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ϕ-Recurrent Sasakian Finsler Structures

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ABSTRACT

In this paper, ϕ-recurrent Sasakian Finsler structures on horizontal and vertical tangent bundle diffusions and their various geometric properties are studied.

Keywords: *ϕ-recurrent; ϕ-symmetric; sectional curvature; Ricci tensor; tangent bundle.*

AMS Subject Classification (2010): *Primary:53D15; Secondary: 53C05; 53C15; 53C60.*

1. Introduction

The notion of locally symmetry is a strong notion in contact geometry, so it is weakened such as semi-symmetric, recurrent, pseudo-symmetric, weakly symmetric, locally ϕ-symmetric cases. In this regard, some conditions for ϕ-recurrency of several manifolds are obtained. De et al. [3] studied the notion of ϕ-recurrency in the sense of Takahashi for Sasakian manifolds in [7]. Additionally, De et al. [4] and Peyghan et al. [5] have studies for this notion defined for Kenmotsu manifolds and contact metric manifolds, respectively. In Prakash's paper [6], ϕ-recurrency is discussed via concircular curvature tensor for para-Sasakian manifolds. In this paper, ϕ-recurrent Sasakian Finsler structures on tangent bundle diffusions are studied.

2. Preliminaries

Suppose M be an $m = (2n+1)$-dimensional smooth manifold. Besides, $x = (x^1, \dots, x^m)$ are the local coordinates of M , T_xM is an m-dimensional tangent space at $x \in M$ and $y = y^i \frac{\partial}{\partial x^i} \in T_xM$. So, TM denotes $2m$-dimensional tangent bundle of M and $u = (x, y) \in TM$.

If $F : TM \to [0, \infty[$ is a Finsler function with the following properties;

1. F is C^∞ on slit tangent bundle,
2. $F(x^i, \lambda y^i) = \lambda, F(x^i, y^i)$ for $\lambda > 0$,
3. The $n \times n$ Hessian $g_{ij}(x, y) = \frac{1}{2}[\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite on slit tangent bundle,

then $F^m = (M, F)$ is called a Finsler manifold and g is a Finsler metric tensor with g_{ij} coefficients [2].

The bundle projection $\pi : TM \to M$ and its differential map $\pi : T_uTM \to T_{\pi(u)}M$ satisfy the transformations $u = (x^1, \dots, x^m, y^1, \dots, y^m) \mapsto (x^1, \dots, x^m)$ and $X_u \mapsto \pi_*(X_u)$, respectively. Hence, the vertical subbundle VTM is derived from $ker(\pi_*)$. The horizontal subbundle $HTM = (N_i^j(x, y))$ is a non-linear connection on TM where $N_i^j = \frac{\partial N^j}{\partial y^i}$ are obtained by using spray coefficients $N^j = \frac{1}{4} g^{jk} (\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k})$.

Provided that $\triangledown$ is the linear connection on the vertical tangent bundle distribution VTM, the pair $(HTM, \triangledown)$ is called a Finsler connection on M [1].

For $X \in T_uTM$ and $X = X^i(\frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}) + (N_i^j(x, y)X^i + X^j)\frac{\partial}{\partial y^j} = X^i \frac{\delta}{\delta x^i} + X^j \frac{\partial}{\partial y^j}$ is obtained. Here, $\frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial y^j}$ are the bases of $T_u^H TM$ and $T_u^V TM$ at $u \in TM$, respectively and their dual bases are dx^i and $\delta y^j = dy^j + N_i^j dx^i$, respectively. The complementary distributions HTM and VTM consist of the base spaces H_{TM} and V_{TM} and their tangent spaces $T_u^H TM$ and $T_u^V TM$ at $u \in TM$, respectively where $T_uTM = T_u^H TM \oplus T_u^V TM$.

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By decomposing $\eta = \eta_i dx^i + \eta_j \delta y^j \in (T_u TM)^*$ to horizontal and vertical parts, it is possible to have $\eta^H \in (T_u^H TM)^*$ and $\eta^V \in (T_u^V TM)^*$.

The Sasaki-Finsler metric G on TM is defined as follows:

$$G(X, Y) = G(X^H, Y^H) + G(X^V, Y^V)$$

where $X, Y \in T_u TM$ and their horizontal and vertical lifts $X^H, Y^H \in T_u^H TM$ and $X^V, Y^V \in T_u^V TM$.

Let $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ be m-dimensional Sasakian structures on H_{TM} and on V_{TM}, respectively. Then following relations hold for Sasakian Finsler structures on H_{TM} and V_{TM} where $X^H, Y^H, Z^H \in T_u^H TM$ and $X^V, Y^V, Z^V \in T_u^V TM$, ξ is the Reeb vector field of type $(\begin{smallmatrix}0 & 0\\ 1 & 1\end{smallmatrix})$, η is the 1-form of type $(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix})$, G is the Finsler metric structure of type $(\begin{smallmatrix}0 & 0\\ 2 & 2\end{smallmatrix})$, ϕ denotes the tensor field of type $(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix})$, N is the Nijenhuis tensor field of the Sasakian Finsler structure on TM, Ω is the fundamental 2-form, $\triangledown$ is the Finsler connection with respect to G on TM, L is the Lie differential operator, R is the Riemann curvature tensor field of type $(\begin{smallmatrix}1 & 1\\ 3 & 3\end{smallmatrix})$, K is the ϕ-sectional curvature, S is the Ricci tensor field of type $(\begin{smallmatrix}0 & 0\\ 2 & 2\end{smallmatrix})$ [9].

$$\phi.\phi = -I + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V \tag{2.1}$$

$$\phi\xi^H = \phi\xi^V = 0 \tag{2.2}$$

$$\eta^H(\xi^H) = \eta^V(\xi^V) = 1 \tag{2.3}$$

$$\eta^H(\phi X^H) = 0, \eta^V(\phi X^V) = 0, \eta^H(\phi X^V) = 0 \tag{2.4}$$

$$\Omega(X^H, Y^H) = 2(\triangledown_X^H \eta)(Y^H) = -2(\triangledown_Y^H \eta)(X^H), \Omega(X^V, Y^V) = 2(\triangledown_X^V \eta)(Y^V) = -2(\triangledown_Y^V \eta)(X^V) \tag{2.5}$$

$$G^H(\phi X, \phi Y) = G^H(X, Y) - \eta^H(X^H)\eta^H(Y^H), G^V(\phi X, \phi Y) = G^V(X, Y) - \eta^V(X^V)\eta^V(Y^V) \tag{2.6}$$

$$G^H(X, \xi) = \eta^H(X^H), G^V(X, \xi) = \eta^V(X^V) \tag{2.7}$$

$$G^H(\phi X, Y) = -G^H(X, \phi Y), G^V(\phi X, Y) = -G^V(X, \phi Y) \tag{2.8}$$

$$N_\phi + d\eta^H \otimes \xi^H + d\eta^V \otimes \xi^V = 0 \tag{2.9}$$

$$\triangledown_\xi^H \phi = 0, \triangledown_\xi^V \phi = 0 \tag{2.10}$$

$$\triangledown_X^H \xi^H = -\frac{1}{2}\phi X^H, \triangledown_X^V \xi^V = -\frac{1}{2}\phi X^V \tag{2.11}$$

$$(\triangledown_X^H \phi)Y^H = \frac{1}{2}[G^H(X^H, Y^H)\xi^H - \eta^H(Y^H)X^H], (\triangledown_X^V \phi)Y^V = \frac{1}{2}[G^V(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V] \tag{2.12}$$

$$R(X^H, Y^H)\xi^H = \frac{1}{4}[\eta^H(Y^H)X^H - \eta^H(X^H)Y^H], R(X^V, Y^V)\xi^V = \frac{1}{4}[\eta^V(Y^V)X^V - \eta^V(X^V)Y^V] \tag{2.13}$$

$$R(X^H, \xi^H)Y^H = \frac{1}{4}[\eta^H(Y^H)X^H - G^H(X, Y)\xi^H], R(X^V, \xi^V)Y^V = \frac{1}{4}[\eta^V(Y^V)X^V - G^V(X, Y)\xi^V] \tag{2.14}$$

$$R(X^H,Y^H)Z^H=\frac{1}{4}[G(Z^H,Y^H)X^H-G(Z^H,X^H)Y^H], R(X^V,Y^V)Z^V=\frac{1}{4}[G(Z^V,Y^V)X^V-G(Z^V,X^V)Y^V] \tag{2.15}$$

$$K(X^H,Y^H)=G(R(X^H,Y^H)Y^H,X^H)=\frac{1}{4}, K(X^V,Y^V)=G(R(X^V,Y^V)Y^V,X^V)=\frac{1}{4} \tag{2.16}$$

$$S(\xi^H,\xi^H)=\frac{n}{2}, S(\xi^V,\xi^V)=\frac{n}{2} \tag{2.17}$$

$$S(X^H,\xi^H)=\frac{n}{2}\eta^H(X^H), S(X^V,\xi^V)=\frac{n}{2}\eta^V(X^V) \tag{2.18}$$

$$S(\phi X^H,\phi Y^H)=S(X^H,Y^H)-\frac{n}{2}\eta^H(X^H)\eta^H(Y^H), S(\phi X^V,\phi Y^V)=S(X^V,Y^V)-\frac{n}{2}\eta^V(X^V)\eta^V(Y^V) \tag{2.19}$$

However Sasakian Finsler structures can be founded both on horizontal and vertical tangent bundle diffusions (H_{TM} and V_{TM}), in this paper; it is studied on H_{TM}, for briefness.

Definition 2.1. A Sasakian Finsler structure $(\phi^H,\xi^H,\eta^H,G^H)$ on H_{TM} is locally $\phi-$ symmetric if and only if

$$\phi^2((\nabla^H_w R)(X^H,Y^H)Z^H)=0 \tag{2.20}$$

for $X^H,Y^H,Z^H,W^H\in T^H_uTM$.

Definition 2.2. A Sasakian Finsler structure $(\phi^H,\xi^H,\eta^H,G^H)$ on H_{TM} is locally ϕ- recurrent if there exists a 1-form $A(\neq 0)$ such that

$$\phi^2((\nabla^H_w R)(X^H,Y^H)Z^H)=A^H(W^H)R(X^H,Y^H)Z^H \tag{2.21}$$

for $X^H,Y^H,Z^H,W^H\in T^H_uTM$.

3. ϕ-recurrent Sasakian Finsler Structures on H_{TM}

Theorem 3.1. *Suppose that H_{TM} have a ϕ-recurrent Sasakian Finsler structure. Then H_{TM} is Einstein.*

Proof. Let H_{TM} be a ϕ-recurrent Sasakian Finsler manifold. By the virtue of (2.1) and (2.21) we have

$$-(\nabla^H_W R)(X^H,Y^H)Z^H+\eta^H((\nabla^H_W R)(X^H,Y^H)Z^H)\xi^H=A^H(W^H)R(X^H,Y^H)Z^H \tag{3.1}$$

□

Inserting G to the above relation we have

$$-G((\nabla^H_W R)(X^H,Y^H)Z^H,U^H)+\eta^H((\nabla^H_W R)(X^H,Y^H)Z^H)\eta^H(U^H)=A^H(W^H)G(R(X^H,Y^H)Z^H,U^H). \tag{3.2}$$

Let $\{E^H_i\}$, $i=1,2,\dots,2n+1$, be the orthonormal basis of T^H_uTM. Then putting $X^H=U^H=E^H_i$and taking summation over i, $1\leq i\leq 2n+1$, we get

$$-(\nabla^H_W S)(Y^H,Z^H)+\sum_{i=1}^{2n+1}\eta^H((\nabla^H_W R)(E^H_i,Y^H)Z^H)\eta^H(E^H_i)=A^H(W^H)S(Y^H,Z^H) \tag{3.3}$$

By putting $Z^H=\xi^H$ in the second term of first part of (3.3), we have the following: $\sum_{i=1}^{2n+1}\eta^H((\nabla^H_W R)(E^H_i,Y^H)Z^H)\eta^H(E^H_i)=G((\nabla^H_W R)(E^H_i,Y^H)\xi^H,\xi^H)G(E^H_i,\xi^H)$

which is equivalent to

$$\begin{aligned}G((\nabla^H_W R)(E^H_i,Y^H)\xi^H,\xi^H)&=G((\nabla^H_W R)(E^H_i,Y^H)\xi^H,\xi^H)-G((\nabla^H_W E^H_i,Y^H)\xi^H,\xi^H)\\&-G(R(E^H_i,\nabla^H_W Y^H)\xi^H,\xi^H)-G(R(E^H_i,Y^H)\nabla^H_W\xi^H,\xi^H)\end{aligned} \tag{3.4}$$

Owing to the fact that E^H_i is the orthonormal basis, it is easily seen that $\nabla^H_W E^H_i=0$.
By virtue of (2.13) we have

$$G(R(E_i^H,\nabla_W^H Y^H)\xi^H,\xi^H)=\tfrac{1}{4}G([\eta^H(\nabla_W^H Y^H)E_i^H-\eta^H(E_i^H)\nabla_W^H Y^H],\xi^H)$$
$$=\tfrac{1}{4}[G(\nabla_W^H Y^H,\xi^H)G(E_i^H,\xi^H)-G(E_i^H,\xi^H)G(\nabla_W^H Y^H,\xi^H)]=0.$$

By using these equalities, second and third terms of the right part of (3.4) vanish. Hence we can state the following:

$$G((\nabla_W^H R)(E_i^H,Y^H)\xi^H,\xi^H)=G((\nabla_W^H R)(E_i^H,Y^H)\xi^H,\xi^H)-G(R(E_i^H,Y^H)\nabla_W^H\xi^H,\xi^H). \tag{3.5}$$

Due to $G((\nabla_W^H R)(E_i^H,Y^H)\xi^H,\xi^H)+G(R(E_i^H,Y^H)\xi^H,\nabla_W^H\xi^H)=0$, we can denote (3.5) as follows:

$$G((\nabla_W^H R)(E_i^H,Y^H)\xi^H,\xi^H)=-G((R)(E_i^H,Y^H)\xi^H,\nabla_W^H\xi^H)-G(R(E_i^H,Y^H)\xi^H,\nabla_W^H\xi^H)=0.$$

In consequence of these calculations and by replacing $Z^H=\xi^H$, (3.3) can be expressed by the following relation:

$$-(\nabla_W^H S)(Y^H,Z^H)=A^H(W^H)S(Y^H,Z^H)=A^H(W^H)\frac{n}{2}\eta^H(Y^H). \tag{3.6}$$

Using (2.11) and (2.18) in $(\nabla_W^H S)(Y^H,Z^H)=\nabla_W^H(S(Y^H,\xi^H))-S(\nabla_W^H Y^H,\xi^H)-S(\nabla_W^H Y^H,\nabla_W^H\xi^H)$ we obtain

$$(\nabla_W^H S)(Y^H,Z^H)=\frac{n}{4}G(W,\phi Y^H)+\frac{1}{2}S(Y^H,\phi W^H). \tag{3.7}$$

By virtue of (3.6) and (3.7) we have

$$-A^H(W^H)\frac{n}{2}\eta^H(Y^H)=\frac{n}{4}G(W,\phi Y^H)+\frac{1}{2}S(Y^H,\phi W^H). \tag{3.8}$$

By putting $Y^H=\phi Y^H$ in (3.8) we get $S(Y^H,W^H)=\frac{n}{2}G(Y^H,W^H)$ for all $Y^H,W^H\in T_u^H TM$.

Corollary 3.1. *A ϕ-recurrent Sasakian Finsler manifold $(H_{TM},\phi^H,\xi^H,\eta^H,G^H)$ is an Einstein manifold such that the Ricci tensor S is proportional to the Sasaki Finsler metric tensor: $S(Y^H,W^H)=\frac{n}{2}G(Y^H,W^H)\in Sym^2(T_u^H TM)$ for all $Y^H,W^H\in T_u^H TM$ by a proportionality constant $\frac{n}{2}\in\mathbb{R}$.*

Theorem 3.2. *Assume that $(H_{TM},\phi^H,\xi^H,\eta^H,G^H)$ be a $(2n+1)$-dimensional $(n>1)$ ϕ-recurrent Sasakian Finsler manifold, then we have the following relation:*

$$A^H(W^H)=\eta^H(W^H)\eta^H(p^H). \tag{3.9}$$

where $W^H,p^H\in T_u^H TM$ and $A^H\in(T_u^H TM)$ with the property of $A^H(\xi^H)=G(\xi^H,p^H)\eta^H(p^H)$.

Proof. By using (2.1) in (2.21) we obtain

$$-(\nabla_W^H R)(X^H,Y^H)Z^H+\eta^H(\nabla_W^H R(X^H,Y^H)Z^H)\xi^H=A^H(W^H)R(X^H,Y^H)Z^H. \tag{3.10}$$

Taking inner product of (3.10) with ξ^H and using the Bianchi identity which is given in the following line;

$$(\nabla_W^H R)(X^H,Y^H)Z^H+(\nabla_Y^H R)(W^H,X^H)Z^H+(\nabla_W^H R)(Y^H,W^H)Z^H=0,$$

it is possible to have

$$A^H(W^H)G(R(X^H,Y^H)Z^H,\xi^H)=A^H(W^H)\eta^H(R(X^H,Y^H)Z^H)=0 \tag{3.11}$$

Similarly, we have the following relations:

$$A^H(Y^H)\eta^H(R(W^H,X^H)Z^H)=0 \tag{3.12}$$

$$A^H(X^H)\eta^H(R(Y^H,W^H)Z^H)=0 \tag{3.13}$$

In consequence of these calculations we obtain

$$A^H(W^H)\eta^H(R(X^H,Y^H)Z^H)+A^H(Y^H)\eta^H(R(W^H,X^H)Z^H)+A^H(X^H)\eta^H(R(Y^H,W^H)Z^H)=0 \tag{3.14}$$

If we use relation (2.15) in each terms for Riemannian curvature tensor in (3.14), we get

$$\frac{1}{4}[A^H(W^H)[G(Y^H,Z^H)\eta^H(X^H)-G(X^H,Z^H)\eta^H(Y^H)]$$
$$+A^H(Y^H)[G(W^H,Z^H)\eta^H(Y^H)-G(Y^H,Z^H)\eta^H(W^H)]$$
$$+A^H(X^H)[G(X^H,Z^H)\eta^H(W^H)-G(W^H,Z^H)\eta^H(X^H)]]=0 \tag{3.15}$$

By putting $Y^H=Z^H=E_i^H$ in (3.15) we have

$$A^H(W^H)\eta^H(X^H)=A^H(X^H)\eta^H(W^H).$$

By replacing $X^H=\xi^H$, we find

$$A^H(W^H)=\eta^H(P^H)\eta^H(W^H).$$

Theorem 3.3. *Suppose $(H_{TM},\phi^H,\xi^H,\eta^H,G^H)$ be a $(2n+1)$- dimensional $(n>1)$ ϕ -recurrent Sasakian Finsler manifold, then it is of constant curvature where $X^H,Y^H\in T_u^HTM$.*

Proof. With the help of (2.5), (2.11) and (2.13), it is possible to get the below relation: □

$$(\nabla_W^H R)(X^H,Y^H)\xi^H=\nabla_W^H(R(X^H,Y^H)\xi^H)-R(\nabla_W^H X^H,Y^H)\xi^H-R(X^H,\nabla_W^H Y^H)\xi^H-R(X^H,Y^H)\nabla_W^H\xi^H \tag{3.16}$$

By using (2.11) and (2.13) in (3.16), we have

$$(\nabla_W^H R)(X^H,Y^H)\xi^H=\frac{1}{4}[\frac{1}{2}\{G(W^H,\phi Y^H)X^H-G(W^H,\phi X^H)Y^H\}]+\frac{1}{2}R(X^H,Y^H)\phi W^H \tag{3.17}$$

Taking inner product of (3.17) with ξ^H, we have the following:

$$G((\nabla_W^H R)(X^H,Y^H)\xi^H,\xi^H)=\frac{1}{8}G(W^H,\phi Y^H)G(X^H,\xi^H)$$
$$-\frac{1}{8}G(W^H,\phi X^H)G(Y^H,\xi^H)+\frac{1}{2}G(R(X^H,Y^H)\phi W^H,\xi^H) \tag{3.18}$$

because of, $\eta^H(X^H)=0$ and $\eta^H(Y^H)=0$ for $X^H,Y^H\in T_u^HTM$, so we can calculate (3.17) in the following way:

$$(\eta^H(\nabla_W^H R)(X^H,Y^H)\xi^H)=0. \tag{3.19}$$

If we use the following equality in (3.17) [8],

$$R(X^H,Y^H)\phi W^H=$$
$$\phi R(X^H,Y^H)Z^H+\tfrac{1}{4}\{G(\phi X^H,W^H)Y^H-G(Y^H,W^H)\phi X^H+G(X^H,W^H)\phi Y^H-G(\phi Y^H,W^H)X^H\}$$

we have

$$(\nabla_W^H R)(X^H,Y^H)\xi^H=\frac{1}{2}[\phi R(X^H,Y^H)W^H+\frac{1}{4}\{G(X^H,W^H)\phi Y^H-G(Y^H,W^H)\phi X^H\}]. \tag{3.20}$$

By replacing $Z^H=\xi^H$ in (3.1), we get

$$(\nabla_W^H R)(X^H,Y^H)\xi^H=-A^H(W^H)R(X^H,Y^H)\xi^H. \tag{3.21}$$

From the equations (3.20) and (3.21), we obtain

$$\tfrac{1}{2}\{\phi R(X^H,Y^H)W^H+\tfrac{1}{4}[G(X^H,W^H)\phi Y^H-G(Y^H,W^H)\phi X^H]\}=$$
$$\eta^H(W^H)\eta^H(W^H)\{\tfrac{1}{4}[\eta^H(p^H)X^H-\eta^H(X^H)Y^H]\}.$$

Here, $\eta^H(X^H)=0$ and $\eta^H(Y^H)=0$. This means:

$$\phi R(X^H,Y^H)W^H=\tfrac{1}{4}\{-G(X^H,W^H)\phi Y^H+G(Y^H,W^H)\phi X^H\}.$$

By applying ϕ to the last equation, we have the following relation that means H_{TM} is of constant curvature:

$$R(X^H, Y^H)W^H = \frac{1}{4}\{G(Y^H, W^H)X^H - G(X^H, W^H)Y^H\} \tag{3.22}$$

□

In addition, by the virtue of (3.1), we find

$$(\nabla^H_W R)(X^H, Y^H)Z^H = -A^H(W^H)R(X^H, Y^H)Z^H + \frac{1}{2}\{G(\phi R(X^H, Y^H)W^H, Z^H) + \frac{1}{4}[G(X^H, W^H)G(\phi Y^H, Z^H)G(Y^H, W^H)G(\phi X^H, Z^H)]\}\xi^H. \tag{3.23}$$

So, it is possible to have the below corollary:

Corollary 3.2. *Let $(H_{TM}, \phi^H, \xi^H, \eta^H, G^H)$ be a $(2n+1)$-dimensional $(n > 1)$ ϕ -recurrent Sasakian Finsler manifold then the relation (3.23) holds.*

$$\phi^2((\nabla^H_W R)(X^H, Y^H)Z^H) =$$
$$A^H(W^H)R(X^H, Y^H)Z^H - (\eta^H(\nabla^H_W R)(X^H, Y^H)\xi^H) + (\eta^H(\nabla^H_W R)(X^H, Y^H)\xi^H) - A^H(W^H)\eta^H(R(X^H, Y^H)Z^H)$$
$$= A^H(W^H)R(X^H, Y^H)Z^H - \tfrac{1}{4}A^H(W^H)\{G(Y^H, Z^H)\eta^H(X^H) - G(X^H, Z^H)\eta^H(Y^H)\}\xi^H.$$

Namely, the following relation is found

$$\phi^2((\nabla^H_W R)(X^H, Y^H)Z^H) = A^H(W^H)R(X^H, Y^H)Z^H$$

which enables to get following corollary:

Corollary 3.3. *Assume $(H_{TM}, \phi^H, \xi^H, \eta^H, G^H)$ be a $(2n+1)$-dimensional $(n > 1)$ ϕ -recurrent Sasakian Finsler manifold that satisfies the relation (3.23) then it is ϕ-recurrent.*

Theorem 3.4. *Suppose $(H_{TM}, \phi^H, \xi^H, \eta^H, G^H)$ be a $(2n+1)$-dimensional $(n > 1)$ ϕ -recurrent Sasakian Finsler manifold then there is a non-zero constant sectional curvature of it and it is locally ϕ-symmetric.*

Proof. If we suppose that $\pi = Sp\{X^H, Y^H\}$ and $X^H, Y^H \in T^H_u TM$ so $\pi \in T^H_u TM$, in this case we have $K(\pi) = G(R(X^H, Y^H)Y^H, X^H) = \lambda$, here λ is a non-zero constant and X^H, Y^H are orthonormal basis of π. Then we have

$$G((\nabla^H_Z R)(X^H, Y^H)Y^H, X^H) = 0 \tag{3.24}$$

$$G((\nabla^H_Z R)(X^H, Y^H)Y^H, \xi^H) = \eta^H((\nabla^H_Z R)(X^H, Y^H)Y^H)\eta^H(\xi^H) - A^H(Z^H)G(R(X^H, Y^H)Y^H, \xi^H)$$

$$G((\nabla^H_Z R)(X^H, Y^H)Y^H, \xi^H)\eta^H(X^H) = G((\nabla^H_Z R)(X^H, Y^H)Y^H, X^H) - A^H(Z^H)G(R(X^H, Y^H)Y^H, \xi^H)$$

$$\eta^H(X^H)[\tfrac{1}{8}\{G(Y^H, Z^H)G(\phi X^H, Y^H) - G(X^H, Z^H)G(\phi Y^H, Y^H)\} - \tfrac{1}{2}G(\phi R(X^H, Y^H)Z^H, Y^H) - A^H(Z^H)G(R(X^H, Y^H)Y^H, \xi^H)] = -\lambda A^H(\xi^H)$$

By replacing $Z^H = \xi^H$ in the last relation, we get

$$\eta^H(p^H)[G(Y^H, Y^H)\eta^H(X^H) - G(X^H, Y^H)\eta^H(Y^H)] = -\lambda\eta^H(p^H),$$

$$\eta^H(p^H)[\tfrac{1}{4}\{G(Y^H, Y^H)\eta^H(X^H) - G(X^H, Y^H)\eta^H(Y^H)\}] - \lambda = 0,$$

$$\eta^H(p^H) = 0.$$

Consequently;

$$\phi^2((\nabla^H_W R)(X^H, Y^H)Z^H) = 0.$$

□

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