A δ-Invariant for QR-Submanifolds in Quaternion Space Forms

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ABSTRACT

Starting from an inequality involving the invariant $\delta(\mathcal{D})$ for an anti-holomorphic submanifold of a complex space form [1] and using optimization methods on Riemannian manifolds, we establish a corresponding inequality for the invariant $\delta(\mathcal{D}^\perp)$ defined on QR-submanifolds in quaternion space forms, in terms of the squared mean curvature. We obtain a relationship between intrinsic and extrinsic invariants for QR-submanifolds of quaternion space forms.

1. Introduction

The fact that every Riemannian manifold can be regarded as a Riemannian submanifold isometrically embedded in some Euclidean space with sufficiently high codimension (according to the embedding theorem of J.F. Nash [6]) gives the opportunity to use the extrinsic help in Riemannian geometry. One of the most fundamental problems in the theory of submanifold is to find simple relationships between intrinsic and extrinsic invariants of a submanifold.

In this paper, we consider δ-invariants of QR-submanifolds of a quaternion space forms; they are very important among intrinsic invariants, being different in nature from the classical Ricci and scalar curvature. The non-trivial δ-invariants are obtained from scalar curvature by substracting a certain amount of sectional curvatures.

Let $\tilde{M}$ be a Kaehler manifold with complex structure J and let N be a Riemannian manifold isometrically immersed in $\tilde{M}$. One denotes by $\mathcal{D}_x$, $x \in N$, the maximal complex subspace $T_x N \cap J(T_x N)$ of the tangent space $T_x N$ of N. If the dimension of $\mathcal{D}_x$ is constant for all $x \in N$, then $\mathcal{D} : x \to \mathcal{D}_x$ defines a *holomorphic distribution $\mathcal{D}$* on N. A subspace ν of $T_x N$, $x \in N$, is called *totally real* if $J(\nu)$ is a subspace of the normal space $T_x^\perp N$ at x. If each tangent space of N is totally real, then N is called a *totally real submanifold* of the Kaehler manifold $\tilde{M}$.

If the orthogonal complementary distribution $\mathcal{D}^\perp$ of the holomorphic distribution $\mathcal{D}$ is totally real, i.e., $TN = \mathcal{D} \oplus \mathcal{D}^\perp$, $J\mathcal{D}_x^\perp \subset T_x^\perp N$, $x \in N$, then the submanifold N is called a *CR-submanifold*.

The totally real distribution $\mathcal{D}^\perp$ of every CR-submanifold of a Kaehler manifold is an integrable distribution ([3]).

In order to give some answers to an open question concerning minimal immersions proposed by S. S. Chern in the 1960's and to provide some applications of the Nash embedding theorem, B.-Y. Chen introduced in early 1990's the notion of *δ-invariants*. In the case of a CR-submanifold N of a Kaehler manifold, Chen introduced two δ-invariants $\delta(\mathcal{D})$ and $\delta(\mathcal{D}^\perp)$, called *CR δ-invariants*, defined by Chen in [4]:

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x),$$

$$\delta(\mathcal{D}^\perp)(x) = \tau(x) - \tau(\mathcal{D}_x^\perp),$$

where τ is the scalar curvature of N and $\tau(\mathcal{D}_x)$ and $\tau(\mathcal{D}^\perp)$ are the scalar curvature of the holomorphic distribution $\mathcal{D}$ and totally real distribution $\mathcal{D}^\perp$ of N, respectively.

In [1], Al-Solamy, Chen and Deshmukh proved an inequality involving the δ-invariant $\delta(\mathcal{D})$, for an anti-holomorphic submanifold in a complex space form, in terms of the squared mean curvature.

In 1986, A. Bejancu [2] introduced the notion of QR-submanifolds as a generalization of real hypersurfaces of a quaternion Kaehler manifold (see also [8]).

Let $\tilde{M}$ be a quaternion Kaehler manifold and N be a real submanifold of $\tilde{M}$. N is called a *QR-submanifold* if there exists a vector subbundle ν of the normal bundle such that we have

$$J_\alpha(\nu_x) = \nu_x \text{ and } J_\alpha(\nu_x^\perp) \subset T_x N, \ x \in N, \ \alpha = \overline{1,3},$$

where $\nu^\perp$ is the complementary orthogonal bundle.

Taking into account the research done until now ([5]), we remark that quaternion CR-submanifolds and QR-submanifolds have very little in common (see also section 2).

In the present paper, we give a corresponding inequality to the inequality given in [1], for $\delta(\mathcal{D}^\perp)$ in the case of a QR-submanifold of a quaternion space form with minimal codimension, i.e., $\dim \nu_x = 0$.

2. Basics on quaternion manifolds and submanifolds

Let $\tilde{M}$ be a Riemannian manifold and $N \subset \tilde{M}$ a Riemannian submanifold of $\tilde{M}$ with the induced Riemannian metric. We denote by TN and $T^\perp N$ the tangent bundle, respectively the normal bundle of N, and by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of N and $\tilde{M}$, respectively.

The Gauss and Weingarten formulae are given by:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

$\forall X, Y \in \Gamma(TN), \ V \in \Gamma(T^\perp N)$, where $\nabla^\perp$ is the normal connection on $T^\perp N$.

One has $g(h(X,Y),V) = g(A_V X, Y)$.

If $\tilde{M}$ is a $4n$-dimensional manifold with the Riemannian metric g, then $\tilde{M}$ is called a *quaternion Kaehler manifold* if there exists a 3-dimensional vector bundle σ of local basis of almost Hermitian structures J_1, J_2, J_3 such that

$$J_\alpha \circ J_{\alpha+1} = -J_{\alpha+1} \circ J_\alpha = J_{\alpha+2}, \ J_\alpha^2 = -\operatorname{Id},$$

where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

In this case, σ is called an *almost quaternion structure* on $\tilde{M}$, $\{J_1, J_2, J_3\}$ is the *canonical local basis of* σ and $(\tilde{M}, \sigma)$ is called an *almost quaternion manifold*, with $\dim \tilde{M} = 4m, \ m \geq 1$.

A Riemannian metric $\tilde{g}$ on $\tilde{M}$ is said to be *adapted to the almost quaternion structure* σ if it satisfies

$$\tilde{g}(J_\alpha X, J_\alpha Y) = \tilde{g}(X,Y), \ \forall \alpha = \overline{1,3}.$$

Then $(\tilde{M}, \sigma, \tilde{g})$ is called an *almost quaternion Hermitian manifold*.

If σ is parallel with respect to $\tilde{\nabla}$, then $(\tilde{M}, \sigma, \tilde{g})$ is called a *quaternion Kaehler manifold*. Equivalently, there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that $\forall \alpha = \overline{1,3}, \ (\tilde{\nabla}_X J_\alpha)(X) = \omega_{\alpha+2}(X)J_{\alpha+1} - \omega_{\alpha+1}J_{\alpha+2}$, where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

Remark 2.1. Any quaternion Kaehler manifold is an Einstein manifold (for $\dim \tilde{M} \geq 4$).

Let $(\tilde{M}, \sigma, \tilde{g})$ be a quaternion Kaehler manifold and X be a non-null vector on $\tilde{M}$. Then the 4-plane spanned by $\{X, J_1 X, J_2 X, J_3 X\}$, denoted by $Q(X)$, is called a *quaternion 4-plane*. Any 2-plane in $Q(X)$ is called a *quaternion plane*. The sectional curvature of a quaternion plane is called a *quaternion sectional curvature*.

A quaternion Kaehler manifold is called a *quaternion space form* if its quaternion sectional curvature is constant, say c. So, $(\tilde{M}, \sigma, \tilde{g})$ is a quaternion space form if and only if

$$\tilde{R}(X,Y)Z = \frac{c}{4}\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y +$$

$$+ \sum_{\alpha=1}^{3} [\tilde{g}(Z, J_\alpha Y)J_\alpha X - \tilde{g}(Z, J_\alpha X)J_\alpha Y + 2\tilde{g}(X, J_\alpha Y)J_\alpha Z]\},$$

$\forall X, Y, Z \in \Gamma(T\tilde{M})$.

For a submanifold N of $\tilde{M}$, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pN and $\{e_{n+1}, \ldots, e_{4m}\}$ an orthonormal basis of $T_p^\perp N$, $p \in N$, the *mean curvature vector* is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

One denotes by

$$\|h\|^2(p) = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For a quaternion Kaehler manifold, we have

$$(\tilde{\nabla}_X J_\alpha)(X) = \sum_{\beta=1}^{3} Q_{\alpha\beta}(X)J_\beta, \ \alpha = \overline{1,3}, \ \forall X \in \Gamma(T\tilde{M}),$$

where $Q_{\alpha\beta}$ are certain 1-forms locally defined on $\tilde{M}$ such that $Q_{\alpha\beta} + Q_{\beta\alpha} = 0$.

Let $\tilde{M}$ be a quaternion Kaehler manifold and N be a real submanifold of $\tilde{M}$. N is called a *QR-submanifold* if there exists a vector subbundle ν of the normal bundle such that

$$J_\alpha(\nu_x) = \nu_x \ \text{ and } \ J_\alpha(\nu_x^\perp) \subset T_xN, \ x \in N, \ \alpha = \overline{1,3},$$

where $\nu^\perp$ is the complementary orthogonal bundle.

Let $\mathcal{D}_{\alpha x} = J_\alpha(\nu_x^\perp)$, $\mathcal{D}_x^\perp = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}$ a $3q$-dimensional distribution $\mathcal{D}^\perp : x \to \mathcal{D}_x^\perp$ globally defined on N, where $q = \dim \nu_x^\perp$. One has

$$J_\alpha(\mathcal{D}_{\alpha x}) = \nu_x^\perp, \ J_\alpha(\mathcal{D}_{\beta x}) = \mathcal{D}_{\gamma x}, \ \forall x \in N,$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

$\mathcal{D}$ is the orthogonal complementary distribution of $\mathcal{D}^\perp$ in TN and $J_\alpha(\mathcal{D}_x) = \mathcal{D}_x$. $\mathcal{D}$ is called the *quaternion distribution*.

So

$$T\tilde{M} = TN \oplus T^\perp N, \ TN = \mathcal{D} \oplus \mathcal{D}^\perp,$$

$$T^\perp N = \nu \oplus \nu^\perp, \ \nu, \nu^\perp \subset T^\perp N, \ \mathcal{D}_x^\perp = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}.$$

Recall that N is a *quaternion CR-submanifold* (see [5]) if it admits a differential quaternion distribution $\mathcal{D}$ such that its orthogonal complementary distribution $\mathcal{D}^\perp$ is totally real, i.e. $J_\alpha(\mathcal{D}_x^\perp) \subset T_x^\perp N$, for $\alpha = 1, 2, 3$ and $\forall x \in N$.
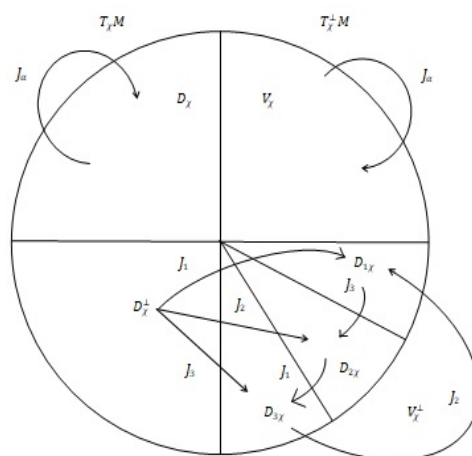

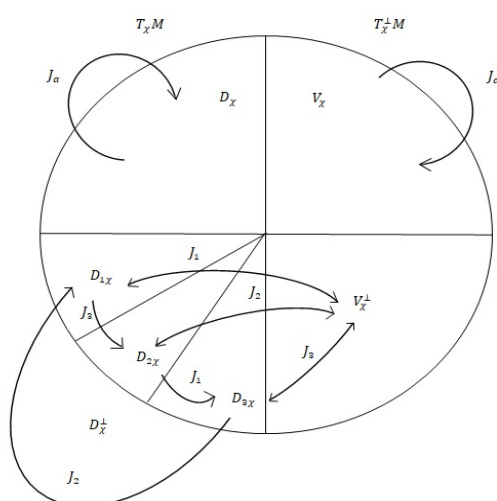**Figure 1.** Quaternion CR-submanifolds

Figure 2. QR-submanifolds

The differences between the quaternion CR-submanifolds and QR-submanifolds in quaternion space forms can be represented in the Figure 1 and Figure 2.

For $Y \in \Gamma(TN)$ we consider the decomposition $J_\alpha Y = \Phi_\alpha Y + F_\alpha Y$, $\alpha = \overline{1,3}$; $\Phi_\alpha Y$, $F_\alpha Y$ are the tangential and normal components of $J_\alpha Y$, respectively.

For $V \in \Gamma(T^\perp N)$ we consider the decomposition $J_\alpha V = t_\alpha V + f_\alpha V$, $\alpha = \overline{1,3}$; $t_\alpha V$, $f_\alpha V$ are the tangential and normal components of $J_\alpha V$, respectively.

N is called *mixed geodesic* if $h(X,Y) = 0$, $\forall X \in \Gamma(\mathcal{D})$, $Y \in \Gamma(\mathcal{D}^\perp)$.

Let $\pi = \mathrm{sp}\{X,Y\}$ be a tangent plane to $\tilde{M}$ at a point $p \in \tilde{M}$. The sectional curvature of π is

$$\tilde{K}(\pi) = \frac{\tilde{R}(X,Y,X,Y)}{\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}^2(X,Y)}.$$

From

$$\tilde{R}(X,Y)Z = \frac{c}{4}\{\tilde{g}(Z,Y)X - \tilde{g}(X,Z)Y +$$

$$+ \sum_{\alpha=1}^{3}[\tilde{g}(Z,J_\alpha Y)J_\alpha X - \tilde{g}(Z,J_\alpha X)J_\alpha Y + 2\tilde{g}(X,J_\alpha Y)J_\alpha Z]\},$$

we obtain

$$\tilde{K}(X \wedge Y) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3}\tilde{g}^2(J_\alpha X, Y)\right],$$

$\forall X,Y \in \Gamma(T_p\tilde{M})$, $p \in \tilde{M}$, unit vector fields.

By the Gauss equation, we have

$$K(X \wedge Y) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3}\tilde{g}^2(J_\alpha X, Y)\right] + \tilde{g}(h(X,X),h(Y,Y)) - \tilde{g}(h(X,Y),h(X,Y)).$$

We recall the following result.

Let (N,g) be a Riemannian submanifold of a Riemannian manifold $(\tilde{M},\tilde{g})$ and $f \in \mathcal{C}^\infty(\tilde{M})$. We attach the following *Optimum Problem*:

(2.1)
$$\min_{x \in N} f(x).$$

Theorem 2.1. *[7] If $x_0 \in N$ is a solution of the problem* (2.1), *then*
  a) $(\operatorname{grad} f)(x_0) \in T_{x_0}^{\perp} N$;
  b) the bilinear form $\beta : T_{x_0} N \times T_{x_0} N \to \mathbb{R}$,

$$\beta(X, Y) = \operatorname{Hess}_f(X, Y) + \tilde{g}(h(X, Y), (\operatorname{grad} f)(x_0))$$

is positive semidefinite, where h is the second fundamental form of the submanifold N in $\tilde{M}$.

Remark 2.2. If β is negative semidefinite, then we have a solution of $\max_{x \in N} f(x)$.

3. An inequality for a new δ-invariant

If $N \subset \tilde{M}$ is a QR-submanifold of minimal codimension, i.e., $\dim \nu_x = 0$ for $x \in M$, we consider the following orthonormal bases:

$$\{e_1, \ldots, e_n\} \subset \mathcal{D}_x;$$

$$\{J_1 e_{n+1}, \ldots, J_1 e_{n+q}; \ J_2 e_{n+1}, \ldots, J_2 e_{n+q}; \ J_3 e_{n+1}, \ldots, J_3 e_{n+q}\} \subset \mathcal{D}_x^{\perp};$$

$$\{ e_{n+1}, \ldots, e_{n+q}\} \subset T_x^{\perp} N.$$

For $x \in N$, we have

$$\dim \mathcal{D}_x = n; \ \dim \mathcal{D}_x^{\perp} = 3q; \ \dim T_x N = n + 3q;$$

$$\dim \nu_x = 0, \ \dim T_x^{\perp} N = q = \dim \nu_x^{\perp}.$$

We define the following QR δ-*invariant* $\delta(\mathcal{D}^{\perp})$ by

$$\delta(\mathcal{D}^{\perp})(x) = \tau(x) - \tau(\mathcal{D}_x^{\perp}), \ x \in \tilde{M},$$

where τ and $\tau(\mathcal{D}^{\perp})$ denote the scalar curvature of N and the scalar curvature of the distribution $\mathcal{D}^{\perp} \subset TN$, respectively.

In the following, we will use the convention on range of indices, unless mentioned otherwise:

$$i, j, k = \overline{1, n} \ ; \ \alpha, \beta, \gamma = \overline{1, 3} \ ; \ r, s, t = \overline{n + 1, n + q} \ ; \ A, B, C = \overline{1, n + q}.$$

In [1], the authors proved an inequality for $\delta(\mathcal{D})$ for an anti-holomorphic submanifold of a complex space form:

Theorem 3.1. *[1] Let N be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(c)$ with $h = \operatorname{rank}_{\mathbf{C}} \mathcal{D} \geq 1$ and $p = \operatorname{rank} \mathcal{D}^{\perp} \geq 2$. Then we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(2h+p)^2}{2(p+2)} \cdot \|H\|^2 + \frac{p}{2}(4h + p - 1) \cdot \frac{c}{4}.$$

The equality sign holds identically if and only if the following three conditions are satisfied:
(a) N is $\mathcal{D}$-minimal,
(b) N is mixed geodesic, and
(c) there exists an orthonormal frame $\{e_{2h+1}, \ldots, e_n\}$ of $\mathcal{D}^{\perp}$ such that the second fundamental form σ of N satisfies $\sigma_{rr}^r = 3\sigma_{ss}^r$, for $2h + 1 \leq r \neq s \leq 2h + p$, and
$\sigma_{rs}^t = 0$ for distinct $r, s, t \in \{2h + 1, \ldots, 2h + p\}$.

The main result of our study is the following inequality involving $\delta(\mathcal{D}^{\perp})$ for a QR-submanifold of a quaternion space form.

Theorem 3.2. *Let N be a QR-submanifold of minimal codimension of a quaternion space form $\tilde{M}(c)$, $\dim \mathcal{D}_x = n, \dim \mathcal{D}_x^{\perp} = 3q, \dim \nu_x = 0, \ \dim \nu_x^{\perp} = q, \ x \in N$. Then we have:*

$$(*) \qquad\qquad \delta(\mathcal{D}^{\perp}) \leq \frac{n(n+3q)^2}{2(n+1)} \cdot \|H\|^2 + \frac{n(n+6q+8)}{2} \cdot \frac{c}{4}.$$

The equality sign holds identically if and only if the following three conditions are satisfied:
(a) N is mixed geodesic,
(b) the distribution $\mathcal{D}$ is totally umbilical, and
(c) there exists an orthonormal frame

$$\{J_1 e_{n+1}, \ldots, J_1 e_{n+q};\ J_2 e_{n+1}, \ldots, J_2 e_{n+q};\ J_3 e_{n+1}, \ldots, J_3 e_{n+q}\}$$

of $\mathcal{D}_x^\perp$ such that the second fundamental form σ of N satisfies

$$h_{ij}^r = 0,\ i,j = \overline{1,n},\ i \neq j,\ r = \overline{n+1, n+q}.$$

Proof. With the above notations, for $x \in N$ we have

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) + \sum_{\alpha,\beta=1}^{3} \sum_{r,s=n+1}^{n+q} K(J_\alpha e_r \wedge J_\beta e_s) + \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} K(e_i \wedge J_\alpha e_r).$$

$$\tau(\mathcal{D}_x^\perp) = \sum_{\alpha,\beta=1}^{3} \sum_{r,s=n+1}^{n+q} K(J_\alpha e_r \wedge J_\beta e_s).$$

From these two relations, we obtain

$$(3.1) \qquad \delta(\mathcal{D}^\perp)(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) + \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} K(e_i \wedge J_\alpha e_r).$$

Applying the Gauss equation for $X = e_i$, $Y = e_j$, $i,j = \overline{1,n}$, $i \neq j$, we get

$$(3.2) \qquad K(e_i \wedge e_j) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3} \tilde{g}^2(J_\alpha e_i, e_j)\right] + \tilde{g}(h(e_i, e_i), h(e_j, e_j)) - \tilde{g}(h(e_i, e_j), h(e_i, e_j)).$$

Because $J_\alpha e_i \in \mathcal{D}$ and $J_\alpha e_r \in \mathcal{D}^\perp$, by applying the Gauss equation for $X = e_i$ and $Y = J_\alpha e_r$, $i = \overline{1,n}$, $r = \overline{n+1, n+q}$, $\alpha = \overline{1,3}$, we have

$$(3.3) \qquad K(e_i \wedge J_\alpha e_r) = \frac{c}{4} + \tilde{g}(h(e_i, e_i), h(J_\alpha e_r, J_\alpha e_r)) - \tilde{g}(h(e_i, J_\alpha e_r), h(e_i, J_\alpha e_r)).$$

Using the relations (3.2) and (3.3) in (3.1), it follows that

$$\delta(\mathcal{D}^\perp)(x) = \sum_{1 \leq i < j \leq n} \left\{\frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3} \tilde{g}^2(J_\alpha e_i, e_j)\right] + \tilde{g}(h(e_i, e_i), h(e_j, e_j)) - \tilde{g}(h(e_i, e_j), h(e_i, e_j))\right\} +$$

$$+ \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} \left[\frac{c}{4} + \tilde{g}(h(e_i, e_i), h(J_\alpha e_r, J_\alpha e_r)) - \tilde{g}(h(e_i, J_\alpha e_r), h(e_i, J_\alpha e_r))\right] =$$

$$= \frac{n(n-1)}{2} \cdot \frac{c}{4} + \frac{3c}{4} \sum_{\alpha=1}^{3} \sum_{1 \leq i < j \leq n} \tilde{g}^2(J_\alpha e_i, e_j) +$$

$$+ \sum_{1 \leq i < j \leq n} [\tilde{g}(h(e_i, e_i), h(e_j, e_j)) - \tilde{g}(h(e_i, e_j), h(e_i, e_j))] +$$

$$+ 3nq \cdot \frac{c}{4} + \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} [\tilde{g}(h(e_i, e_i), h(J_\alpha e_r, J_\alpha e_r)) - \tilde{g}(h(e_i, J_\alpha e_r), h(e_i, J_\alpha e_r))];$$

thus we obtain

$$(3.4) \qquad \delta(\mathcal{D}^\perp)(x) = \frac{n(n+6q-1)}{2} \cdot \frac{c}{4} + \frac{3c}{4} \sum_{\alpha=1}^{3} \sum_{1 \leq i < j \leq n} \tilde{g}^2(J_\alpha e_i, e_j) +$$

$$+ \sum_{1 \leq i < j \leq n} \tilde{g}(h(e_i, e_i), h(e_j, e_j)) + \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} \tilde{g}(h(e_i, e_i), h(J_\alpha e_r, J_\alpha e_r)) -$$

$$- \sum_{1 \leq i < j \leq n} \tilde{g}(h(e_i, e_j), h(e_i, e_j)) - \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} \tilde{g}(h(e_i, J_\alpha e_r), h(e_i, J_\alpha e_r)).$$

Obviously

$$(3.5) \qquad \|P_\alpha\|^2 = \sum_{i,j=1}^{n} \tilde{g}^2(J_\alpha e_i, e_j) = n.$$

Taking into account that the term $\sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} \tilde{g}(h(e_i, J_\alpha e_r), h(e_i, J_\alpha e_r))$ is positive (being a sum of squares), the relations (3.4) and (3.5) imply

$$(3.6) \qquad \delta(\mathcal{D}^\perp)(x) \leq \frac{n(n+6q-1)}{2} \cdot \frac{c}{4} + 9n\frac{c}{8} + \sum_{1 \leq i < j \leq n} \tilde{g}(h(e_i, e_i), h(e_j, e_j)) +$$

$$+ \sum_{\alpha=1}^{3} \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} \tilde{g}(h(e_i, e_i), h(J_\alpha e_r, J_\alpha e_r)) - \sum_{1 \leq i < j \leq n} \tilde{g}(h(e_i, e_j), h(e_i, e_j)) =$$

$$= \frac{n(n+6q+8)}{2} \cdot \frac{c}{4} + \sum_{1 \leq i < j \leq n} \sum_{r=n+1}^{n+q} h_{ii}^r h_{jj}^r +$$

$$+ \sum_{i=1}^{n} \sum_{r,s=n+1}^{n+q} h_{ii}^s \left[\tilde{h}_{rr}^s + \tilde{\tilde{h}}_{rr}^s + \tilde{\tilde{\tilde{h}}}_{rr}^s \right] - \sum_{1 \leq i < j \leq n} \sum_{r=n+1}^{n+q} \left(h_{ij}^r \right)^2,$$

where

$$(3.7.1) \qquad h_{ij}^r = \tilde{g}(h(e_i, e_j), e_r),$$

$$(3.7.2) \qquad \tilde{h}_{rs}^t = \tilde{g}(h(J_1 e_r, J_1 e_s), e_t),$$

$$(3.7.3) \qquad \tilde{\tilde{h}}_{rs}^t = \tilde{g}(h(J_2 e_r, J_2 e_s), e_t),$$

$$(3.7.4) \qquad \tilde{\tilde{\tilde{h}}}_{rs}^t = \tilde{g}(h(J_3 e_r, J_3 e_s), e_t),$$

with $i, j = \overline{1, n}$, $r, s, t = \overline{n+1, n+q}$.

Using the fact that $\sum_{1 \leq i < j \leq n} \sum_{r=n+1}^{n+q} \left(h_{ij}^r \right)^2$ is positive as a sum of squares, from (3.6), we get

$$(3.8) \qquad \delta(\mathcal{D}^\perp)(x) \leq \frac{n(n+6q+8)}{2} \cdot \frac{c}{4} +$$

$$+ \sum_{1 \leq i < j \leq n} \sum_{r=n+1}^{n+q} h_{ii}^r h_{jj}^r + \sum_{i=1}^{n} \sum_{r,s=n+1}^{n+q} h_{ii}^r \left[\tilde{h}_{ss}^r + \tilde{\tilde{h}}_{ss}^r + \tilde{\tilde{\tilde{h}}}_{ss}^r \right].$$

We consider the following quadratic forms $f_t : \mathbf{R}^{n+3q} \to \mathbf{R}$,

$$(3.9) \qquad f_t(h_{11}^t, \ldots, h_{nn}^t; \tilde{h}_{n+1;n+1}^t, \ldots, \tilde{h}_{n+q;n+q}^t; \tilde{\tilde{h}}_{n+1;n+1}^t, \ldots, \tilde{\tilde{h}}_{n+q;n+q}^t; \tilde{\tilde{\tilde{h}}}_{n+1;n+1}^t, \ldots, \tilde{\tilde{\tilde{h}}}_{n+q;n+q}^t) =$$

$$= \sum_{1 \leq i < j \leq n} h_{ii}^t h_{jj}^t + \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} h_{ii}^t \left[\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t \right], \quad t = \overline{n+1, n+q}.$$

For $f_t(h_{11}^t, \ldots, \tilde{\tilde{\tilde{h}}}_{n+q;n+q}^t)$ we must find an upper bound, subject to

(3.10)
$$P: h_{11}^t + \ldots + h_{nn}^t + \tilde{h}_{n+1;n+1}^t + \ldots + \tilde{h}_{n+q;n+q}^t + \tilde{\tilde{h}}_{n+1;n+1}^t + \ldots + \tilde{\tilde{h}}_{n+q;n+q}^t +$$

$$+\tilde{\tilde{\tilde{h}}}_{n+1;n+1}^t + \ldots + \tilde{\tilde{\tilde{h}}}_{n+q;n+q}^t = c^t,$$

where c^t is a real constant.

For this, we calculate the partial derivatives of f_t:

(3.11.1)
$$\frac{\partial f_t}{\partial h_{ii}^t} = \sum_{\substack{1 \le j \le n \\ j \ne i}}^{} h_{jj}^t + \sum_{r=n+1}^{n+q} (\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t), \ i = \overline{1, n},$$

(3.11.2)
$$\frac{\partial f_t}{\partial \tilde{h}_{ss}^t} = \frac{\partial f_t}{\partial \tilde{\tilde{h}}_{ss}^t} = \frac{\partial f_t}{\partial \tilde{\tilde{\tilde{h}}}_{ss}^t} = \sum_{i=1}^{n} h_{ii}^t, \ s = \overline{n+1, n+q}.$$

In the standard frame of $\mathbf{R}^{n+3q}$, the Hessian of f_t has the matrix:

$$\begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where $B \in \mathcal{M}_{n,3q}(\mathbf{R})$, with all the elements equal to 1, $C \in \mathcal{M}_{3q}(\mathbf{R})$, with all the elements equals to 0 and A is the matrix :

$$A = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \hdotsfor{5} \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}, \quad A \in \mathcal{M}_n(\mathbf{R}).$$

We obtain

$$\beta(X, X) = 2 \sum_{1 \le i < j \le n} X_i X_j + 2 \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} X_i(\tilde{X}_r + \tilde{\tilde{X}}_r + \tilde{\tilde{\tilde{X}}}_r) =$$

$$= \left[\sum_{i=1}^{n} X_i + \sum_{r=n+1}^{n+q} (\tilde{X}_r + \tilde{\tilde{X}}_r + \tilde{\tilde{\tilde{X}}}_r) \right]^2 - \sum_{i+1}^{n} (X_i)^2 - \sum_{r=n+1}^{n+q} \left[(\tilde{X}_r)^2 + (\tilde{\tilde{X}}_r)^2 + (\tilde{\tilde{\tilde{X}}}_r)^2 \right] -$$

$$-2 \sum_{r,s=n+1}^{n+q} \left(\tilde{X}_r \tilde{\tilde{X}}_s + \tilde{X}_r \tilde{\tilde{\tilde{X}}}_s + \tilde{\tilde{X}}_r \tilde{\tilde{\tilde{X}}}_s \right) =$$

$$= \left[\sum_{i=1}^{n} X_i + \sum_{r=n+1}^{n+q} (\tilde{X}_r + \tilde{\tilde{X}}_r + \tilde{\tilde{\tilde{X}}}_r) \right]^2 - \sum_{i+1}^{n} (X_i)^2 - \left[\sum_{n+1}^{n+q} (\tilde{X}_r + \tilde{\tilde{X}}_r + \tilde{\tilde{\tilde{X}}}_r) \right]^2 =$$

$$= - \sum_{i+1}^{n} (X_i)^2 - \left[\sum_{n+1}^{n+q} (\tilde{X}_r + \tilde{\tilde{X}}_r + \tilde{\tilde{\tilde{X}}}_r) \right]^2 < 0,$$

because $\left[\sum_{i=1}^{n} X_i + \sum_{r=n+1}^{n+q} (\tilde{X}_r + \tilde{\tilde{X}}_r + \tilde{\tilde{\tilde{X}}}_r) \right]^2 = 0$, P being totally geodesic in $\mathbb{R}^{n+3q}$. Then the Hessian of f_t is negative semidefinite, so f_t reaches its maximum (see Remark 2.2.).

Searching for the critical points $(h_{11}^t, \ldots, \tilde{\tilde{\tilde{h}}}_{n+q;n+q}^t)$ of f_t, we find:

$$\frac{\partial f_t}{\partial h_{11}^t} = \frac{\partial f_t}{\partial h_{22}^t} \implies$$

$$\sum_{j=2}^{n} h_{jj}^t + \sum_{r=n+1}^{n+q} (\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t)) = \sum_{\substack{1 \le j \le n \\ j \ne 2}}^{} h_{jj}^t + \sum_{r=n+1}^{n+q} (\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t),$$

which gives

$$(3.12) \qquad h_{11}^t = h_{22}^t = \ldots = h_{nn}^t = \lambda.$$

Also

$$\frac{\partial f_t}{\partial h_{11}^t} = \frac{\partial f_t}{\partial \tilde{h}_{n+1;n+1}^t} \implies$$

$$(3.13) \qquad h_{11}^t = \sum_{r=n+1}^{n+q} (\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t) = \lambda.$$

From (3.10), (3.12) and (3.13) we obtain

$$n\lambda + \lambda = c^t \implies \lambda = \frac{c^t}{n+1},$$

which gives

$$(3.14) \qquad h_{11}^t = h_{22}^t = \ldots = h_{nn}^t = \sum_{r=n+1}^{n+q} (\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t) = \frac{c^t}{n+1}.$$

Using the relations (3.14) in the expression of f_t from (3.9) we have

$$f_t(h_{11}^t, \ldots, \tilde{\tilde{\tilde{h}}}_{n+q;n+q}^t) \le \frac{n(n-1)}{2} \cdot \left(\frac{c^t}{n+1}\right)^2 + n \cdot \left(\frac{c^t}{n+1}\right)^2 =$$

$$= \left(\frac{c^t}{n+1}\right)^2 \cdot \left[\frac{n(n-1)}{2} + n\right] = \left(\frac{c^t}{n+1}\right)^2 \cdot \left[\frac{n^2 - n + 2n}{2}\right] =$$

$$= \left(\frac{c^t}{n+1}\right)^2 \cdot \left(\frac{n^2 + n}{2}\right) = \left(\frac{c^t}{n+1}\right)^2 \cdot \frac{n(n+1)}{2} = \frac{(c^t)^2}{2} \cdot \frac{n}{n+1},$$

and then

$$(3.15) \qquad f_t \le \frac{n}{2(n+1)} \cdot (n+3q)^2 \cdot \|H^t\|^2,$$

where

$$H^t = \frac{1}{n+3q} \cdot \left[\sum_{i=1}^{n} h_{ii}^t + \sum_{r=n+1}^{n+q} (\tilde{h}_{rr}^t + \tilde{\tilde{h}}_{rr}^t + \tilde{\tilde{\tilde{h}}}_{rr}^t)\right].$$

From (3.8) and (3.15) we obtain the relation (*).
The relations (3.4), (3.6) and (3.12) imply the conditions for the equality case.

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