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# On $f$-Biharmonic Curves

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**ABSTRACT**

**We study $f$-biharmonic curves in Sol spaces, Cartan-Vranceanu 3-dimensional spaces, homogeneous contact 3-manifolds and we analyze non-geodesic $f$-biharmonic curves in these spaces.**

***Keywords:*** *$f$-biharmonic curves; Sol spaces; Cartan-Vranceanu 3-dimensional spaces; homogeneous contact 3-manifolds.*

***AMS Subject Classification (2010):*** *Primary: 53C25 ; Secondary: 53C40; 53A04.*

## 1. Introduction

Harmonic maps between Riemannian manifolds were first introduced by Eells and Sampson in [8]. Let $(M,g)$ and $(N,h)$ be two Riemannian manifolds. $\varphi : M \to N$ is called a *harmonic map* if it is a critical point of the *energy functional*

$$E(\varphi) = \frac{1}{2}\int_{\Omega} \|d\varphi\|^2 \, d\nu_g,$$

where $\Omega$ is a compact domain of $M$. Let $\{\varphi_t\}_{t\in I}$ be a differentiable variation of $\varphi$ and $V = \frac{\partial}{\partial t}\mid_{t=0}$, we have critical points of energy functional (see [8])

$$\frac{\partial}{\partial t}E(\varphi_t)\mid_{t=0} = \frac{1}{2}\int_{\Omega}\left\{\frac{\partial}{\partial t}\langle d\varphi_t, d\varphi_t\rangle\right\}_{t=0} d\nu_g$$

$$= \int_{\Omega}\langle tr(\nabla d\varphi), V\rangle \, d\nu_g$$

Hence, the Euler-Lagrange equation of $E(\varphi)$ is

$$\tau(\varphi) = tr(\nabla d\varphi) = 0,$$

where $\tau(\varphi)$ is the *tension field* of $\varphi$ [8]. The map $\varphi$ is said to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2}\int_{\Omega}\|\tau(\varphi)\|^2 \, d\nu_g,$$

where $\Omega$ is a compact domain of $M$. In [11], the Euler-Lagrange equation for the bienergy functional is obtained by

$$\tau_2(\varphi) = tr(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.1}$$

where $\tau_2(\varphi)$ is the *bitension field* of $\varphi$ and $R^N$ is the curvature tensor of $N$.

The map $\varphi$ is a *$f$-harmonic map* with a function $f : M \overset{C^\infty}{\to} \mathbb{R}$, if it is a critical point of $f$-energy

$$E_f(\varphi) = \frac{1}{2}\int_{\Omega} f\,\|d\varphi\|^2 \, d\nu_g,$$

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where $\Omega$ is a compact domain of $M$. The Euler-Lagrange equation of $E_f(\varphi)$ is

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\operatorname{grad} f) = 0,$$

where $\tau_f(\varphi)$ is the $f$-*tension field* of $\varphi$ (see [6] and [13]). The map $\varphi$ is said to be $f$-*biharmonic*, if it is a critical point of the $f$-bienergy functional

$$E_{2,f}(\varphi) = \frac{1}{2}\int_\Omega f\,\|\tau(\varphi)\|^2\,d\nu_g,$$

where $\Omega$ is a compact domain of $M$ [12]. The Euler-Lagrange equation for the $f$-bienergy functional is given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla^{\varphi}_{\operatorname{grad} f}\tau(\varphi) = 0, \tag{1.2}$$

where $\tau_{2,f}(\varphi)$ is the $f$-bitension field of $\varphi$ [12]. If an $f$-biharmonic map is neither harmonic nor biharmonic then we call it by *proper* $f$-*biharmonic* and if $f$ is a constant, then an $f$-biharmonic map turns into a biharmonic map [12].

In [4], Caddeo, Montaldo and Piu considered biharmonic curves on a surface. In [2], Caddeo, Montaldo and Oniciuc classified biharmonic submanifolds in 3-sphere $S^3$. More generally, in [3], the same authors studied biharmonic submanifolds in spheres. In [7], Caddeo, Oniciuc and Piu considered the biharmonicity condition for maps and studied non-geodesic biharmonic curves in the Heisenberg group $H_3$. They proved that all of curves are helices in $H_3$. In [16], Ou and Wang studied linear biharmonic maps from Euclidean space into Sol, Nil, and Heisenberg spaces using the linear structure of the target manifolds. In [5], Caddeo, Montaldo, Oniciuc and Piu characterized all biharmonic curves of Cartan-Vranceanu 3-dimensional spaces and they gave their explicit parametrizations. In [10], Inoguchi considered biminimal submanifolds in contact 3-manifolds. In [14], Ou derived equations for $f$-biharmonic curves in a generic manifold and he gave characterization of $f$-biharmonic curves in $n$-dimensional space forms and a complete classification of $f$-biharmonic curves in 3-dimensional Euclidean space. In [9], Güvenç and the second author studied $f$-biharmonic Legendre curves in Sasakian space forms.

Motivated by the above studies, in the present paper, we consider $f$-biharmonicity condition for the Sol space, Cartan-Vranceanu 3-dimensional space and homogeneous contact 3-manifold. We find the necessary and sufficient conditions for the curves in these spaces to be $f$-biharmonic.

## 2. $f$-Biharmonicity Conditions For Curves

### 2.1. $f$-*Biharmonic curves of Sol space*

Sol space can be seen as $\mathbb{R}^3$ with respect to Riemannian metric

$$g_{sol} = ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where $(x, y, z)$ are standard coordinates in $\mathbb{R}^3$ [16], [18]. In [16] and [18], the Levi-Civita connection $\nabla$ of the metric $g_{sol}$ with respect to the orthonormal basis is given by

$$e_1 = e^{-z}\frac{\partial}{\partial x}, e_2 = e^{z}\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}.$$

In terms of the basis $\{e_1, e_2, e_3\}$, they obtained as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1 = -e_3, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_3, & \nabla_{e_2}e_3 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0, \end{array}$$

(see [18]). Now we assume that $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a curve in Sol space $(\mathbb{R}^3, g_{sol})$ parametrized by arc length and let $\{T, N, B\}$ be orthonormal frame field tangent to Sol space along $\gamma$, where $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = B_1e_1 + B_2e_2 + B_3e_3$.

Now, we state the $f$-biharmonicity condition for curves of Sol space $(\mathbb{R}^3, g_{sol})$:

**Theorem 2.1.** *Let $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a curve parametrized by arc length in Sol space $(\mathbb{R}^3, g_{sol})$. Then $\gamma$ is $f$-biharmonic if and only if the following equations hold:*

$$
\begin{aligned}
-3f\kappa\kappa' - 2f'\kappa^2 &= 0, \\
f\kappa'' - f\kappa^3 - f\kappa\tau^2 + 2f\kappa B_3^2 - f\kappa + 2f'\kappa' + f''\kappa &= 0, \\
2f\kappa'\tau + f\kappa\tau' - 2f\kappa N_3 B_3 + 2f'\kappa\tau &= 0.
\end{aligned} \tag{2.1}
$$

*Proof.* Let $\{e_i\}$, $1 \le i \le 3$ be an orthonormal basis. Let $\gamma = \gamma(s)$ be a curve parametrized by arc length. Then we have

$$
\begin{aligned}
\tau(\gamma) = tr(\nabla d\varphi) &= \nabla^{\gamma}_{\frac{\partial}{\partial s}} \left( d\gamma \left( \frac{\partial}{\partial s} \right) \right) - d\gamma \left( \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} \right) \\
&= \nabla^{\gamma}_{\frac{\partial}{\partial s}} \left( d\gamma \left( \frac{\partial}{\partial s} \right) \right) = \nabla_{\gamma'} \gamma' = \kappa N.
\end{aligned} \tag{2.2}
$$

From [15] or [16], we know that

$$R(T, N, T, N) = 2B_3^2 - 1 \tag{2.3}$$

$$R(T, N, T, B) = -2N_3 B_3. \tag{2.4}$$

Using the equation (2.2) in (1.1), we can write

$$
\begin{aligned}
\tau_2(\gamma) &= (-3\kappa\kappa')\, T + \left(\kappa'' - \kappa^3 - \kappa\tau^2\right) N \\
&\quad + \kappa R(T, N)\, T + (2\kappa'\tau + \kappa\tau')\, B.
\end{aligned} \tag{2.5}
$$

On the other hand, an easy calculation gives us

$$\nabla^{\gamma}_{\text{grad } f} \tau(\gamma) = \nabla^{\gamma}_{\text{grad } f} \kappa N = f' \nabla_T(\kappa N) = f' \left( -\kappa^2 T + \kappa' N + \kappa\tau B \right) \tag{2.6}$$

In view of equations (2.2), (2.5) and (2.6) into equation (1.2), we have

$$
\begin{aligned}
\tau_{2,f}(\gamma) &= (-3f\kappa\kappa')\, T + \left(f\kappa'' - f\kappa^3 - f\kappa\tau^2\right) N + (2f\kappa'\tau + f\kappa\tau')\, B \\
&\quad + f\kappa R(T, N)\, T + f''\kappa N + 2f' \left( -\kappa^2 T + \kappa' N + \kappa\tau B \right) = 0.
\end{aligned} \tag{2.7}
$$

Finally, taking the scalar product of equation (2.7) with $T, N$ and $B$, respectively and using the equations (2.3) and (2.4) we obtain (2.1). □

In the following four cases, we find necessary and sufficient conditions for curves of Sol space to be $f$-biharmonic:

**Case 2.1.** *If $\kappa = constant \neq 0$, then we have the following corollary:*

**Corollary 2.1.** *Let $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a differentiable $f$-biharmonic curve parametrized by arc length in Sol space $(\mathbb{R}^3, g_{sol})$. If $\kappa = constant \neq 0$, then $\gamma$ is biharmonic.*

*Proof.* We assume that $\kappa =$ constant $\neq 0$. By the use of equations (2.1), we find

$$f' = 0.$$

Hence, $\gamma$ is a biharmonic curve. □

**Case 2.2.** *If $\tau = constant \neq 0$, then we have the following corollaries:*

**Corollary 2.2.** *Let $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a differentiable $f$-biharmonic curve parametrized by arc length in Sol space $(\mathbb{R}^3, g_{sol})$. If $\tau = constant \neq 0$ and $N_3 B_3 = 0$, then $\gamma$ is biharmonic.*

*Proof.* We assume that $\tau =$ constant $\neq 0$ and $N_3 B_3 = 0$. By the use of equations (2.1), we have

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{2.8}$$

and

$$\tau \left( \frac{\kappa'}{\kappa} + \frac{f'}{f} \right) = 0. \tag{2.9}$$

Then, substituting the equation (2.8) into (2.9), we obtain $f =$ constant and $\gamma$ is a biharmonic curve. □

**Corollary 2.3.** *Let $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a differentiable $f$-biharmonic curve parametrized by arc length in Sol space $(\mathbb{R}^3, g_{sol})$. If $\tau = constant \neq 0$, then $f = e^{\int \frac{3N_3B_3}{\tau}}$.*

*Proof.* Using the equations (2.1), we obtain

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{2.10}$$

and

$$2f\kappa'\tau - 2f\kappa N_3B_3 + 2f'\kappa\tau = 0. \tag{2.11}$$

Then, putting the equation (2.10) into (2.11), we get the result. □

**Case 2.3.** *If $\tau = 0$, then we have the following corollary:*

**Corollary 2.4.** *Let $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a differentiable non-geodesic curve parametrized by arc length in Sol space $(\mathbb{R}^3, g_{sol})$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are satisfied:*

$$f^2\kappa^3 = c_1^2, \tag{2.12}$$

$$(f\kappa)'' = f\kappa\left(\kappa^2 - 2B_3^2 + 1\right) \tag{2.13}$$

*and*

$$N_3B_3 = 0, \tag{2.14}$$

*where $c_1 \in \mathbb{R}$.*

*Proof.* We assume that $\tau = 0$. Then using the equations (2.1), integrating the first equation, we find the desired result. □

**Case 2.4.** *If $\kappa \neq constant \neq 0$ and $\tau \neq constant \neq 0$, then we have the following corollary:*

**Corollary 2.5.** *Let $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ be a differentiable non-geodesic curve parametrized by arc length in Sol space $(\mathbb{R}^3, g_{sol})$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are hold:*

$$f^2\kappa^3 = c_1^2, \tag{2.15}$$

$$(f\kappa)'' = f\kappa\left(\kappa^2 + \tau^2 - 2B_3^2 + 1\right) \tag{2.16}$$

*and*

$$f^2\kappa^2\tau = e^{\int \frac{2N_3B_3}{\tau}}, \tag{2.17}$$

*where $c_1 \in \mathbb{R}$.*

*Proof.* We suppose that $\kappa \neq$ constant $\neq 0$ and $\tau \neq$ constant $\neq 0$. Then using equations (2.1), integrating the first and third equations, the proof is completed. □

From Corollary 2.4 and Corollary 2.5, we can state the following theorem:

**Theorem 2.2.** *An arc length parametrized curve $\gamma : I \longrightarrow (\mathbb{R}^3, g_{sol})$ in Sol space $(\mathbb{R}^3, g_{sol})$ is proper $f$-biharmonic if and only if one of the following cases happens:*

$(i)$ $\tau = 0$, $f = c_1\kappa^{-\frac{3}{2}}$ and the curvature $\kappa$ solves the following equation

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2 - 2B_3^2 + 1\right).$$

$(ii)$ $\tau \neq 0$, $\frac{\tau}{\kappa} = \frac{e^{\int \frac{2N_3B_3}{\tau}}}{c_1^2}$, $f = c_1\kappa^{-\frac{3}{2}}$ and the curvature $\kappa$ solves the following equation

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2\left(1 + \frac{e^{\int \frac{4N_3B_3}{\tau}}}{c_1^4}\right) - 2B_3^2 + 1\right).$$

*Proof.* $(i)$ Using the equation (2.12), we have

$$f = c_1\kappa^{-\frac{3}{2}}. \tag{2.18}$$

Putting the equation (2.18) into (2.13), we get the result.
$(ii)$ Solving the equation (2.15), we get

$$f = c_1\kappa^{-\frac{3}{2}}. \tag{2.19}$$

Putting the equation (2.19) into (2.17), we have

$$\frac{\tau}{\kappa} = \frac{e^{\int \frac{2N_3B_3}{\tau}}}{c_1^2}. \tag{2.20}$$

Finally, substituting the equations (2.19) and (2.20) into (2.16), we obtain

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2\left(1 + \frac{e^{\int \frac{4N_3B_3}{\tau}}}{c_1^4}\right) - 2B_3^2 + 1\right).$$

This completes the proof of the theorem. □

As an immediate consequence of the above theorem, we have:

**Corollary 2.6.** *An arc length parametrized $f$-biharmonic curve $\gamma : I \longrightarrow \left(\mathbb{R}^3, g_{sol}\right)$ in Sol space $\left(\mathbb{R}^3, g_{sol}\right)$ with constant geodesic curvature is biharmonic.*

### 2.2. $f$-*Biharmonic curves of Cartan-Vranceanu* 3-*dimensional space*

The Cartan-Vranceanu metric is the following two parameter family of Riemannian metrics

$$ds^2_{\ell,m} = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left(dz + \frac{\ell}{2}\frac{yd_x - xd_y}{[1 + m(x^2 + y^2)]}\right),$$

where $\ell, m \in \mathbb{R}$ defined on $M = \mathbb{R}^3$ if $m \geq 0$ and on $M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < -\frac{1}{m}\right\}$ [5]. The Levi-Civita connection $\nabla$ of the metric $ds^2_{\ell,m}$ with respect to the orthonormal basis

$$e_1 = [1 + m(x^2 + y^2)]\frac{\partial}{\partial x} - \frac{\ell y}{2}\frac{\partial}{\partial z},\ e_2 = [1 + m(x^2 + y^2)]\frac{\partial}{\partial y} + \frac{\ell x}{2}\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}$$

is

$$\begin{array}{ccc} \nabla_{e_1}e_1 = 2mye_2, & \nabla_{e_1}e_2 = -2mye_1 + \frac{\ell}{2}e_3, & \nabla_{e_1}e_3 = -\frac{\ell}{2}e_2, \\ \nabla_{e_2}e_1 = -2mxe_2 - \frac{\ell}{2}e_3, & \nabla_{e_2}e_2 = 2mxe_1, & \nabla_{e_2}e_3 = \frac{\ell}{2}e_1, \\ \nabla_{e_3}e_1 = -\frac{\ell}{2}e_2, & \nabla_{e_3}e_2 = \frac{\ell}{2}e_1, & \nabla_{e_3}e_3 = 0, \end{array}$$

(see [5]).

Now assume that $\gamma : I \longrightarrow \left(M, ds^2_{\ell,m}\right)$ be a curve on Cartan-Vranceanu 3-dimensional space $\left(M, ds^2_{\ell,m}\right)$ parametrized by arc length and let $\{T, N, B\}$ be orthonormal frame field tangent to Cartan-Vranceanu 3-dimensional space along $\gamma$, where $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = B_1e_1 + B_2e_2 + B_3e_3$.

In this part, we investigate $f$-biharmonic curves of Cartan-Vranceanu 3-dimensional space. Firstly, we have the following theorem:

**Theorem 2.3.** *Let $\gamma : I \longrightarrow \left(M, ds^2_{\ell,m}\right)$ be a curve parametrized by arc length in Cartan-Vranceanu 3-dimensional space $\left(M, ds^2_{\ell,m}\right)$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are satisfied:*

$$\begin{gathered} -3f\kappa\kappa' - 2f'\kappa^2 = 0, \\ f\kappa'' - f\kappa^3 - f\kappa\tau^2 - (\ell^2 - 4m)f\kappa B_3^2 + \frac{\ell^2}{4}f\kappa + 2f'\kappa' + f''\kappa = 0, \\ 2f\kappa'\tau + f\kappa\tau' + (\ell^2 - 4m)f\kappa N_3B_3 + 2f'\kappa\tau = 0. \end{gathered} \tag{2.21}$$

*Proof.* From [5], we have

$$R(T,N,T,N)=\frac{\ell^2}{4}-(\ell^2-4m)B_3^2, \tag{2.22}$$

$$R(T,N,T,B)=(\ell^2-4m)N_3B_3. \tag{2.23}$$

Using the bitension field from [5], we can write

$$\begin{aligned}\tau_2(\gamma)&=(-3\kappa\kappa')T+\left(\kappa''-\kappa^3-\kappa\tau^2\right)N\\&+\kappa R(T,N)T+(2\kappa'\tau+\kappa\tau')B.\end{aligned} \tag{2.24}$$

Substituting equations (2.2), (2.24) and (2.6) into equation (1.2), we obtain

$$\begin{aligned}\tau_{2,f}(\gamma)&=(-3f\kappa\kappa')T+\left(f\kappa''-f\kappa^3-f\kappa\tau^2\right)N+(2f\kappa'\tau+f\kappa\tau')B\\&+f\kappa R(T,N)T+f''\kappa N+2f'\left(-\kappa^2T+\kappa'N+\kappa\tau B\right)=0.\end{aligned} \tag{2.25}$$

Finally, taking the scalar product of equation (2.25) with $T,N$ and $B$, respectively and using equations (2.22) and (2.23) we have the desired result. □

*Remark* 2.1.
- If $\ell=m=0$, $\left(M,ds^2_{\ell,m}\right)$ is the Euclidean space and $\gamma$ is a $f$-biharmonic curve [14].
- If $\ell^2=4m$ and $\ell\neq 0$, $\left(M,ds^2_{\ell,m}\right)$ is locally the 3-dimensional sphere with sectional curvature $\frac{\ell^2}{4}$ and $\gamma$ is a proper $f$-biharmonic curve.
- If $m=0$ and $\ell\neq 0$, $\left(M,ds^2_{\ell,m}\right)$ is the Heisenberg space $H_3$ endowed with a left invariant metric and $\gamma$ is a $f$-biharmonic curve in $H_3$.
- If $\ell=1$, $\left(M,ds^2_{\ell,m}\right)$ is a 3-dimensional Sasakian space form [5] and $\gamma$ is a $f$-biharmonic curve in a 3-dimensional Sasakian space form.

Now, we shall assume that $\ell^2\neq 4m$ and $m\neq 0$. As in the following cases we have $f$-biharmonicity conditions:

**Case 2.5.** *If $\kappa=constant\neq 0$, then we have the following corollary:*

**Corollary 2.7.** *Let $\gamma:I\longrightarrow\left(M,ds^2_{\ell,m}\right)$ be a differentiable $f$-biharmonic curve parametrized by arc length in Cartan-Vranceanu 3-dimensional space $\left(M,ds^2_{\ell,m}\right)$. If $\kappa=constant\neq 0$, then $\gamma$ is biharmonic.*

*Proof.* Putting $\kappa=$ constant $\neq 0$ into the equations (2.21), $\gamma$ is biharmonic. □

**Case 2.6.** *If $\tau=constant\neq 0$, then we have the following corollaries:*

**Corollary 2.8.** *Let $\gamma:I\longrightarrow\left(M,ds^2_{\ell,m}\right)$ be a differentiable $f$-biharmonic curve parametrized by arc length in Cartan-Vranceanu 3-dimensional space $\left(M,ds^2_{\ell,m}\right)$. If $\tau=constant\neq 0$ and $N_3B_3=0$, then $\gamma$ is a biharmonic curve.*

*Proof.* Using the same method in the proof of Corollary 2.2, we obtain $f=$ constant and $\gamma$ is a biharmonic curve. □

**Corollary 2.9.** *Let $\gamma:I\longrightarrow\left(M,ds^2_{\ell,m}\right)$ be a differentiable $f$-biharmonic curve parametrized by arc length in Cartan-Vranceanu 3-dimensional space $\left(M,ds^2_{\ell,m}\right)$. If $\tau=constant\neq 0$, then $f=e^{\int\frac{3(\ell^2-4m)N_3B_3}{2\tau}}$.*

*Proof.* By the same method in the proof of Corollary 2.3, we get the result. □

**Case 2.7.** *If $\tau=0$, then we have the following corollary:*

**Corollary 2.10.** *Let $\gamma:I\longrightarrow\left(M,ds^2_{\ell,m}\right)$ be a differentiable non-geodesic curve parametrized by arc length in Cartan-Vranceanu 3-dimensional space $\left(M,ds^2_{\ell,m}\right)$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are satisfied:*

$$f^2\kappa^3=c_1^2, \tag{2.26}$$

$$(f\kappa)''=f\kappa\left(\kappa^2+(\ell^2-4m)B_3^2-\frac{\ell^2}{4}\right) \tag{2.27}$$

*and*

$$N_3B_3=0, \tag{2.28}$$

*where $c_1\in\mathbb{R}$.*

*Proof.* Suppose that $\tau = 0$. By the use of equations (2.21) and integrating the first equation, we find the desired result. $\square$

**Case 2.8.** *If $\kappa \neq constant \neq 0$ and $\tau \neq constant \neq 0$, then we have the following corollary:*

**Corollary 2.11.** *Let $\gamma : I \longrightarrow \left(M, ds^2_{\ell,m}\right)$ be a differentiable non-geodesic curve parametrized by arc length in Cartan-Vranceanu 3-dimensional space $\left(M, ds^2_{\ell,m}\right)$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are fulfilled:*

$$f^2\kappa^3 = c_1^2, \tag{2.29}$$

$$(f\kappa)'' = f\kappa\left(\kappa^2 + \tau^2 + (\ell^2 - 4m)B_3^2 - \frac{\ell^2}{4}\right) \tag{2.30}$$

*and*

$$f^2\kappa^2\tau = e^{\int \frac{-(\ell^2-4m)N_3B_3}{\tau}}, \tag{2.31}$$

*where $c_1 \in \mathbb{R}$.*

*Proof.* We suppose that $\kappa \neq$ constant $\neq 0$ and $\tau \neq$ constant $\neq 0$. Then using the equations (2.21) and integrating the first and third equations, the proof is completed. $\square$

Using Corollary 2.10 and Corollary 2.11, we find the following theorem:

**Theorem 2.4.** *An arc length parametrized curve $\gamma : I \longrightarrow \left(M, ds^2_{\ell,m}\right)$ in Cartan-Vranceanu 3-dimensional space is proper $f$-biharmonic if and only if one of the following cases happens:*

$(i)$ $\tau = 0$, $f = c_1\kappa^{-\frac{3}{2}}$ and the curvature $\kappa$ solves the following equation

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2 + (\ell^2 - 4m)B_3^2 - \frac{\ell^2}{4}\right).$$

$(ii)$ $\tau \neq 0$, $\frac{\tau}{\kappa} = \frac{e^{\int \frac{-(\ell^2-4m)N_3B_3}{\tau}}}{c_1^2}$, $f = c_1\kappa^{-\frac{3}{2}}$ and the curvature $\kappa$ solves the following equation

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2\left(1 + \frac{e^{\int \frac{-2(\ell^2-4m)N_3B_3}{\tau}}}{c_1^4}\right) + (\ell^2 - 4m)B_3^2 - \frac{\ell^2}{4}\right).$$

*Proof.* $(i)$ From the equation (2.26), we can write

$$f = c_1\kappa^{-\frac{3}{2}}. \tag{2.32}$$

Then, putting equation (2.32) into (2.27), we obtain the result.
$(ii)$ From the equation (2.29), we have

$$f = c_1\kappa^{-\frac{3}{2}}. \tag{2.33}$$

Putting the equation (2.33) into (2.31), we find

$$\frac{\tau}{\kappa} = \frac{e^{\int \frac{-(\ell^2-4m)N_3B_3}{\tau}}}{c_1^2}. \tag{2.34}$$

Then substituting the equations (2.33) and (2.34) into (2.30), we get

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2\left(1 + \frac{e^{\int \frac{-2(\ell^2-4m)N_3B_3}{\tau}}}{c_1^4}\right) + (\ell^2 - 4m)B_3^2 - \frac{\ell^2}{4}\right).$$

This completes the proof of the theorem. $\square$

From the above theorem, we have the following corollary:

**Corollary 2.12.** *An arc length parametrized $f$-biharmonic curve $\gamma : I \longrightarrow \left(M, ds^2_{\ell,m}\right)$ in Cartan-Vranceanu 3-dimensional space $\left(M, ds^2_{\ell,m}\right)$ with constant geodesic curvature is biharmonic.*

*2.3. $f$-Biharmonic curves of homogeneous contact 3-manifolds*

A contact Riemannian 3-manifold is said to be *homogeneus* if there is a connected Lie group $G$ acting transitively as a group of isometries on it which preserve the contact form, (see [10] and [17]). The simply connected homogeneous contact Riemannian 3-manifolds are Lie groups together with a left invariant contact Riemannian structure [17].

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional unimodular Lie group with left invariant Riemannian metric $g$. Then $M$ admits its compatible left-invariant contact Riemannian structure if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = c_2 e_1, \quad [e_3, e_1] = c_3 e_2$$

[17]. Let $\varphi$ be the $(1,1)$-tensor field defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$ and $\varphi(e_3) = 0$. Then using the linearity of $\varphi$ and $g$ we have

$$\eta(e_3) = 1, \quad \varphi^2(X) = -X + \eta(X)e_3, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

In [17], Perrone calculated the Levi-Civita connection of homogeneous contact 3-manifolds as follows:

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, \\ \nabla_{e_2} e_1 &= \tfrac{1}{2}(c_3 - c_2 - 2)e_3, \\ \nabla_{e_3} e_1 &= \tfrac{1}{2}(c_3 + c_2 - 2)e_2, \end{aligned}$$

$$\begin{aligned} \nabla_{e_1} e_2 &= \tfrac{1}{2}(c_3 - c_2 + 2)e_3, & \nabla_{e_1} e_3 &= -\tfrac{1}{2}(c_3 - c_2 + 2)e_2, \\ \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= -\tfrac{1}{2}(c_3 - c_2 - 2)e_1, \\ \nabla_{e_3} e_2 &= -\tfrac{1}{2}(c_3 + c_2 - 2)e_1, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

A 1-dimensional integral submanifold of a homogeneous contact Riemannian manifold $M$ is called a *Legendre curve* of $M$ [1].

Let $\gamma : I \longrightarrow M$ be a Legendre curve on homogeneous contact 3-manifold parametrized by arc length and let $\{T, N, B\}$ be orthonormal frame field tangent to homogeneous contact 3-manifold along $\gamma$ where $T = T_1 e_1 + T_2 e_2 + T_3 e_3$, $N = N_1 e_1 + N_2 e_2 + N_3 e_3$ and $B = B_1 e_1 + B_2 e_2 + B_3 e_3$.

Now, we obtain the $f$-biharmonicity condition for Legendre curves of homogeneous contact 3-manifold:

**Theorem 2.5.** *Let $\gamma : I \longrightarrow M$ be a Legendre curve parametrized by arc length in a homogeneous contact 3-manifold $M$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are satisfied:*

$$\begin{gathered} -3f\kappa\kappa' - 2f'\kappa^2 = 0, \\ f\kappa'' - f\kappa^3 - f\kappa\tau^2 + fk\left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_2 + c_3\right) + 2f'\kappa' + f''\kappa = 0, \\ 2f\kappa'\tau + f\kappa\tau' + 2f'\kappa\tau = 0, \end{gathered} \tag{2.35}$$

*where $c_i \in \mathbb{R}$, $1 \leq i \leq 3$.*

*Proof.* From [10], we have

$$R(T,N,T,N) = \frac{1}{4}(c_3 - c_2)^2 - 3 + c_2 + c_3, \tag{2.36}$$

$$R(T,N,T,B) = 0. \tag{2.37}$$

Using the bitension field from [10], we can write

$$\begin{aligned} \tau_2(\gamma) = {} & (-3\kappa\kappa')T + \left(\kappa'' - \kappa^3 - \kappa\tau^2\right)N \\ & + \kappa R(T,N)T + (2\kappa'\tau + \kappa\tau')B. \end{aligned} \tag{2.38}$$

In view of equations (2.2), (2.38) and (2.6) into equation (1.2), we calculate

$$\begin{aligned} \tau_{2,f}(\gamma) = {} & (-3f\kappa\kappa')T + \left(f\kappa'' - f\kappa^3 - f\kappa\tau^2\right)N + (2f\kappa'\tau + f\kappa\tau')B \\ & + f\kappa R(T,N)T + f''\kappa N + 2f'\left(-\kappa^2 T + \kappa' N + \kappa\tau B\right) = 0. \end{aligned} \tag{2.39}$$

Finally, taking the scalar product of equation (2.39) with $T$, $N$ and $B$, respectively and using the equations (2.36) and (2.37) we obtain the result. □

From the above theorem, we have the following cases:

**Case 2.9.** *If $\kappa = constant \neq 0$, then we have the following corollary:*

**Corollary 2.13.** *Let $\gamma : I \longrightarrow M$ be a differentiable $f$-biharmonic Legendre curve parametrized by arc length in a homogeneous contact 3-manifold $M$. If $\kappa = constant \neq 0$, then $\gamma$ is biharmonic.*

*Proof.* Putting the curvature $\kappa =$ constant $\neq 0$ into the equations (2.35), it is clear that $\gamma$ is a biharmonic curve. □

**Case 2.10.** *If $\tau = constant \neq 0$, then we have the following corollary:*

**Corollary 2.14.** *Let $\gamma : I \longrightarrow M$ be a differentiable $f$-biharmonic Legendre curve parametrized by arc length in a homogeneous contact 3-manifold $M$. If $\tau = constant \neq 0$, then $\gamma$ is biharmonic.*

*Proof.* Putting the curvature $\tau =$ constant $\neq 0$ into the equations (2.35), it is clear that $\gamma$ is a biharmonic curve. □

**Case 2.11.** *If $\tau = 0$, then we have the following corollary:*

**Corollary 2.15.** *Let $\gamma : I \longrightarrow M$ be a differentiable non-geodesic Legendre curve parametrized by arc length in a homogeneous contact 3-manifold $M$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are satisfied:*

$$f^2\kappa^3 = c_1^2, \tag{2.40}$$

*and*

$$(f\kappa)'' = f\kappa\left(\kappa^2 - \frac{1}{4}(c_3 - c_2)^2 + 3 - c_2 - c_3\right) \tag{2.41}$$

*where $c_i \in \mathbb{R}$, $1 \leq i \leq 3$.*

*Proof.* Suppose that $\tau = 0$. Then using the equations (2.35), we find the desired result. □

**Case 2.12.** *If $\kappa \neq constant \neq 0$ and $\tau \neq constant \neq 0$, then we have the following corollary:*

**Corollary 2.16.** *Let $\gamma : I \longrightarrow M$ be a differentiable non-geodesic Legendre curve parametrized by arc length in a homogeneous contact 3-manifold $M$. Then $\gamma$ is $f$-biharmonic if and only if the following equations are satisfied:*

$$f^2\kappa^3 = c_1^2, \tag{2.42}$$

$$(f\kappa)'' = f\kappa\left(\kappa^2 + \tau^2 - \frac{1}{4}(c_3 - c_2)^2 + 3 - c_2 - c_3\right) \tag{2.43}$$

*and*

$$f^2\kappa^2\tau = c_4. \tag{2.44}$$

*where $c_i \in \mathbb{R}$, $1 \leq i \leq 4$.*

*Proof.* Assume that $\kappa \neq$ constant $\neq 0$ and $\tau \neq$ constant $\neq 0$. Then using the equations (2.35) and integrating the first and third equations, we have the result. □

By the use of Corollary 2.15 and Corollary 2.16, we obtain the following theorem:

**Theorem 2.6.** *An arc length parametrized Legendre curve $\gamma : I \longrightarrow M$ in a homogeneous contact 3-manifold $M$ is proper $f$-biharmonic if and only if one of the following cases happens:*

$(i)$ $\tau = 0$, $f = c_1\kappa^{-\frac{3}{2}}$ *and the curvature $\kappa$ solves the following equation*

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2 - \frac{1}{4}(c_3 - c_2)^2 + 3 - c_2 - c_3\right).$$

$(ii)$ $\tau \neq 0$, $\frac{\tau}{\kappa} = c_5$, $f = c_1\kappa^{-\frac{3}{2}}$ *and the curvature $\kappa$ solves the following equation*

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2\left(1 + c_5^2\right) - \frac{1}{4}(c_3 - c_2)^2 + 3 - c_2 - c_3\right),$$

where $c_i \in \mathbb{R}$, $1 \leq i \leq 5$.

*Proof.* $(i)$ Using the equation (2.40), we can write

$$f = c_1 \kappa^{-\frac{3}{2}}. \tag{2.45}$$

Then, substituting the equation (2.45) into (2.41), we find the result.
$(ii)$ From the equation (2.42), we have

$$f = c_1 \kappa^{-\frac{3}{2}}. \tag{2.46}$$

Putting the equation (2.46) into (2.44), we find

$$\frac{\tau}{\kappa} = c_5. \tag{2.47}$$

Then substituting the equations (2.46) and (2.47) into (2.43), we get

$$3\left(\kappa'\right)^2 - 2\kappa\kappa'' = 4\kappa^2\left(\kappa^2\left(1+c_5^2\right) - \frac{1}{4}(c_3-c_2)^2 + 3 - c_2 - c_3\right).$$

□

From the above theorem, we have the following corollary:

**Corollary 2.17.** *An arc length parametrized $f$-biharmonic Legendre curve $\gamma : I \longrightarrow M$ in a homogeneous contact 3-manifold $M$ with constant geodesic curvature is biharmonic.*

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