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Ricci tensor of Hopf hypersurfaces in a complex space form

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ABSTRACT

We classify Hopf hypersurfaces in a non-flat complex space form whose Ricci tensor S satisfies $g((\nabla_X S)X, \xi) = 0$ for any vector field X tangent to ξ, where ξ is the structure vector field. We also classify real hypersurfaces with transversal Killing Ricci tensor satisfying $S\xi = \beta\xi$ for some function β.

Keywords: *Ricci tensor; real hypersurface; complex space form; transversal Killing tensor.*
AMS Subject Classification (2010): *Primary: 53B25 ; Secondary: 53C55; 53C25.*

1. Introduction

For real hypersurfaces in a complex space form $M_n(4c)$ of constant holomorphic sectional curvature $4c \neq 0$, it is an interesting problem to determine real hypersurfaces satisfying an additional condition on the Ricci tensor. Ki [3] showed that there are no real hypersurfaces with parallel Ricci tensor, $\nabla S = 0$, in $M_n(4c)$, $n \geq 3$. Several conditions that weaken the condition $\nabla S = 0$ are studied (cf., [4], [11]). On the other hand, when the structure vector field ξ is principal, then the real hypersurface is said to be Hopf. For Hopf hypersurfaces, fundamental formulas are well-organized form, and it was considered to be a natural condition. So kinds of classification theorems are given under this assumption (see, for example, [10]). If the Ricci tensor S satisfies $g((\nabla_X S)Y, Z) = 0$ for any vector field X, Y and Z orthogonal to ξ, then it is said to be η-parallel (Suh [11]). Suh and Maeda classified Hopf hypersurfaces of $M_n(4c)$ with η-parallel Ricci tensor ([11], [9]). In [8], Maeda gave a classification of Hopf hypersurfaces in $\mathbb{C}P^n$ with $\nabla_\xi S = 0$.

When S satisfies $g((\nabla_X S)X, \xi) = 0$ for any X orthogonal to ξ, we call S the *transversal η-Killing* Ricci tensor. In section 3, we classify Hopf hypersurfaces whose Ricci tensor S is transversal η-Killing.

In [6] and [7], the author showed that If $(\nabla_X S)Y$ is proportional (resp. perpendicular) to the structure vector field ξ for any vector fields X and Y orthogonal to ξ, then M is a Hopf hypersurface (resp. ruled real hypersurface), under an assumption that $S\xi = \beta\xi$, β being a function. On the other hand, for an almost contact metric manifold (M, ϕ, η, ξ, g), Cho [2] considered a condition that a (1,1)-tensor field T on M a *transversal Killing tensor field*, that is, it satisfies $(\nabla_X T)X = 0$ for any vector fields X to ξ.

Combining these with the results in section 3, we classify real hypersurfaces of $M_n(4c)$ whose Ricci tensor S is a transversal Killing tensor field and satisfies $S\xi = \beta\xi$ for some function β, in section 4. We notice that any Hopf hypersurfaces and ruled real hypersurfaces satisfy the condition that $S\xi = \beta\xi$, β beging a function.

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2. Preliminaries

Let $M_n(4c)$ denote the complex space from of complex dimension n (real dimension $2n$) of constant holomorphic sectional curvature $4c$. For the sake of simplicity, if $c > 0$, we only use $c = +1$ and call it the complex projective space $\mathbb{C}P^n$, and if $c < 0$, we just consider $c = -1$, so that we call it the complex hyperbolic space $\mathbb{C}H^n$. Throughout this paper, we suppose that $c \neq 0$. We denote by J the almost complex structure of $M_n(4c)$. The Hermitian metric of $M_n(4c)$ will be denoted by G.

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Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M_n(4c)$. We denote by g the Riemannian metric induced on M from G. We take the unit normal vector field N of M in $M_n(4c)$. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1,1), η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0$$

for any vector field X tangent to M. Moreover, we have

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad \eta(X) = g(X, \xi),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M_n(4c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M. We call A the *shape operator* of M. If the shape operator A of M satisfies $A\xi = \alpha\xi$ for some functions α, then M is said to be *Hopf*. We use the following (cf. [10])

Lemma 2.1. *Let M be a Hopf hypersurface of $M_n(4c)$, $n \geq 2$, $c \neq 0$. If a vector field X is orthogonal to ξ and $AX = \lambda X$, then*

$$(2\lambda - \alpha)A\phi X = (\lambda\alpha + 2c)\phi X,$$

where $\alpha = g(A\xi, \xi)$, and α is constant.

For the almost contact metric structure on M, we have

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X,Y)Z &= c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X \\ &\quad - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

From the equation of Gauss, the Ricci tensor S of M satisfies

$$\begin{aligned} g(SX, Y) &= (2n+1)cg(X,Y) - 3c\eta(X)\eta(Y) \\ &\quad + \mathrm{Tr}A g(AX, Y) - g(AX, AY), \end{aligned} \tag{2.1}$$

where $\mathrm{Tr}A$ is the trace of A. By (2.1), we have

$$\begin{aligned} (\nabla_X S)Y &= -3cg(\phi AX, Y)\xi - 3c\eta(Y)\phi AX \\ &\quad + (X\,\mathrm{Tr}\,A)AY + \mathrm{Tr}A(\nabla_X A)Y - A(\nabla_X A)Y \\ &\quad - (\nabla_X A)AY. \end{aligned} \tag{2.2}$$

We use the following results to prove our theorem (see [1], [5], [10], [12], [13]).

Theorem A. *Let M be a real hypersurface of $M_n(4c)$. Then the principal curvatures of M are constant and ξ is principal, if and only if, M is an open subset of a homogeneous hypersurfaces.*

Theorem B. *Let M be a homogeneous real hypersurface of $\mathbb{C}P^n$. Then M is congruent to one of the following*:

(A_1) *a geodesic sphere of radius* r, *where* $0 < r < \pi/2$,
(A_2) *a tube of radius* r *over a totally geodesic* $\mathbb{C}P^k$ $(1 \leq k \leq n-2)$, *where* $0 < r < \pi/2$,
(B) *A tube of radius* r *around a complex hyperquadric* $\mathbb{C}Q^{n-1}$, *where* $0 < r < \pi/4$.
(C) *a tube of radius* r *over a* $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{n-1}{2}}$, *where* $0 < r < \pi/4$ *and* $n(\geq 5)$ *is odd*,
(D) *a tube of radius* r *over a complex Grassmann* $G_{2,5}(\mathbb{C})$, *where* $0 < r < \pi/4$ *and* $n = 9$,
(E) *a tube of radius* r *over a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \pi/4$ *and* $n = 15$.

The principal curvatures are as follows.

	(A_1)	(A_2)	(B)	$(C,\ D,\ E)$
λ_1	$\cot r$	$\cot r$	$\cot(r-\pi/4)$	$\cot(r-\pi/4)$
λ_2		$-\tan r$	$\cot(r+\pi/4)$	$\cot(r+\pi/4)$
λ_3				$\cot r$
λ_4				$-\tan r$
α	$2\cot(2r)$	$2\cot(2r)$	$2\cot(2r)$	$2\cot(2r)$

The multiplicity $m(\mu)$ of each principal curvature μ of a homogeneous real hypersurface is as follows.

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
λ_1	$2n-2$	$2n-2k-2$	$n-1$	2	4	6
λ_2		$2k$	$n-1$	2	4	6
λ_3				$n-3$	4	8
λ_4				$n-3$	4	8
α	1	1	1	1	1	1

Theorem C. *Let* M *be a Hopf hypersurface of* $\mathbb{C}H^n$, $n \geq 2$. *If all principal curvatures are constant, then* M *is locally congruent to one of the following*:

(A_0) *A horosphere*,
$(A_{1,0})$ *A geodesic sphere of radius* r $(0 < r < \infty)$,
$(A_{1,1})$ *A tube of radius* r *around a totally geodesic* $\mathbb{C}H^{n-1}(c)$, *where* $0 < r < \infty$,
(A_2) *A tube of radius* r *around a totally geodesic* $\mathbb{C}H^l(c)$ $(1 \leq l \leq n-2)$, *where* $0 < r < \infty$,
(B) *A tube of radius* r *around a totally real totally geodesic* $\mathbb{R}H^n(c/4)$, *where* $0 < r < \infty$.

The principal curvatures of these real hypersurfaces are given as follows:

	(A_0)	$(A_{1,0})$	$(A_{1,1})$	(A_2)	(B)
λ_1	1	$\coth r$	$\tanh r$	$\coth r$	$\coth r$
λ_2				$\tanh r$	$\tanh r$
α	2	$2\coth(2r)$	$2\coth(2r)$	$2\coth(2r)$	$2\tanh(2r)$

3. The covariant derivative of the Ricci tensor

Let M be Hopf hypersurface of a complex space form $M_n(4c)$, $c \neq 0$. Then the shape operator A satisfies $Ae_i = a_i e_i$, $1 \leq i \leq 2n-2$, with respect to a suitable orthonormal frame $\{e_1, \cdots, e_{2n-2}, \xi\}$. We remark that if $A_i e_i = a_i e_i$, then

$$(2a_i - \alpha)A\phi e_i = (a_i\alpha + 2c)\phi e_i, \tag{3.1}$$

by Lemma 2.1. In the following, we put $A\phi e_i = \bar{a}_i \phi e_i$. Then we have

$$2a_i\bar{a}_i - a_i\alpha - \bar{a}_i\alpha - 2c = 0. \tag{3.2}$$

Lemma 3.1. *Let* M *be a Hopf hypersurface of* $M_n(4c)$. *The Ricci tensor* S *of* M *is transversal* η*-Killing if and only if*

$$(a_i - a_j)(-3c + \alpha \operatorname{Tr} A - \alpha^2 - a_i a_j)g(\phi e_i, e_j) = 0 \tag{3.3}$$

for $i, j = 1, \cdots, 2n-2$.

Proof. By (2.2), when M is a Hopf hypersurface of $M_n(4c)$, we obtain

$$\begin{aligned}&g((\nabla_{e_i}S)e_j,\xi)\\&=-3ca_ig(\phi e_i,e_j)+(\operatorname{Tr}A-\alpha-a_j)g((\nabla_iA)e_j,\xi)\\&=-3ca_ig(\phi e_i,e_j)+a_i(\operatorname{Tr}A-\alpha-a_j)(\alpha-a_j)g(\phi e_i,e_j)\\&=a_i(-3c+\alpha\operatorname{Tr}A-a_j\operatorname{Tr}A-\alpha^2+a_j^2)g(\phi e_i,e_j).\end{aligned}$$

So we have

$$\begin{aligned}0&=g((\nabla_{e_i}S)e_j,\xi)+g((\nabla_{e_j}S)e_i,\xi)\\&=(a_i-a_j)(-3c+\alpha\operatorname{Tr}A-\alpha^2-a_ia_j)g(\phi e_i,e_j).\end{aligned}$$

First we suppose that $g((\nabla_XS)X,\xi)=0$ for any X orthogonal to ξ. Since $g((\nabla_{X+Y}S)(X+Y),\xi)=0$ for any X and Y orthogonal to ξ, we have

$$g((\nabla_XS)Y,\xi)+g((\nabla_YS)X,\xi)=0.$$

So we have (3.3).

Next we suppose that the Ricci tensor S satisfies (3.3). Then we obtain

$$\begin{aligned}&g((\nabla_{e_i}S)e_j,\xi)+g((\nabla_{e_j}S)e_i,\xi)\\&=(a_i-a_j)(-3c+\alpha\operatorname{Tr}A-\alpha^2-a_ia_j)g(\phi e_i,e_j)=0\end{aligned}$$

for any i and j. Thus we get $g((\nabla_{e_i}S)e_i,\xi)=0$. Any vector field X orthogonal to ξ is represented as $X=\sum_i X_ie_i$. Using $g((\nabla_{e_i}S)e_j,\xi)=-g((\nabla_{e_j}S)e_i,\xi)$, we have

$$\begin{aligned}&g((\nabla_XS)X,\xi)\\&=\sum_{i,j}X_iX_jg((\nabla_{e_i}S)e_j,\xi)\\&=\sum_i {X_i}^2g((\nabla_{e_i}S)e_i,\xi)=0.\end{aligned}$$

So we have our result. □

Lemma 3.2. *Let M be a Hopf hypersurface of $M_n(4c)$. If the Ricci tensor S of M is transversal η-Killing, then M has at most 5 distinct constant principal curvatures.*

Proof. From Lemma 3.1, putting $e_j=\phi e_i$ in (3.3), we have $a_i=\bar{a}_i$ or

$$-3c+\alpha\operatorname{Tr}A-\alpha^2-a_i\bar{a}_i=0. \tag{3.4}$$

If $a_i=\bar{a}_i$, by (3.2), we see that a_i is a solution of the equation

$$x^2-\alpha x-c=0. \tag{3.5}$$

Since α is constant, a_i is also constant.

When $a_i\neq\bar{a}_i$, from (3.1), we have $2a_i=\alpha$ or $\bar{a}_i=\frac{a_i\alpha+2c}{2a_i-\alpha}$. If $2a_i=\alpha$ for some a_i, again from (3.1), we have $a_i\alpha+2c=0$, from which we see that $\alpha^2=-4c$ and $c<0$. Then M has 2 constant principal curvatures (see [1]).

In the following, we suppose $2a_i\neq\alpha$ for any i. From (3.4) and $\bar{a}_i=\frac{a_i\alpha+2c}{2a_i-\alpha}$, we see that a_i is a solution of the following

$$x^2\alpha-2(-4c+\alpha\operatorname{Tr}A-\alpha^2)x+\alpha(-3c+\alpha\operatorname{Tr}A-\alpha^2)=0. \tag{3.6}$$

We remark that $\bar{a}_i$ is also the solution of the above equation since (3.2) and (3.4) is symmetric with respect to a_i and $\bar{a}_i$.

Therefore, we see that the shape operator A has at most 5 distinct principal curvatures. We put λ_1 and $\lambda_2=\bar{\lambda_1}$ are solutions of (3.6), whose multiplicity is k, respectively. We suppose λ_3, λ_4 are solutions of (3.5) with multiplicity l and m, respectively. Then we have

$$\operatorname{Tr}A=k(\lambda_1+\bar{\lambda_1})+l\lambda_3+m\lambda_4+\alpha.$$

When $\alpha \neq 0$, since λ_1 and $\bar{\lambda_1}$ are solutions of (3.6), we have

$$\lambda_1 + \bar{\lambda_1} = \frac{2(-4c + \alpha \operatorname{Tr} A - \alpha^2)}{\alpha}.$$

From these equations, we obtain

$$\alpha(1-2k) \operatorname{Tr} A = (l\lambda_3 + m\lambda_4)\alpha - 8kc - 2k\alpha^2 + \alpha^2.$$

Since $\alpha(1-2k) \neq 0$, we see that $\operatorname{Tr} A$ is constant. By (3.6), λ_1 and $\bar{\lambda_1}$ are also constant. Hence all principal curvatures are constant.

Finally we consider the case that $\alpha = 0$. If $a_i \neq \bar{a}_i$, then a_i and $\bar{a}_i$ are solutions of (3.6). So we have $a_i = \bar{a}_i = 0$. This is a contradiction. So we have $a_i = \bar{a}_i$ for all a_i. Then the principal curvatures are $\sqrt{c}$ and 0 with multiplicities $2n-2$ and 1, respectively.

□

Using these lemmas, we prove the following theorem.

Theorem 3.1. *Let M be a Hopf hypersurface of a complex projective space $\mathbb{C}P^n$. If the Ricci tensor S of M satisfies $g((\nabla_X S)X, \xi) = 0$ for any X orthogonal to ξ, then M is locally congruent to one of the following*:

(A_1) *a geodesic sphere of radius r, where $0 < r < \pi/2$,*
(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$,*
(C) *a tube of radius r over a $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{n-1}{2}}$, where $\cot^2 2r = 5/(2n-6)$ and $n(\geq 5)$ is odd,*
(D) *a tube of radius r over a complex Grassmann $G_{2,5}(\mathbb{C})$, where $\cot^2 2r = 9/8$ and $n = 9$,*
(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $\cot^2 2r = 13/16$ and $n = 15$.*

Proof. By Lemma 3.2, when M is a Hopf hypersurface in $\mathbb{C}P^n$ with at most 5 distinct principal curvatures. Therefore M is locally congruent to one of the list in Theorem B.

When M is locally congruent to type (A_1), then $\lambda_1 = \cot r$ satisfies $\bar{\lambda_1} = \lambda_1$. Thus all principal curvatures satisfy (3.5). From Lemma 3.1, the Ricci operator S of all type (A_1) hypersurfaces satisfy $g((\nabla_X S)X, \xi) = 0$ for X orthogonal to ξ. Similarly, since $\bar{\lambda_1} = \lambda_1$ and $\bar{\lambda_2} = \lambda_2$, type (A_2) hypersurfaces also satisfy that condition.

Next we consider the case that M is locally type (B). The principal curvatures $\lambda_1 = \cot(r - \frac{\pi}{4})$ and $\lambda_2 = \cot(r + \frac{\pi}{4})$ satisfies $\bar{\lambda_1} = \lambda_2$. If λ_1 and λ_2 are solutions of (3.6), then $\lambda_1\lambda_2 = -3 + \alpha \operatorname{Tr} A - \alpha^2 = -1$. Since we have

$$\operatorname{Tr} A = (n-1)\left(\cot(r - \frac{\pi}{4}) + \cot(r + \frac{\pi}{4})\right) + \alpha,$$

we see that

$$\begin{aligned} 1 &= (n-1)\cot 2r \left(\cot(r - \frac{\pi}{4}) + \cot(r + \frac{\pi}{4})\right) \\ &= -2n + 2. \end{aligned}$$

This is a contradiction. So type (B) hypersurfaces do not satisfy $g((\nabla_X S)X, \xi) = 0$, $X \perp \xi$.

Next we consider the case that M has 5 distinct constant principal curvatures. We put

$$\begin{aligned} &\lambda_1 = \cot(r - \frac{\pi}{4}),\ \lambda_2 = \cot(r + \frac{\pi}{4}),\ \lambda_3 = \cot r, \\ &\lambda_4 = -\tan r,\ \alpha = 2\cot(2r), \end{aligned}$$

and their multiplicities are represented by $m(\lambda_1) = m(\lambda_2) = k$, $m(\lambda_3) = m(\lambda_4) = l$. Since λ_1 and λ_2 are solutions of (3.6), similar computation as the case of type (B) shows that $\operatorname{Tr} A \cdot \alpha - \alpha^2 = 2$. On the other hand, we obtain

$$\begin{aligned} \operatorname{Tr} A - \alpha &= k(\lambda_1 + \lambda_2) + l(\lambda_3 + \lambda_4) \\ &= \frac{4k\tan^2 r - l(1 - \tan^2 r)^2}{(\tan^2 r - 1)\tan r}. \end{aligned}$$

Since $\alpha = 2\cot 2r$, we have

$$\alpha(\operatorname{Tr} A - \alpha) = -4k + l\left(\frac{1 - \tan^2 r}{\tan r}\right)^2 = 2,$$

from which we see that

$$\cot^2 2r = \frac{1+2k}{2l}.$$

When M is locally congruent to type (C), then $k = 2$ and $l = n-3$. Thus we have $\cot^2 2r = \frac{5}{2(n-3)}$. Next, when M is locally congruent to (D), we obtain $\cot^2 2r = \frac{9}{8}$. Finally, if M is locally congruent to (E), then we have $\cot^2 2r = \frac{13}{16}$. □

Theorem 3.2. *Let M be a Hopf hypersurface of a complex hyperbolic space $\mathbb{C}H^n$. If the Ricci tensor S of M satisfies $g((\nabla_X S)X, \xi) = 0$ for any X orthogonal to ξ, then M is locally congruent to one of the following*:

(A_0) *A horosphere,*
$(A_{1,0})$ *A geodesic sphere of radius r $(0 < r < \infty)$,*
$(A_{1,1})$ *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$,*
(A_2) *A tube of radius r around a totally geodesic $\mathbb{C}H^l(c)$ $(1 \le l \le n-2)$, where $0 < r < \infty$,*

Proof. Similar argument as the proof of Theorem 3.1 shows that all type (A_0), $(A_{1,0})$ and $(A_{1,1})$ and (A_2) hypersurface satisfies the condition $g((\nabla_X S)X, \xi) = 0$ for any X orthogonal to ξ.

Suppose M is locally congruent to type (B). Then $\lambda_1 = \coth r$ and $\lambda_2 = \tanh r$ are solutions of (3.6). Then we have $\alpha \operatorname{Tr} A - \alpha^2 = -2$. So we have

$$\tanh 2r(n-1)(\coth r + \tanh r) = -1.$$

By the straightforward computation, we have $2(n-1) = -1$. This is a contradiction. □

4. Transversal Killing tensor

For a Riemannian manifold with Riemannian connection ∇, a (1,1)-tensor field T is called a *Killing tensor field* if it satisfies $(\nabla_X T)X = 0$ or $(\nabla_X T)Y + (\nabla_Y T)X = 0$ for any vector fields X and Y. If T is symmetric, then we easily see that T is parallel. For an almost contact metric manifold (M, ϕ, η, ξ, g), we call a (1,1)-tensor field T on M a *transversal Killing tensor field* if it satisfies $(\nabla_X T)X = 0$ or $(\nabla_X T)Y + (\nabla_Y T)X = 0$ for any vector fields X and Y orthogonal to ξ (see Cho[2]). Cho [2] studied a real hypersurfaces in a non-flat complex space form whose shape operator is a transversal Killing tensor field. In this section, we study a real hypersurface M whose Ricci tensor S is a transversal Killing tensor field. We summarize theorems for later use.

Theorem D ([7]). Let M be a connected real hypersurface of $M_n(4c)$, $n \ge 3$, and suppose that the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β.

(1) If $(\nabla_X S)Y$ is proportional to the structure vector field ξ for any vector fields X and Y orthogonal to ξ, then M is a Hopf hypersurface.
(2) If $(\nabla_X S)Y$ is perpendicular to the structure vector field ξ for any vector fields X and Y orthogonal to the structure vector field ξ, then M is a ruled real hypersurface.

When $n = 2$, the author gave a corresponding result in [6].

We use the following theorems for hypersurfaces with η-parallel Ricci tensor (see [9], [11]).

Theorem E. *Let M be a Hopf hypersurface of $\mathbb{C}P^n$, $n \ge 2$ with η-parallel Ricci tensor. Then M is congruent to one of real hypersurfaces of types (A_1), (A_2) and (B) or a non-homogeneous real hypersurface with $A\xi = 0$ in $\mathbb{C}P^2$.*

Theorem F. *Let M be a Hopf hypersurface of $\mathbb{C}H^n$, $n \ge 2$ with η-parallel Ricci tensor. Then M is congruent to one of real hypersurfaces of types (A_0), $(A_{1,0})$, $(A_{1,1})$, (A_2) and (B) or a non-homogeneous real hypersurface with $A\xi = 0$ in $\mathbb{C}H^2$.*

First, we prove the following lemma.

Lemma 4.1. *Let M be a connected real hypersurface of $M_n(4c)$, $n \ge 2$, and suppose that the Ricci tensor S of M is transversal Killing tensor field and satisfies $S\xi = \beta\xi$ for some function β, then M is a Hopf hypersurface with η-parallel Ricci tensor.*

Proof. By the assumption we have $(\nabla_X S)X = 0$ for any X orthogonal to ξ, which is equivalent to $(\nabla_X S)Y + (\nabla_Y S)X = 0$ for any vector fields X and Y orthogonal to ξ. Since S is symmetric, it follows that

$$0 = g((\nabla_X S)X, Y) = -g((\nabla_Y S)X, X).$$

This implies that $g((\nabla_X S)Y, Z) = 0$ for any vector fields X, Y and Z orthogonat to ξ. Hence, the Ricci tensor S is η-parallel. Combining this to Theorem D (1), M is a Hopf hypersurface.

□

If the Ricci tensor S of M is transversal Killing tensor field, then S is transversal η-Killing. Therefore, if a real hypersurface of $M_2(c)$ with $A\xi = 0$ satisfies the condition that the Ricci tensor S of M is transversal Killing tensor field and $S\xi = \beta\xi$ for some function β, then Lemma 2.1 and Lemma 3.1 imply that $a_1 a_2 = c \neq 0$ and $(a_1 - a_2)(a_1 a_2 - 3c) = 0$. Thus M is a totally η-umbilical real hypesurface. Thus a non-homogeneous real hypersurface with $A\xi = 0$ in $M_n(4c)$ does not satisfy the condition that the Ricci tensor S of M is transversal Killing tensor field and $S\xi = \beta\xi$ for some function β.

From Theorems 3.1, 3.2 we also see that real hypersurfaces of type (B) do not satisfy the condition that S is transversal Killing tensor field and $S\xi = \beta\xi$ for some function β. Therefore we have the following theorems.

Theorem 4.1. *Let M be a real hypersurface of $\mathbb{C}P^n$, $n \geq 2$. If the Ricci tensor S of M is transversal Killing tensor field and satisfies $S\xi = \beta\xi$ for some function β, then M is locally congruent to one of the types (A_1) and (A_2).*

Theorem 4.2. *Let M be a real hypersurface of $\mathbb{C}H^n$, $n \geq 2$. If the Ricci tensor S of M is transversal Killing tensor field and satisfies $S\xi = \beta\xi$ for some function β, then M is locally congruent to one of the types (A_0), $(A_{1,0})$, $(A_{1,1})$ and (A_2).*
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