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A Ladder of Curvatures in the Geometry of Surfaces

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ABSTRACT

Many investigations in the local differential geometry of surfaces focused on Gaussian curvature and mean curvature. Besides these classical curvature invariants, are there any other geometric quantities that deserve to be investigated? In the recent decades, there have been important developments in the area of new curvature invariants for submanifolds, mostly included in Bang-Yen Chen's important monograph *Pseudo-Riemannian geometry, $\delta-$invariants and applications*, **World Scientific, 2011. These developments are inviting us to look at the classical content from a different perspective, exploring other quantities that might be of interest.**

Keywords: *principal curvatures; Gaussian curvature; mean curvature; Casorati curvature; smooth surfaces; conformal Gauss map.*

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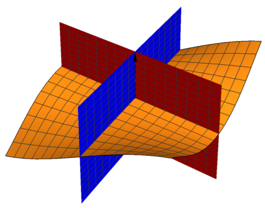
CLASSICAL CURVATURE INVARIANTS IN THE GEOMETRY OF SMOOTH SURFACES. The history of local differential geometry of smooth surfaces can be traced back to Leonhard Euler's 1760 work [9] and is an interesting casebook in the history of mathematics (see, e.g. [13]). For our present investigation, we start by reminding the classical curvature invariants for smooth surfaces.

Consider a smooth surface S lying in $\mathbb{R}^3$, and an arbitrary point $P \in S$ (see e.g. [12, 14]). Consider N_P the normal to the surface at P. Consider the family of all planes passing through P that contain the line through P with the same direction as N_P. These planes yield a family of curves on S called *normal sections*. Consider now the curvature $\kappa(P)$ of the normal sections, viewed as planar curves. Then $\kappa(P)$ has a maximum, denoted κ_1, and a minimum, denoted κ_2. The curvatures κ_1 and κ_2 are called the *principal curvatures*. Using these principal curvatures, one may define the Gaussian curvature [10] as $K(P) = \kappa_1(P) \cdot \kappa_2(P)$, and the mean curvature [11] as $H(P) = \frac{1}{2}\left[\kappa_1(P) + \kappa_2(P)\right]$.



Complementing a century of investigations on what the correct definition for curvature should be, Felice Casorati introduced in 1890 what is today called the Casorati curvature [1]. In his paper, Casorati argues that there are important geometric reasons why one should investigate $C(P) = \frac{1}{2}\left[\kappa_1^2(P) + \kappa_2^2(P)\right]$. (See also [18].) Several recent works pursue the study of Casorati curvatures and its generalizations see e.g. [6, 7] as well as their references.

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TANGENTIAL CURVATURE. With the notations presented above, we propose the investigation of the following curvature invariant, which we call *tangential curvature:*

$$\tau(p) = \frac{|\kappa_1(p)\kappa_2(p)| - 1 + \sqrt{(\kappa_1^2(p)+1)(\kappa_2^2(p)+1)}}{|\kappa_1(p)| + |\kappa_2(p)|}.$$

For a Riemannian manifold in general there are three curvature invariants: the sectional curvature, the Ricci curvature and the scalar curvature [3, 8]. In the last decade, a new important development was the investigation of B.-Y.Chen's δ-curvature invariants [3]. Our main motivation to investigate this invariant is suggested by our main result on the tangential curvature, which will be presented below. Before approaching it, we would like to recall the amalgamatic curvature introduced and investigated first in [5], then in [4] and [16]. For an investigation of $|\kappa_1(p) - \kappa_2(p)|$, inspired by the developments in [2], see [15].

In [5], the following questions is posed: In the classical geometry of curves, a curve satisfying the property that the ratio between curvature and torsion is constant is called a generalized helix. It's natural to think if it is possible to extend this idea to higher dimensional geometric objects; would it make sense to study surfaces that are satisfying a similar relationship between the mean curvature H and the Gaussian curvature K? One could consider both the ratio $\frac{K}{H}$ or $\frac{K}{H^2}$ and derive some analogies with the theory of curves. The history of the original idea can be traced back to Weingarten's original papers [19, 20]. One might look at the idea of the ratio $\frac{K}{H}$ seen in the geometry of surfaces. Suppose we work on a surface patch where $H(p) \neq 0$, for all points p. In general, the ratio $\frac{K}{H}$ is a function that depends on the point of the surface, everywhere where it is defined. If we denote the principal curvatures by κ_1 and κ_2, then

$$\frac{K}{H} = \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}.$$

Note that this term could be also viewed as the harmonic ratio of the real numbers κ_1 and κ_2. The idea is to define a geometric quantity that encodes the same information as this ratio. This motivated in [5] the following.

Definition 1. Let $\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface given by the smooth map σ. Then the amalgamatic curvature at point p is

$$A(p) = \frac{2|\kappa_1||\kappa_2|}{|\kappa_1| + |\kappa_2|}.$$

Additionally, one may look at the absolute mean curvature, denoted by $\bar{H}$ i.e.

$$\bar{H} = \frac{1}{2}\left(|\kappa_1| + |\kappa_2|\right).$$

Also, we refer to [5, 4] for the following.

Definition 2. Let $\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth surface given by the smooth map σ. The point p on the surface is called *absolutely umbilical* if the principal curvatures satisfy $|\kappa_1| = |\kappa_2|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

The absolute umbilical hypersurfaces are hypersurfaces with conformal Gauss map. We can see this in the following way. Let (M, g) be a hypersurface of the Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. Then the Gauss map ν of M is defined to be the map which carries each point $p \in M$ to the unit normal vector $\xi = e_{n+1}$. Then Weingarten's formula gives

$$d\nu(X) = \tilde{\nabla}_X \xi = -A_\xi(X),$$

where A_ξ denotes the shape operator at p. Hence the induced metric g_ν on M via Gauss map is given by

$$g_\nu(X, Y) = g(A_\xi X, A_\xi Y).$$

Now, if $e_1, \ldots, e_n$ are eigenvectors of A_ξ with $A_\xi e_i = \kappa_i e_i$, then we have $g_\nu(e_i, e_i) = \kappa_i^2 g(e_i, e_i)$. Hence the Gauss map is conformal if and only if $\kappa_1^2 = \ldots = \kappa_n^2$.

We prove the following result.

Theorem 1. *Let $\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface given by the smooth map σ. Denote by κ_1 and κ_2 the principal curvatures on $\sigma(U)$. Then the amalgamatic curvature, the tangential curvature, the absolute mean curvature and the Casorati curvature satisfy the inequality*

$$A(p) \leq \tau(p) \leq \bar{H}(p) \leq \sqrt{C(p)},$$

with equality holding at the points where the Gauss map is conformal (i.e. the equality holds at the absolute umbilical points, where $|\kappa_1(p)| = |\kappa_2(p)|$).

Proof: To simplify our presentation, denote $|\kappa_1| = x$, and $|\kappa_2| = y$. Consider $x, y > 0$. The first inequality we have to prove is $A \leq \tau$, which converts into

$$\frac{2xy}{x+y} \leq \frac{xy - 1 + \sqrt{x^2+1}\sqrt{y^2+1}}{x+y}.$$

This reduces immediately to

$$xy + 1 \leq \sqrt{x^2+1}\sqrt{y^2+1},$$

and by squaring both sides we get

$$x^2y^2 + 2xy + 1 \leq x^2y^2 + x^2 + y^2 + 1$$

which leads to a complete square $(x-y)^2 \geq 0$. This also shows that the equality holds when $x = y$, that is at an absolute umbilical point.

The second inequality we have to prove is $\tau \leq \bar{H}$, which means

$$\frac{xy - 1 + \sqrt{x^2+1}\sqrt{y^2+1}}{x+y} \leq \frac{x+y}{2}.$$

A direct cross-multiplication leads to

$$2\sqrt{x^2y^2 + x^2 + y^2 + 1} \leq x^2 + y^2 + 2,$$

which by squaring again yields

$$4x^2y^2 + 4x^2 + 4y^2 + 4 \leq x^4 + y^4 + 4 + 2x^2y^2 + 4x^2 + 4y^2,$$

which reduces to

$$0 \leq x^4 + y^4 - 2x^2y^2 = (x^2 - y^2)^2.$$

This also shows that the equality case holds in $\tau \leq \bar{H}$, when $x^2 = y^2$, that is when the point is absolutely umbilical. This is consistent with the fact noted in [5], that $A = \bar{H}$ when $x = y$, that is the equality holds when the harmonic mean of the numbers x and y equals their arithmetic mean. This represents the algebraic description of absolutely umbilical points.

The last inequality, namely $\bar{H}(p) \leq \sqrt{C(p)}$ represents nothing else but the elementary AM-QM inequality (between the arithmetic mean and the quadratic mean), applied to the positive real numbers x and y, with $|\kappa_1| = x$, and $|\kappa_2| = y$. The equality holds at the points p where $|\kappa_1| = |\kappa_2|$. □

Note that the ladder of curvatures presented in this theorem is different from the ladder of curvatures for hypersurfaces investigated in [4].

Remark: It is known that the principal curvatures are the roots of the quadratic equation

$$\kappa^2 - 2H\kappa + K = 0,$$

and hence $\kappa_1 = H + \sqrt{H^2 - K}$, and $\kappa_1 = H - \sqrt{H^2 - K}$. If κ_1 and κ_2 can be expressed in terms of the mean curvature H and of the Gaussian curvature K, it is natural to investigate how is the tangential curvature τ related to the classical curvature invariants.

Proposition 1. *Let $\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface given by the smooth map σ. Denote by κ_1 and κ_2 the principal curvatures on $\sigma(U)$. Then the tangential curvature τ, the absolute mean curvature $\bar{H}$, the mean curvature H and the Gaussian curvature K satisfy the equality*

$$\tau = \frac{|K| - 1 + \sqrt{(K-1)^2 + 4H^2}}{\bar{H}},$$

at every point p of the surface.

Proof: Denote $|\kappa_1| = a$ and $|\kappa_2| = b$, then $|K| = |ab|$. We obtain immediately

$$\tau = \frac{|K| - 1 + \sqrt{K^2 + a^2 + b^2 + 1}}{\bar{H}}.$$

Since $a^2 + b^2 = \kappa_1^2 + \kappa_2^2 = (2H)^2 - 2K$, we obtain immediately the claimed equality. □

THE SURFACE ASSOCIATED TO THE TANGENTIAL CURVATURE. All these investigations led us to consider the surface associated to the tangential curvature defined as $\tau : \mathbb{R}^2 \to \mathbb{R}^3$, defined by the vectorial equation

$$\tau(x, y) = \left\langle x, y, \frac{|xy| - 1 + \sqrt{x^2+1}\sqrt{y^2+1}}{|x| + |y|} \right\rangle.$$

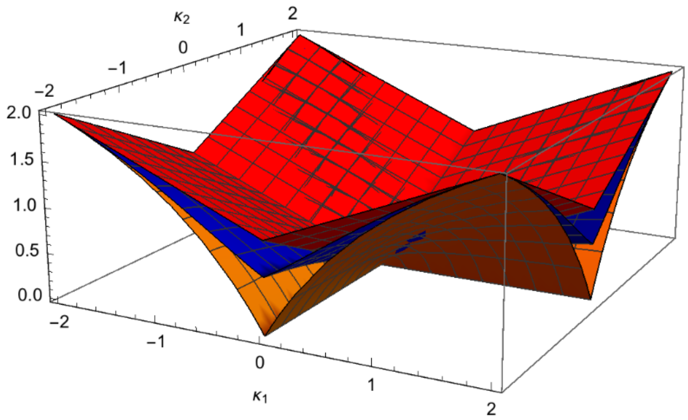
It is interesting to notice that this surface τ lies in $\mathbb{R}^3$ between the surfaces associated to the amalgamatic curvature

$$a(x, y) = \left\langle x, y, \frac{2|x||y|}{|x| + |y|} \right\rangle,$$

and the absolute mean curvature

$$b(x, y) = \left\langle x, y, \frac{|x| + |y|}{2} \right\rangle,$$

as illustrated in the following figure.



EXAMPLES. It is interesting to see what happens with the tangential curvature and with the inequality $A \le \tau \le \bar{H}$ for various surfaces.

Example 1. Consider a sphere of radius R in $\mathbb{R}^3$. Then the principal curvatures are $|\kappa_1| = |\kappa_2| = \frac{1}{R}$, which yields the amalgamatic curvature $A = \frac{1}{R}$, the tangential curvature $\tau = \frac{1}{R}$, and the mean curvature equal to absolute mean curvature $H = \bar{H} = \frac{1}{R}$. All the points of the sphere are absolutely umbilical.

Example 2. Consider the cylinder $\mathbb{R} \times S^1(r)$. Its principal curvatures are $\kappa_1 = 0$, and $\kappa_2 = \frac{1}{r}$. Then the amalgamatic curvature is $A = 0$, the tangential curvature $\tau = \sqrt{r^2+1} - r$, and the mean curvature equal to absolute mean curvature $H = \bar{H} = \frac{1}{2r}$. Since the equality case does not hold, we do not have on the cylinder any absolutely umbilical points.

Note that as $r \to 0$, we have $H = \bar{H} \to \infty$, while $\tau \to 1$.

Example 3. Consider the hyperbolic paraboloid $z = xy$. Its first fundamental form is $E = 1 + y^2$, $F = xy$, $G = 1 + x^2$, and its second fundamental form is $L = N = 0$, $M = \frac{1}{\sqrt{x^2+y^2+1}}$. The Gaussian curvature is $K = -\frac{1}{(x^2+y^2+1)^2}$, and the mean curvature is $H = -\frac{xy}{(x^2+y^2+1)^{3/2}}$. The principal curvatures for the hyperbolic paraboloid are

$$\kappa_1 = \frac{-xy + \sqrt{(x^2+1)(y^2+1)}}{(x^2+y^2+1)^{3/2}} > 0,$$

and

$$\kappa_2 = \frac{-xy - \sqrt{(x^2+1)(y^2+1)}}{(x^2+y^2+1)^{3/2}} < 0.$$

In conclusion since $\kappa_2 < 0 < \kappa_1$, it is known we can not have umbilical points on the hyperbolic paraboloid.

Can we have absolute umbilical points on this hyperbolic paraboloid? This can happen whenever

$$| - xy + \sqrt{(x^2+1)(y^2+1)}| = | - xy - \sqrt{(x^2+1)(y^2+1)}|.$$

This equality can hold in the following two cases.

Case 1. $xy - \sqrt{(x^2+1)(y^2+1)} = xy + \sqrt{(x^2+1)(y^2+1)}$. This case does not yield any solution.

Case 2. $xy - \sqrt{(x^2+1)(y^2+1)} = -xy - \sqrt{(x^2+1)(y^2+1)}$. In this case we obtain $xy = 0$, which yields the solution $x = 0$ or $y = 0$.

Thus, the points $(a, 0, 0)$ and the points $(0, a, 0)$ are absolute umbilical points on the hyperbolic paraboloid $z = xy$. At these points we have $A = \tau = \bar{H} = \sqrt{C}$.

Example 4. Consider the elliptic cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$. As an implicit quadric surface, it is known that its classical curvature invariants are

$$K = 0, \quad H = -\frac{x^2+y^2+z^2}{2a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{3/2}},$$

with the principal curvatures

$$\kappa_1 = 0, \quad \kappa_2 = -\frac{x^2+y^2+z^2}{a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{3/2}}.$$

If we except the vertex $(0, 0, 0)$ from this analysis, we note that on the elliptic cone we can not have $|\kappa_1| = |\kappa_2|$, hence on this smooth surface there are no absolute umbilical points and the inequality $A \leq \bar{H}$ is strict.

Example 5. It is well-known that the helicoid $\sigma(u, v) = \langle v\cos u, v\sin u, \lambda u\rangle$, with $\lambda > 0$, is a minimal surface, i.e. $H = 0$ at every point. Its principal curvatures are $\kappa = \pm\frac{\lambda}{\lambda^2+v^2}$, and all its points are absolutely umbilical points. Also, to illustrate what happens with the quantities studied in Theorem 1:

$$A = \tau = \bar{H} = \sqrt{C} = \frac{\lambda}{\lambda^2+v^2}.$$

Example 6. Another classical example of minimal surface is the catenoid given by the smooth parametrization $\sigma(u, v) = \langle\cosh u\cos v, \cosh u\sin v, u\rangle$. Its principal curvatures are $\kappa = \pm\frac{1}{\cosh^2 u}$, and we have

$$A = \tau = \bar{H} = \sqrt{C} = \frac{1}{\cosh^2 u}.$$

CONCLUSION. There may be geometric quantities of interest in the differential geometry of surfaces of low-dimensional hypersurfaces. We thought it useful to investigate one such quantity. It is hard to expect that other concepts will ever have the mathematical importance of Gaussian curvature and mean curvature, but nevertheless their investigation may be worthy of interest. To better illustrate our thought, in dimension $n = 3$, see [16, 17]. It is natural to ask whether there are any other quantities of interest expressing curvature in various geometric ways? What can we say about them?

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