity which are limit points of trajectories of the flow. In fact, there is a Laurent decomposition of such asymptotic solutions,

$$x\left(t\right) \equiv \left(y_1\left(t\right), y_2\left(t\right), x_1\left(t\right), x_2\left(t\right)\right), \tag{6.1}$$

$$y_1 = \frac{y_1^{(0)}}{t} + y_1^{(1)} + y_1^{(2)}t + y_1^{(3)}t^2 + y_1^{(4)}t^3 + \cdots,$$

$$y_2 = \frac{y_2^{(0)}}{t} + y_2^{(1)} + y_2^{(2)}t + y_2^{(3)}t^2 + y_2^{(4)}t^3 + \cdots,$$

$x_1 = \dot{y}_1, x_2 = \dot{y}_2$, and which depend on $\dim(phase\ space) - 1 = 3$ free parameters. Putting (6.1) into (5.2), solving inductively for the $x^{(k)}$, one finds at the 0th step a non-linear equation,

$$x^{(0)} + f(x^{(0)}) = 0, \tag{6.2}$$

and at the kth step, a linear system of equations,

$$(\mathrm{J} - k\mathrm{I})\, x^{(k)} = \begin{cases} 0 \text{ for } k = 1 \\ \text{quadratic polynomial in } x^{(1)}, ..., x^{(k-1)} \text{ for } k \rangle 1 \end{cases}, \tag{6.3}$$

where J denotes the Jacobian map of (6.2). One parameter α appear at the 0th step, i.e., in the resolution of (6.2) and the two remaining ones β, γ, at the kth step, $k = 3, k = 4$ i.e., in the resolution of (6.3). These Laurent series solutions are explicitly given by

$$y_1 = \frac{\alpha}{t} + \frac{1}{6}\left(-a_2\alpha^2 + \alpha^2 a_1 - 2a_2 + 3a_1\right)\alpha t + \beta t^2 + y_1^{(4)}t^3 + \cdots \tag{6.4}$$

$$y_2 = \frac{\varepsilon i\sqrt{2 + \alpha^2}}{t} + \frac{\varepsilon i}{6}\left(-a_2\alpha^2 + a_2 + \alpha^2 a_1\right)\sqrt{2 + \alpha^2}t + \frac{\varepsilon i\alpha\beta}{\sqrt{2 + \alpha^2}}t^2 + \gamma t^3 + \cdots$$

$$x_1 = -\frac{\alpha}{t^2} + \frac{1}{6}\left(-a_2\alpha^2 + \alpha^2 a_1 - 2a_2 + 3a_1\right)\alpha + 2\beta t + 3y_1^{(4)}t^2 + \cdots$$

$$x_2 = -\frac{\varepsilon i\sqrt{2 + \alpha^2}}{t^2} + \frac{\varepsilon i}{6}\left(-a_2\alpha^2 + a_2 + \alpha^2 a_1\right)\sqrt{2 + \alpha^2} + \frac{2\varepsilon i\alpha\beta}{\sqrt{2 + \alpha^2}}t + 3\gamma t^2 + \cdots$$

with $y_1^{(4)}$ given by

$$y_1^{(4)} = -\frac{\alpha}{24(2 + \alpha^2)}(2a_2^2 - 8a_2a_1 + 6a_1^2 + 2\alpha^4 a_1^2 + 2a_2^2\alpha^4 + 5a_2^2\alpha^2 - 12a_1a_2\alpha^2$$

$$+ 7\alpha^2 a_1^2 + 24\varepsilon i\sqrt{(2 + \alpha^2)}\gamma - 4a_2\alpha^4 a_1),$$

and $\varepsilon = \pm 1$. Using the majorant method, we can show that the formal Laurent series solution are convergent. We have

Theorem 6.1. *a) The non identically zero solutions of (6.2) define a sphere. The system (6.3) has 1 degree of freedom for $k = 3$ and $k = 4$.*

b) The differential equations (5.2) admit Laurent series solutions depending on 3 free parameters α, β and γ. The first free parameter α appears in the resolution of (6.2) and the two remaining ones β, γ at the kth step, $k = 3$, $k = 4$. These Laurent series solutions are explicitly given above.

Now, taking into account only solutions trajectories lying on the surface V_c (3.5), we show that the Laurent solutions are parameterized by two copies C_ε of the same hyperelliptic curve C for $\varepsilon = \pm 1$. To be precise we search for the set of Laurent solutions which remain confined to a fixed affine invariant surface, related to specific values of c_1, c_2, i.e.,

$$
\begin{aligned}
C &= \text{closure of the continuous components of} \\
&\quad \{\text{Laurent series solutions } x(t) \text{ such that } H_i(x(t)) = c_i,\ 1 \le i \le 2\}, \\
&= \bigcap_{i=1}^{2} \{t^0 - \text{coefficient of } H_i(x(t)) = c_i\}, \\
&= \text{an algebraic curve :}
\end{aligned}
$$

$$
\beta^2 = \left(2 + \alpha^2\right)\left(A_1\alpha^6 + A_2\alpha^4 + A_3\alpha^2 + A_4\right), \tag{6.5}
$$

where $A_1 = \frac{(a_2-a_1)^3}{72}$, $A_2 = \frac{(a_2-a_1)^2(a_2-2a_1)}{36}$, $A_3 = \frac{(a_2-a_1)\left(a_1^2 - a_1 a_2 - c_1\right)}{18}$ and $A_4 = \frac{(a_1 c_1 - c_2)}{9}$. This is a eighth-order equation, so C is an hyperelliptic curve of genus 3 and the Laurent solutions (6.4) restricted to the surface V_c (3.5) are parameterized by two copies C_{+1} and C_{-1} of the same hyperelliptic curve C. Moreover, the map

$$
\sigma : C \longrightarrow C,\ (\alpha, \beta) \longmapsto (-\alpha, \beta), \tag{6.6}
$$

is an involution on C and the quotient $C_0 = C/\sigma$ is an elliptic curve :

$$
C_0 : \beta^2 = (2 + \zeta)\left(A_1\zeta^3 + A_2\zeta^2 + A_3\zeta + A_4\right). \tag{6.7}
$$

The curve C is a double ramified covering of C_0,

$$
\varphi : C \longrightarrow C_0, (\alpha, \beta) \longmapsto (\zeta, \beta), \tag{6.8}
$$

ramified at the four points covering $\zeta = 0$ and ∞. The curve C can also be seen as a $2-$sheeted unramified cover

$$
\pi : C \longrightarrow \mathcal{H}, (\alpha, \beta) \longmapsto (\zeta, \eta), \tag{6.9}
$$

of the following hyperelliptic curve $\mathcal{H}$ of genus 2 :

$$
\mathcal{H} : \eta^2 = \zeta(2 + \zeta)\left(A_1\zeta^3 + A_2\zeta^2 + A_3\zeta + A_4\right). \tag{6.10}
$$

Thus, we have proved

Theorem 6.2. *The Laurent solutions (6.4) restricted to the surface V_c (3.5) are parameterized by two copies C_{+1} and C_{-1} of the same hyperelliptic curve C (6.5) of genus 3. The latter is a double ramified cover of an elliptic curve C_0 (6.7) and a 2-1 unramified cover of an hyperelliptic curve $\mathcal{H}$ (6.10) of genus 2.*

7. Affine parts of abelian surfaces as complete intersection of two quartics

We recall some various useful facts about abelian varieties which will be needed (details can be found in [18, 31]). Let $\widetilde{V}_c$ be a smooth surface compactifying V_c (3.5) and D the embedding of $C_{+1} + C_{-1}$ into $\mathbb{CP}^N$. Consider a basis $1, f_1, ..., f_N$ of the vector space $\mathcal{L}(D)$ of meromorphic functions on $\widetilde{V}_c$ with at worst a simple pole along D and the map $\widetilde{V}_c \longrightarrow \mathbb{CP}^N$, $p \longmapsto [1, f_1(p), ..., f_N(p)]$, considered projectively, because if at p some $f_i(p) = \infty$, we divide by f_i having the highest order pole near p, which makes every element finite. The Kodaira embedding theorem tells us that if the line bundle associated with the divisor is positive, then for $k \in \mathbb{N}$, the functions of $\mathcal{L}(kD)$ embed smoothly $\widetilde{V}_c$ into $\mathbb{CP}^N$ and then by Chow's theorem, $\widetilde{V}_c = \bigcap_i \left\{X \in \mathbb{CP}^N : P_i(X) = 0\right\}$, where $P_i(X)$ are homogeneous polynomials, i.e., $\widetilde{V}_c$ can be realized as an algebraic variety. In fact in our case, $k = 2$ suffices, i.e., the divisor $2D$ provides a smooth embedding into $\mathbb{CP}^{15}$, via the meromorphic sections of $L(2D)$. Put $S \equiv 2D$ and let

$$
\chi(S) = \dim H^0\left(\widetilde{V}_c, \mathcal{O}_{\widetilde{V}_c}(S)\right) - \dim H^1\left(\widetilde{V}_c, \mathcal{O}_{\widetilde{V}_c}(S)\right),
$$

be the Euler characteristic of S. The adjunction formula and the Riemann-Roch theorem for divisors on $\widetilde{V}_c$ imply that

$$g\left(S\right) = \frac{K_{\widetilde{V}_c}.S + S.S}{2} + 1, \qquad \chi\left(S\right) = p_a\left(\widetilde{V}_c\right) + 1 + \frac{1}{2}\left(S.(S - K_{\widetilde{V}_c})\right),$$

$g\left(S\right)$ is the geometric genus of S, $p_a\left(\widetilde{V}_c\right)$ is the arithmetic genus of $\widetilde{V}_c$ and $S.S$ denote the number of intersection points of S with $a + S$ where $a + S$ is a small translation by a of S on $\widetilde{V}_c$. Using Kodaira-Serre duality :

$$\begin{aligned} \chi\left(S\right) &= \dim \mathcal{L}(S) - \dim H^1\left(\widetilde{V}_c, \Omega^2\left(S \otimes \mathcal{O}_{\widetilde{V}_c}\left(-S\right)\right)\right), \\ &= \dim \mathcal{L}(S) - \dim H^1\left(\widetilde{V}_c, \mathcal{O}_{\widetilde{V}_c}\left(K_{\widetilde{V}_c} \setminus S\right)\right), \end{aligned}$$

Kodaira-Nakano vanishing theorem : $\dim H^1\left(\widetilde{V}_c, \mathcal{O}_{\widetilde{V}_c}\left(S\right)\right) = 0$, then it is easy to see that if $\widetilde{V}_c$ is an abelian surface $\left(\text{i.e., } K_{\widetilde{V}_c} = 0, \ p_a\left(\widetilde{V}_c\right) = -1\right)$,

$$g\left(S\right) - 1 = \frac{S.S}{2} = \chi\left(S\right) = \dim \mathcal{L}(S) \ \left(\equiv h^0\left(\mathcal{L}\right)\right) = N + 1. \tag{7.1}$$

Let $\delta_1, \delta_2 \in \mathbb{N}^*$, $\delta_1 \mid \delta_2$, be the elementary divisors. Recall that a polarization of type (δ_1, δ_2) on $\widetilde{V}_c$ is defined by an ample line bundle $\mathcal{L}$ on $\widetilde{V}_c$, such that the group of translation t on $\widetilde{V}_c$ with $t^*\mathcal{L} \simeq \mathcal{L}$ is isomorphic with $(\mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_2})^2$. The integers δ_1 and δ_2 are related as follows :

$$\dim \mathcal{L}(S) = \delta_1 \delta_2. \tag{7.2}$$

Theorem 7.1. *a) The* 16 *functions* $1, f_1, ..., f_{15}$ *defined in* (7.4) *provide the embedding of* $\widetilde{V}_c$ *in projective space* $\mathbb{CP}^{15}$. *The divisor* $D = C_{+1} + C_{-1}$ *obtained has genus* 5 *and* $S \equiv 2D$ *has genus* 17.

b) The orbits of the vector field (5.2) *running through* S *form a smooth surface* Σ *near* S *such that* $\Sigma \setminus S \subseteq \widetilde{V}_c$. *Moreover, the variety* $\widetilde{V}_c = V_c \bigcup \Sigma$ *is smooth, compact and connected.*

c) The variety $\widetilde{V}_c$ *comes equipped with two everywhere independent commuting vector fields, which extend holomorphically on* $\widetilde{V}_c$. *The flow* (5.2) *evolves on abelian surface* $\widetilde{V}_c = \mathbb{T}^2_c \subset \mathbb{CP}^{15}$ *with the polarization of type* $(2, 2)$ *and it will be expressed in terms of abelian integrals, involving the differentials* (7.5).

Proof. a) Based on the motivation above, it is easy to find a set of polynomial functions $\{1, f_1, ..., f_N\}$ in $\mathcal{L}(S)$, i.e., $\mathcal{L}(2D)$ such that the embedding of S with those functions into $\mathbb{CP}^N$ yields a curve of genus $N + 2$. Straightforward calculation, using asymptotic expansions, shows that the space $\mathcal{L}(S)$ is spanned by the following functions

$$\mathcal{L}(S) = \{1, f_1, ..., f_{15}\}, \tag{7.3}$$

where $f_1 = y_1$, $f_2 = y_2$, $f_3 = x_1$, $f_4 = x_2$, $f_5 = f_1^2$, $f_6 = f_2^2$, $f_7 = f_1 f_2$, $f_8 = f_1 f_4 - f_2 f_3$, $f_9 = f_1 f_8 =$, $f_{10} = f_2 f_8$, $f_{11} = f_8^2$, $f_{12} = (f_5 + f_6) f_8 + 2(a_1 f_1 f_4 - a_2 f_2 f_3)$, $f_{13} = (f_5 + f_6) f_1 f_2 + 2 f_3 f_4$, $f_{14} = f_3 f_8 + (a_2 - a_1) f_1 f_7$, $f_{15} = f_4 f_8 + (a_2 - a_1) f_2 f_7$. Using these functions $1, f_1, ..., f_{15}$, one embeds each of the curves C_{+1} and C_{-1} into $\mathbb{CP}^{15}$. Let us look more closely at certain points of interest on the curve C. In a neighborhood of the point $s_k \equiv (\alpha = \pm i\sqrt{2}, \beta = 0)$, $k = 1, 2$, it is convenient to divide the vector in (7.2) evaluated on C, by f_{12}. Then, upon putting

$$\left(\frac{1}{f_{12}}, \frac{f_1}{f_{12}}, ..., \frac{f_{15}}{f_{12}}\right) \equiv F_k^{(0)} + F_k^{(1)} t + \cdots, 0 \leq k \leq 15,$$

we find

$$F_k^{(0)}(s_k) = (0, 0, 0, \frac{\mp\sqrt{2}}{4\varepsilon\sqrt{a_2 c_1 - c_2}}, 0, \frac{\pm i}{2\varepsilon\sqrt{a_2 c_1 - c_2}}, 0, 0, 0, 0, 0, 1, 0, \frac{\mp i\sqrt{2}}{2}, 0),$$

which leads to eight different points in $\mathbb{CP}^{15}$ according to the sign of $\varepsilon = \pm 1$. Using the transformation $C_{+1} \to C_{-1} : (\alpha, \beta) \mapsto (-\alpha, \beta)$, we have $F_k^{(0)}(s_k)\mid_{C_{+1}} = F_k^{(0)}(s_k)\mid_{C_{-1}}$, implying that the two points s_k on one curve are identified pairwise with the two corresponding points on the other curve. Also, we have

$$\frac{\partial F_k^{(0)}(s_k)}{\partial \alpha}\mid_{C_{+1}} = \frac{\partial F_k^{(0)}(s_k)}{\partial \alpha}\mid_{C_{-1}},$$

then the curves C_{+1} and C_{-1} have two points in common $s_1 \equiv (\alpha = i\sqrt{2}, \beta = 0)$ and $s_2 \equiv (\alpha = -i\sqrt{2}, \beta = 0)$, at which they are tangent to each other. In a neighborhood of the point $(\alpha = 0,\ \beta = \pm\frac{\sqrt{2}}{3}\sqrt{a_1 c_1 - c_2})$, dividing the vector in (7.2) by f_{12} and taking the limit when α tends to 0, we find the following four points in $\mathbb{CP}^{15}$:

$$(0,0,0,0,\mp\frac{3\sqrt{2}}{\sqrt{a_1 c_1 - c_2}},0,\pm\frac{i}{2\varepsilon\sqrt{a_1 c_1 - c_2}},0,0,0,0,0,1,0,0,-\frac{\sqrt{2}}{2i\varepsilon}),\ \varepsilon = \pm 1.$$

Also, in a neighborhood of the point $\alpha = \infty$, dividing the vector in (7.2) by f_{11} and letting $\alpha \nearrow \infty$, we find

$$F_k^{(0)}(\alpha \nearrow \infty) = (0,...,1,0,-\frac{i\varepsilon}{a_2 - a_1},\mp\frac{i\varepsilon\sqrt{2}}{2\sqrt{a_2 - a_1}},\pm\frac{\sqrt{2}}{2\sqrt{a_2 - a_1}}),$$

which leads to four different points according to the sign of $\varepsilon = \pm 1$. The divisor $D = C_{+1} + C_{-1}$ obtained in this way has genus 5 and thus $S = 2D$ has genus 17, satisfying the requirement (7.1).

b) Let $\psi(t,p) = \{x(t) = (y_1(t), y_2(t), x_1(t), x_2(t)) : t \in \mathbb{C}, 0\langle|t|\langle\varepsilon\}$, be the orbit of the vector field (5.2) going through the point $p \in S$. Let $\Sigma_p \subset \mathbb{CP}^{15}$ be the surface element formed by the divisor S and the orbits going through p, and set $\Sigma \equiv \bigcup_{p\in S}\Sigma_p$. Consider the curve $S' = \mathcal{H}\bigcap\Sigma$ where $\mathcal{H} \subset \mathbb{CP}^{15}$ is a hyperplane transversal to the direction of the flow. If S' is smooth, then using the implicit function theorem the surface Σ is smooth. But if S' is singular at 0, then Σ would be singular along the trajectory ($t-$axis) which go immediately into the affine part V_c. Hence, V_c would be singular which is a contradiction because V_c is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ and so smooth for almost all the two constants of the motion c_i. Next, let $\overline{V}_c$ be the projective closure of V_c into $\mathbb{CP}^4$, let $X = (X_0, X_1, X_2, Y_1, Y_2) \in \mathbb{CP}^4$ and let $I = \overline{V}_c\bigcap\{X_0 = 0\}$ be the locus at infinity (see theorem 4). Consider the map $\overline{V}_c \subseteq \mathbb{CP}^4 \to \mathbb{CP}^{15}$, $X \mapsto f(X)$, where $f = (1, f_1, ..., f_{15}) \in L(S)$ and let $\widetilde{V}_c = f(\overline{V}_c)$. In a neighborhood $V(p) \subseteq \mathbb{CP}^{15}$ of p, we have $\Sigma_p = \widetilde{V}_c$ and $\Sigma_p \setminus S \subseteq V_c$. Otherwise there would exist an element of surface $\Sigma'_p \subseteq \widetilde{V}_c$ such that $\Sigma_p \cap \Sigma'_p = t-$axis, orbit $\psi(t,p) = t-$axis $\setminus p \subseteq V_c$ and hence V_c would be singular along the $t-$axis which is impossible. Since the variety $\overline{V}_c \cap \{X_0 \neq 0\}$ is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\overline{V}_c$ is also irreducible, all hyperplane sections are connected and hence I is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{CP}^4 \times \mathbb{CP}^{15}$ of the map f, which is irreducible together with $\overline{V}_c$. It follows from the irreducibility of I that a generic hyperplane section $\Gamma_f \cap \{\mathcal{H}_{gen.} \times \mathbb{CP}^{15}\}$ is irreducible, hence the special hyperplane section $\Gamma_f \cap \{\{X_0 = 0\} \times \mathbb{CP}^{15}\}$ is connected and therefore the projection map $proj_{\mathbb{CP}^{15}}[\Gamma_f \cap \{\{X_0 = 0\} \times \mathbb{CP}^{15}\}] = f(I) \equiv S$, is connected. Hence, the variety $V_c \cup \Sigma = \widetilde{V}_c$ is compact, connected and embeds smoothly into $\mathbb{CP}^{15}$ via f.

c) Let t_1 and t_2 be the time variables respectively of the vector field $X_1 : \dot{x} = J\frac{\partial H_1}{\partial x} = \frac{\partial x}{\partial t_1}$ (13) and the second vector field $X_2 : \dot{x} = J\frac{\partial H_2}{\partial x} = \frac{\partial x}{\partial t_2}$ commuting with the first one as well. Let g^{t_1} and g^{t_2} be the flows generated respectively by the first and second vector fields respectively. For $p \in S$ and for small $\varepsilon > 0$, $g^{t_1}(p)$, $\forall t_1 \in \mathbb{C}$, $0 < |t_1| < \varepsilon$, is well defined and $g^{t_1}(p) \in V_c$. We define g^{t_2} on V_c by $g^{t_2}(q) = g^{-t_1}g^{t_2}g^{t_1}(q)$, $q \in U(p) = g^{-t_1}(U(g^{t_1}(p)))$, where $U(p)$ is a neighborhood of p. By commutativity one can see that g^{t_2} is independent of t_1; $g^{-t_1-\varepsilon_1}g^{t_2}g^{t_1+\varepsilon_1}(q) = g^{-t_1}g^{-\varepsilon_1}g^{t_2}g^{t_1}g^{\varepsilon_1} = g^{-t_1}g^{t_2}g^{t_1}(q)$. We affirm that $g^{t_2}(q)$ is holomorphic away from S. This because $g^{t_2}g^{t_1}(q)$ is holomorphic away from S and that g^{t_1} is holomorphic in $U(p)$ and maps bi-holomorphically $U(p)$ onto $U(g^{t_1}(p))$. Since the flows g^{t_1} and g^{t_2} are holomorphic and independent on S, we can show along the same lines as in the Arnold-Liouville theorem [5] that $\widetilde{V}_c$ is a torus. And that will done, by considering the holomorphic map $\mathbb{C}^2 \longrightarrow \widetilde{V}_c, (t_1, t_2) \longmapsto g^{t_1}g^{t_2}(p)$, for a fixed origin $p \in V_c$. Then $lattice = \{(t_1, t_2) \in \mathbb{C}^2 : g^{t_1}g^{t_2}(p) = p\}$, is a lattice of $\mathbb{C}^2$ and hence $\mathbb{T}_c^2 = \mathbb{C}^2/lattice \to \widetilde{V}_c$ is a biholomorphic diffeomorphism. Therefore $\widetilde{V}_c \subseteq \mathbb{CP}^{15}$ is conformal to a complex torus $\mathbb{T}_c^2$ and an abelian surface as a consequence of Chow theorem. We now prove that the divisor $S \equiv 2D$ defines on the abelian surface $\widetilde{V}_c$ a polarization either of type $(2,8)$. Note that the affine invariant surface V_c, has the following involution $\sigma : (y_1, y_2, x_1, x_2) \mapsto (y_1, -y_2, -x_1, x_2)$, which amounts to a reflection about some appropriately chosen origin on V_c. This map acts on the parameters of the Laurent solution (19) as follows $(t, \alpha, \beta) \mapsto (-t, -\alpha, \beta)$. Since $\mathcal{L}$ is symmetric ($\sigma^*\mathcal{L}\simeq\mathcal{L}$), σ can be lifted to $\mathcal{L}$ as an involution $\widetilde{\sigma}$ in two ways differing in sign and for each section (theta-function) $s \in H^0(\mathcal{L})$, we therefore have $\widetilde{\sigma}s = \pm s$. Recall that a section $s \in H^0(\mathcal{L})$ is called even (resp. odd) if $\widetilde{\sigma}s = +s$ (resp. $\widetilde{\sigma}s = -s$). Under $\widetilde{\sigma}$ the vector space $H^0(\mathcal{L})$ splits into an even and odd subspace $H^0(\mathcal{L}) = H^0(\mathcal{L})^{even} \oplus H^0(\mathcal{L})^{odd}$, with $H^0(\mathcal{L})^{even}$ containing all the even sections and $H^0(\mathcal{L})^{odd}$ all odd ones. Using the inverse formula [35, p.331], we see after a small

computation that

$$h^0\left(\mathcal{L}\right)^{even} \equiv \dim H^0\left(\mathcal{L}\right)^{even} = \frac{\delta_1\delta_2}{2} + 2^{-1+\sharp \text{ even } \delta_k}, k = 1, 2 \tag{7.4}$$

$$h^0\left(\mathcal{L}\right)^{odd} \equiv \dim H^0\left(\mathcal{L}\right)^{odd} = \frac{\delta_1\delta_2}{2} - 2^{-1+\sharp \text{ even } \delta_k}, k = 1, 2$$

By the classification theory of ample line bundles on abelian varieties, we have $\widetilde{V}_c \simeq \mathbb{C}^2/L_\Omega$ with period lattice given by the columns of the matrix $\Omega = \begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}$, $\text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$, according to (7.1) and (7.2) with $\delta_1\delta_2 = h^0\left(\mathcal{L}\right) = g(S) - 1 = 16$, $\delta_1 \mid \delta_2$, $\delta_i \in \mathbb{N}^*$. Hence we have three possibilities : (i) $\delta_1 = 1, \delta_2 = 16$. (ii) $\delta_1 = 2$, $\delta_2 = 8$, (iii) $\delta_1 = 4$, $\delta_2 = 4$. From formula (31), the corresponding line bundle $\mathcal{L}$ has in case (i), 9 even sections, 7 odd ones and in cases $(ii), (iii)$, 10 even sections, 6 odd ones. Among the functions of $\mathcal{L}$, there are 10 even and 6 odd functions for the involution σ, showing that cases $(ii), (iii)$ are the only alternative and the period matrices have the form $\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} Z$, $Z^\mathsf{T} = Z$, $\text{Im} Z > 0$ for the case (ii) and $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} Z$, $Z^\mathsf{T} = Z$, $\text{Im} Z > 0$ for the case (iii). On the other hand, the squares y_1^2, y_2^2 are meromorphic on $Jac(\Gamma)$, whereas the coordinates y_1, y_2 themselves are meromorphic on a 4-fold cover of $Jac(\Gamma)$. Hence $\widetilde{V}_c$ must be a 4-fold cover of $Jac(\Gamma)$ obtained by doubling two of the four period vectors of $Jac(\Gamma)$. As a result, $\widetilde{V}_c$ must have polarization $(2,2)$. To give a more explicit description of the periods of motion, observe that the variables t_1 and t_2 form a local system of coordinates on the torus $\widetilde{V}$ and consider their differentials dt_1 and dt_2 defined by $dt_i\left(X_j\right) = \delta_{ij}$, $1 \le i, j \le 2$. Then we show, using the Laurent solutions, that the differentials dt_1 and dt_2 restricted to the curve C (6.7), descend to two differentials on C :

$$dt_1\left.\right|_C = \frac{\alpha^2 d\alpha}{\beta}, = \omega_1, \qquad dt_2\left.\right|_C = \frac{d\alpha}{\beta} = \omega_2. \tag{7.5}$$

Clearly both ω_1 and ω_2 are holomorphic differentials. This concludes the proof of theorem 7.1.

8. Additional note and Conclusion

At the end of this paper, It is worth to mention some similar problems. Abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can then be translated into differential equations and their Laurent solutions. Among the results presented in this paper, there is an explicit calculation of invariants for hamiltonian systems which cut out an open set in an abelian variety and various algebraic curves related to these systems are given explicitly. The integrable systems presented here are interesting problems, particular to experts of abelian varieties who may want to see explicit examples of a correspondence for varieties defined by different algebraic curves. The methods used are primarily analytical but heavily inspired by algebraic geometrical methods. Strange as it may seem, the use of the Lax spectral curve technique may not give the tori correctly, but perhaps with period doubling, in contrast with the statement that the correct tori would be obtained by the Kowalewski-Painlevé analysis. This indicated a need for caution in interpretation of the result for tori calculated from use of the Lax spectral curve technique. Moser [34] was aware of a similar situation in the context of the Jacobi's geodesic flow problem on ellipsoids. Another striking example of this phenomenon already appears in the Euler-equations associated to a class of geodesic flow on $SO(4)$ for a left-invariant diagonal metric. Haine [19] has given a proof using Kowalewski-Painlevé analysis that the four quadrics in this problem intersect in the affine part of an abelian surface characterized as the Prym variety $Prym(C/C_0)$ where C is a genus 3 curve and is a double covering of an elliptic curve C_0. In [1], Adler and van Moerbeke linearize the problem, using the method of isospectral deformations, on a 2-dimensional Prym variety $Prym(K/K_0)$ of another genus 3 curve over an elliptic curve K_0. However, the two abelian surfaces $Prym(C/C_0)$ and $Prym(K/K_0)$ are not ismorphic but only isogenous, they intimately related in the precise sense that they are dual of each other. It turns out that these two abelian surfaces are isomorphic only up to multiplication of some (not all) periods by 2. The natural coordinates are not meromorphic on the tori but only their squares, if one linearizes the problem via Lax spectral curve technique. In contrast these natural coordinates are meromorphic on the invariant surfaces and on their natural completion into tori using the Kowalewski-Painlevé analysis. The relation between the two sets of tori is as follows : one set can be obtained from the other by doubling some but not all periods. The relationship between the curves C and K

is complicated : $Prym(K/K_0)\backslash\Pi = \Theta \cap Prym(K/K_0) = C$ where Θ is a translate of the Riemann's theta divisor on $Jac(K)$ et $\Pi \subset Prym(K/K_0)$ is a Zariski-open set and $\Theta \cap Prym^*(K/K_0) = \Theta \cap Prym(C/C_0) = K$ where Θ is a translate of the Riemann's theta divisor on $Jac(C)$.

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