

Elliptic Matrix Representations of Elliptic Biquaternions and Their Applications

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(Communicated by Yusuf Yaylı)

ABSTRACT

In this study, we obtain the 4×4 elliptic matrix representations of elliptic biquaternions with the aid of the left and right Hamilton operators. Afterwards, we show that the space of 4×4 matrices generated by left Hamilton operator is isomorphic to the space of elliptic biquaternions. Then, we study the De-Moivre's and Euler formulas for the matrices of this matrix space. Additionally, the powers of these matrices are obtained with the aid of the De-Moivre's formula.

Keywords: Complex quaternions; Hamilton matrices; Matrix representations; Elliptic biquaternions.

AMS Subject Classification (2010): Primary: 11R52 ; Secondary: 15B99;

1. Introduction

The set of quaternions was introduced by Sir W. R. Hamilton in 1843 [1]. The set of quaternions can be represented as

$$H = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where the quaternionic units \mathbf{i} , \mathbf{j} and \mathbf{k} satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The real quaternion algebra has an important place in kinematic, differential geometry, quantum physics and image processing etc.

We know that real quaternion algebra is isomorphic to a real 4×4 matrix algebra. The real matrix representations of quaternions are investigated in [2, 3, 4]. On the other hand, the Euler's and De-Moivre's formulas for the complex numbers are generalized to quaternions in [5]. These formulas are investigated for real matrices associated with real quaternions in [6].

W. R Hamilton also introduced the complex quaternion algebra within ten years of the discovery of quaternions [7]. Complex quaternion algebra $H_{\mathbb{C}}$ is a generalization of real quaternion algebra by permitting q_0, q_1, q_2 and q_3 to be complex numbers where \mathbf{i} , \mathbf{j} and \mathbf{k} are exactly the same in real quaternions. That is, the set of complex quaternions can be represented as

$$H_{\mathbb{C}} = \{Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} : Q_0, Q_1, Q_2, Q_3 \in \mathbb{C}\}.$$

Euler formula and De-Moivre's formula for complex quaternions are discussed by M. Jafari in [8]. M. Jafari in this study, also give these formulas for 4×4 complex matrices generated by left Hamilton operator. Hamilton operators have an important role in the quaternion theory. Some algebraic properties of Hamilton operators are taken into consideration in [9]. Also, some studies which are performed by using the Hamilton operators can be found in [10, 11, 12, 13].

Recently, we have presented the elliptic biquaternions and investigated the terms; inverse, inner product, seminorm, modulus and conjugate for them in [14]. Also, some properties of elliptic biquaternions relating to seminorm and the three types of conjugate have been given and the idempotent elements in the elliptic biquaternion

algebra have been discussed in [15]. Lastly, some algebraic relations between elliptic biquaternions and extended p -trigonometric functions have been discussed in [16].

In this paper, firstly, we present an introduction to elliptic biquaternions and recall some information on elliptic matrices for the necessary background. Afterwards, with the help of the Hamilton operators, we obtain 4×4 elliptic matrix representations of elliptic biquaternions. Also, we show that the matrix space, which includes the matrix representations corresponding to left Hamilton operator, is isomorphic to the space of elliptic biquaternions. Then, we call the matrix, which corresponds to an elliptic biquaternion Q under this isomorphism, as the elliptic matrix associated with Q . Finally, we express the De-Moivre's and Euler Formulas for these matrices by writing them in the p -trigonometric form and obtain the powers of these matrices by means of the De-Moivre's formula.

Throughout this paper, the following notations are used. Let \mathbb{C}_p , HC_p and $M_{m \times n}(\mathbb{C}_p)$ denote the elliptic number field, the elliptic biquaternion algebra and the set of all $m \times n$ elliptic matrices, respectively.

2. Preliminaries

In this section, firstly, we recall the fundamental concepts of elliptic biquaternions (see [14] and [16] for more details.)

The set of elliptic biquaternions is given in the cartesian form as in the following

$$HC_p = \{Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} : A_0, A_1, A_2, A_3 \in \mathbb{C}_p\}$$

where $\mathbb{C}_p = \{x + Iy : x, y \in \mathbb{R}, I^2 = p < 0, p \in \mathbb{R}\}$ states the system of elliptic numbers and \mathbf{i}, \mathbf{j} and \mathbf{k} are the quaternionic units which satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

For any two elliptic biquaternions $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, addition and multiplication by a scalar $\lambda \in \mathbb{C}_p$ are defined as

$$Q + R = (A_0 + B_0) + (A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k}$$

$$\lambda Q = (\lambda A_0) + (\lambda A_1)\mathbf{i} + (\lambda A_2)\mathbf{j} + (\lambda A_3)\mathbf{k}.$$

and also, the product of Q and R is given as follows:

$$QR = [(A_0B_0) - (A_1B_1) - (A_2B_2) - (A_3B_3)] + [(A_0B_1) + (A_1B_0) + (A_2B_3) - (A_3B_2)]\mathbf{i} + [(A_0B_2) - (A_1B_3) + (A_2B_0) + (A_3B_1)]\mathbf{j} + [(A_0B_3) + (A_1B_2) - (A_2B_1) + (A_3B_0)]\mathbf{k}, [14].$$

Moreover, for an elliptic biquaternion $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, the Hamiltonian conjugate, complex conjugate and total conjugate of Q are

$$\bar{Q} = A_0 - A_1\mathbf{i} - A_2\mathbf{j} - A_3\mathbf{k}$$

$$Q^* = A_0^* + A_1^*\mathbf{i} + A_2^*\mathbf{j} + A_3^*\mathbf{k}$$

$$Q^\dagger = A_0^* - A_1^*\mathbf{i} - A_2^*\mathbf{j} - A_3^*\mathbf{k}$$

respectively. Here, superscript stars which are on A_0, A_1, A_2 and A_3 denote the usual complex conjugation. If $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ are any two elliptic biquaternions then the inner product of Q and R is given as in the following:

$$\langle Q, R \rangle = \frac{1}{2} (\bar{Q}R + R\bar{Q}) = \frac{1}{2} (Q\bar{R} + R\bar{Q}) = A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3.$$

The semi-norm of an elliptic biquaternion $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ is defined as

$$N_Q = \langle Q, Q \rangle = A_0^2 + A_1^2 + A_2^2 + A_3^2 = Q\bar{Q} = \bar{Q}Q$$

via aforementioned inner product. In addition, an elliptic biquaternion which has a unit real semi norm is called unit elliptic biquaternion, [14]. The set of all unit elliptic biquaternions is denoted by η and is given as in the following

$$\eta = \{Q \in HC_p : N_Q = 1\} \subset HC_p, [16].$$

Also, an elliptic biquaternion whose scalar part is zero called pure elliptic biquaternion, [14]. The set of all pure elliptic biquaternions whose semi-norm is equal to $|p|$ is denoted by ζ and is given as in the following

$$\zeta = \{ \mathbf{V} = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} : N_{\mathbf{V}} = |p|, \bar{\mathbf{V}} = -\mathbf{V}, \mathbf{V}^2 = p \}, [16].$$

On the other hand, the multiplicative inverse of an elliptic biquaternion Q which has a nonzero semi-norm is defined as $Q^{-1} = \frac{\bar{Q}}{N_Q}$. Also, for an elliptic biquaternion Q , its modulus is denoted by $|Q|$ and satisfies the equality $N_Q = |Q|^2$, [14].

In elliptic plane, due to [17], it is well known that the p trigonometric functions; p -cosine (cosp) and p -sine (sinp) are defined as

$$\text{cosp } \theta_p = \cos \left(\theta_p \sqrt{|p|} \right), \quad \text{sinp } \theta_p = \frac{1}{\sqrt{|p|}} \sin \left(\theta_p \sqrt{|p|} \right)$$

where $\theta_p = \frac{1}{\sqrt{|p|}} \tan^{-1} \left(\frac{y}{x} \sqrt{|p|} \right)$ is p -argument of an elliptic number $z = x + Iy$. On the other hand, the functions p -cosine (cosp) and p -sine (sinp) of an elliptical complex variable φ_p are given by the equalities

$$\text{cosp } \varphi_p = \cos \left(\varphi_p \sqrt{|p|} \right) = \frac{e^{I\varphi_p} + e^{-I\varphi_p}}{2} = \text{cosp } x \cosh (py) + I \text{sinp } x \sinh (py) = \sum_{n=0}^{\infty} \frac{p^n \varphi_p^{2n}}{(2n)!} \quad (2.1)$$

and

$$\text{sinp } \varphi_p = \frac{\sin \left(\varphi_p \sqrt{|p|} \right)}{\sqrt{|p|}} = \frac{e^{I\varphi_p} - e^{-I\varphi_p}}{2I} = \text{sinp } x \cosh (py) + I \frac{1}{p} \text{cosp } x \sinh (py) = \sum_{n=0}^{\infty} \frac{p^n \varphi_p^{2n+1}}{(2n+1)!} \quad (2.2)$$

in which case, the angle $\varphi_p = x + Iy \in \mathbb{C}_p$ is called as elliptical complex angle. For any two elliptical complex angles φ_p and ψ_p , the following addition formulas hold [16]:

$$\begin{aligned} (a) \quad & \text{cosp} (\varphi_p + \psi_p) = \text{cosp } \varphi_p \text{cosp } \psi_p + p \text{sinp } \varphi_p \text{sinp } \psi_p, \\ (b) \quad & \text{sinp} (\varphi_p + \psi_p) = \text{sinp } \varphi_p \text{cosp } \psi_p + \text{cosp } \varphi_p \text{sinp } \psi_p. \end{aligned} \quad (2.3)$$

Let $Q = A_0 + A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}$ be any elliptic biquaternion with a non-zero modulus while $\sqrt{A_1^2 + A_2^2 + A_3^2}$ is non-zero. In this case, Q can be given in the p -trigonometric form:

$$Q = R (\text{cosp } \varphi_p + \mathbf{w}_Q \text{sinp } \varphi_p) \quad (2.4)$$

with

$$(a) R = \sqrt{A_0^2 + A_1^2 + A_2^2 + A_3^2}, \quad (b) \text{cosp } \varphi_p = \frac{A_0}{R}, \quad (c) \text{sinp } \varphi_p = \frac{\sqrt{A_1^2 + A_2^2 + A_3^2}}{R\sqrt{|p|}}, \quad (d) \mathbf{w}_Q = \sqrt{|p|} \frac{A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}}{\sqrt{A_1^2 + A_2^2 + A_3^2}} \quad (2.5)$$

where φ_p is an elliptical complex angle in the form $\varphi_p = x + Iy \in \mathbb{C}_p$, [16].

Also, a natural generalization of Euler's formula for elliptic biquaternions is given as

$$e^{\mathbf{w}_Q \varphi_p} = \text{cosp } \varphi_p + \mathbf{w}_Q \text{sinp } \varphi_p \quad (2.6)$$

for any elliptic number φ_p . An elliptic biquaternion $Q = R (\text{cosp } \varphi_p + \mathbf{w}_Q \text{sinp } \varphi_p) \in H\mathbb{C}_p$ can be given in the elliptical exponential form as in the following:

$$Q = R e^{\mathbf{w}_Q \varphi_p} \quad (2.7)$$

by means of Euler's formula, [16]. Moreover, Q satisfies the property which is given in the next theorem.

Theorem 2.1. For a given elliptic biquaternion $Q = R (\text{cosp } \varphi_p + \mathbf{w}_Q \text{sinp } \varphi_p) = R e^{\mathbf{w}_Q \varphi_p} \in H\mathbb{C}_p$, the equality

$$Q^n = R^n [\text{cosp} (n\varphi_p) + \mathbf{w}_Q \text{sinp} (n\varphi_p)] = R^n e^{\mathbf{w}_Q n\varphi_p}$$

is satisfied where n is any integer, [16].

Lastly, we recall some information on the elliptic matrices to disambiguate the ensuing sections (See [18] for more details).

In the matrix set $M_{m \times n}(\mathbb{C}_p)$, which includes $m \times n$ matrices with elliptic number entries, the ordinary matrix addition and multiplication are defined. Also, the scalar multiplication is defined as

$$\lambda A = \lambda(a_{ij}) = (\lambda a_{ij}) \in M_{m \times n}(\mathbb{C}_p)$$

where $A = (a_{ij}) \in M_{m \times n}(\mathbb{C}_p)$ and $\lambda \in \mathbb{C}_p$.

For an elliptic matrix $A = (a_{ij}) \in M_{m \times n}(\mathbb{C}_p)$, it is known that the matrix $\bar{A} = (a_{ij}^*) \in M_{m \times n}(\mathbb{C}_p)$ is the complex conjugate of A where a_{ij}^* is the usual complex conjugation of $a_{ij} \in \mathbb{C}_p$. Also, the matrix $A^* = (\bar{A})^T \in M_{n \times m}(\mathbb{C}_p)$ is the conjugate transpose of A , [18].

3. Hamilton Matrices of Elliptic Biquaternions

Let Q be an elliptic biquaternion. We define the linear maps

$$\begin{aligned} h^+ : HC_p &\rightarrow HC_p & h^- : HC_p &\rightarrow HC_p \\ R &\rightarrow h^+(R) = QR & R &\rightarrow h^-(R) = RQ. \end{aligned}$$

These operators are called Hamilton operators. Using our newly defined operators and the base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of HC_p , we can write

$$\begin{aligned} h^+(1) &= Q1 = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \\ h^+(\mathbf{i}) &= Qi = -A_1 + A_0\mathbf{i} + A_3\mathbf{j} - A_2\mathbf{k} \\ h^+(\mathbf{j}) &= Qj = -A_2 - A_3\mathbf{i} + A_0\mathbf{j} + A_1\mathbf{k} \\ h^+(\mathbf{k}) &= Qk = -A_3 + A_2\mathbf{i} - A_1\mathbf{j} + A_0\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} h^-(1) &= 1Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \\ h^-(\mathbf{i}) &= \mathbf{i}Q = -A_1 + A_0\mathbf{i} - A_3\mathbf{j} + A_2\mathbf{k} \\ h^-(\mathbf{j}) &= \mathbf{j}Q = -A_2 + A_3\mathbf{i} + A_0\mathbf{j} - A_1\mathbf{k} \\ h^-(\mathbf{k}) &= \mathbf{k}Q = -A_3 - A_2\mathbf{i} + A_1\mathbf{j} + A_0\mathbf{k}. \end{aligned}$$

Then, we find the following elliptic matrix representations

$$H^+(Q) = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix} \tag{3.1}$$

and

$$H^-(Q) = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{pmatrix} \tag{3.2}$$

which correspond to Hamilton operators h^+ and h^- , respectively.

By looking at an elliptic biquaternion as a four-dimensional vector, the multiplications of two elliptic biquaternions $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ can be described by matrix-vector products as:

$$QR = H^+(Q)R = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{3.3}$$

and

$$RQ = H^-(Q)R = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}. \quad (3.4)$$

As already discussed up to now, the meanings of the symbols; over bar, dagger and superscript star vary according to terms which they are applied to. For the rest of the paper, we need to warn the readers about these cases.

Theorem 3.1. Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ be two elliptic biquaternions in the space HC_p and λ be an elliptic number. Then the following properties hold:

- (1) $Q = R \Leftrightarrow H^+(Q) = H^+(R) \Leftrightarrow H^-(Q) = H^-(R)$,
- (2) $H^+(Q + R) = H^+(Q) + H^+(R)$, $H^-(Q + R) = H^-(Q) + H^-(R)$,
- (3) $H^+(\lambda Q) = \lambda H^+(Q)$, $H^-(\lambda Q) = \lambda H^-(Q)$,
- (4) $H^+(QR) = H^+(Q)H^+(R)$, $H^-(QR) = H^-(R)H^-(Q)$,
- (5) $H^+(Q^{-1}) = [H^+(Q)]^{-1}$, $H^-(Q^{-1}) = [H^-(Q)]^{-1}$,
- (6) $H^+(\overline{Q}) = [H^+(Q)]^T$, $H^-(\overline{Q}) = [H^-(Q)]^T$,
- (7) $H^+(Q^*) = \overline{H^+(Q)}$, $H^-(Q^*) = \overline{H^-(Q)}$,
- (8) $H^+(Q^\dagger) = [H^+(Q)]^*$, $H^-(Q^\dagger) = [H^-(Q)]^*$,
- (9) $\det H^+(Q) = (N_Q)^2$, $\det H^-(Q) = (N_Q)^2$, $\text{Trace } [H^+(Q)] = \text{Trace } [H^-(Q)] = 4A_0$.

Proof. We will prove 4 and 5 the other properties can be easily shown.

(4) For convenience, let us show the components of the elliptic biquaternion QR with A, B, C, D . That is QR can be written in the following form:

$$QR = A + B\mathbf{i} + C\mathbf{j} + D\mathbf{k} \quad (3.5)$$

where $A = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$, $B = A_0B_1 + A_1B_0 + A_2B_3 - A_3B_2$, $C = A_0B_2 - A_1B_3 + A_2B_0 + A_3B_1$ and $D = A_0B_3 + A_1B_2 - A_2B_1 + A_3B_0$. Then we obtain

$$H^+(QR) = \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix} = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix} \begin{pmatrix} B_0 & -B_1 & -B_2 & -B_3 \\ B_1 & B_0 & -B_3 & B_2 \\ B_2 & B_3 & B_0 & -B_1 \\ B_3 & -B_2 & B_1 & B_0 \end{pmatrix} = H^+(Q)H^+(R)$$

and

$$H^-(QR) = \begin{pmatrix} A & -B & -C & -D \\ B & A & D & -C \\ C & -D & A & B \\ D & C & -B & A \end{pmatrix} = \begin{pmatrix} B_0 & -B_1 & -B_2 & -B_3 \\ B_1 & B_0 & B_3 & -B_2 \\ B_2 & -B_3 & B_0 & B_1 \\ B_3 & B_2 & -B_1 & B_0 \end{pmatrix} \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{pmatrix} = H^-(R)H^-(Q).$$

(5) From the inverse property, we can write

$$QQ^{-1} = Q^{-1}Q = 1.$$

Then, the equations

$$H^+(Q)H^+(Q^{-1}) = H^+(QQ^{-1}) = H^+(1) = I_4$$

and

$$H^+(Q^{-1})H^+(Q) = H^+(Q^{-1}Q) = H^+(1) = I_4$$

are satisfied. It means that $H^+(Q^{-1}) = [H^+(Q)]^{-1}$. In the same manner, one can see that $H^-(Q^{-1}) = [H^-(Q)]^{-1}$. \square

Now we will define a function γ from the elliptic biquaternion space HC_p to matrix space M as in the following

$$\begin{aligned} \gamma : HC_p &\rightarrow M \\ Q &\rightarrow \gamma(Q) = H^+(Q) \end{aligned}$$

where

$$M = \left\{ \left(\begin{array}{cccc} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{array} \right) : A_0, A_1, A_2, A_3 \in \mathbb{C}_p \right\}.$$

The function γ is bijection and it comprises the following properties

$$\begin{aligned} \gamma(QR) &= \gamma(Q)\gamma(R) \\ \gamma(Q + R) &= \gamma(Q) + \gamma(R) \end{aligned}$$

where Q and R are any elliptic biquaternions. Therefore, γ is a linear isomorphism. But, as a consequence of theorem 3.1 (4), a map such $\delta : Q \rightarrow \delta(Q) = H^-(Q)$ is not a linear isomorphism. Considering the linear isomorphism γ , we call the matrix $H^+(Q)$ as 4×4 elliptic matrix associated with Q .

4. Applications of 4×4 Elliptic Matrices Associated with the Elliptic Biquaternions

Let $Q = R(\cos \varphi_p + \mathbf{w}_Q \sin \varphi_p)$ be an elliptic biquaternion in the p -trigonometric form. Since Q can be written as

$$Q = (R \cos \varphi_p) + (Rw_1 \sin \varphi_p) \mathbf{i} + (Rw_2 \sin \varphi_p) \mathbf{j} + (Rw_3 \sin \varphi_p) \mathbf{k}$$

where $\mathbf{w}_Q = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$, we get the elliptic matrix $H^+(Q)$ in the p -trigonometric form as

$$H^+(Q) = \begin{pmatrix} R \cos \varphi_p & -Rw_1 \sin \varphi_p & -Rw_2 \sin \varphi_p & -Rw_3 \sin \varphi_p \\ Rw_1 \sin \varphi_p & R \cos \varphi_p & -Rw_3 \sin \varphi_p & Rw_2 \sin \varphi_p \\ Rw_2 \sin \varphi_p & Rw_3 \sin \varphi_p & R \cos \varphi_p & -Rw_1 \sin \varphi_p \\ Rw_3 \sin \varphi_p & -Rw_2 \sin \varphi_p & Rw_1 \sin \varphi_p & R \cos \varphi_p \end{pmatrix} = R [\cos \varphi_p I_4 + \sin \varphi_p H^+(\mathbf{w}_Q)].$$

Note that

$$H^+(\mathbf{w}_Q) = \begin{pmatrix} 0 & -w_1 & -w_2 & -w_3 \\ w_1 & 0 & -w_3 & w_2 \\ w_2 & w_3 & 0 & -w_1 \\ w_3 & -w_2 & w_1 & 0 \end{pmatrix}. \tag{4.1}$$

Lemma 4.1. Let $Q = A_0 + A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} = R(\cos \varphi_p + \mathbf{w}_Q \sin \varphi_p) \in HC_p$ be given. Then, the equality $w_1^2 + w_2^2 + w_3^2 = -p$ is satisfied where $\mathbf{w}_Q = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$.

Proof. From (2.5)d, we know that

$$w_i = \sqrt{|p|} \frac{A_i}{\sqrt{A_1^2 + A_2^2 + A_3^2}}, \quad i = 1, 2, 3.$$

Thus we get

$$w_1^2 + w_2^2 + w_3^2 = |p| \frac{A_1^2 + A_2^2 + A_3^2}{A_1^2 + A_2^2 + A_3^2} = |p| = -p. \quad \square$$

From the equation (4.1) and lemma 4.1, one immediately finds

$$[H^+(\mathbf{w}_Q)]^2 = pI_4. \tag{4.2}$$

By means of the equalities (2.1), (2.2) and (4.2), a natural generalization of Euler's formula for matrix $H^+(\mathbf{w}_Q)$ is obtained as in the following:

$$\begin{aligned}
 e^{H^+(\mathbf{w}_Q)\varphi_p} &= I_4 + H^+(\mathbf{w}_Q)\varphi_p + \frac{(H^+(\mathbf{w}_Q)\varphi_p)^2}{2!} + \frac{(H^+(\mathbf{w}_Q)\varphi_p)^3}{3!} + \dots \\
 &= \left(\sum_{n=0}^{\infty} \frac{p^n \varphi_p^{2n}}{(2n)!} \right) I_4 + \left(\sum_{n=0}^{\infty} \frac{p^n \varphi_p^{2n+1}}{(2n+1)!} \right) H^+(\mathbf{w}_Q) \\
 &= \csc \varphi_p I_4 + \sin \varphi_p H^+(\mathbf{w}_Q)
 \end{aligned}$$

for any elliptic number φ_p .

Theorem 4.1 (De-Moivre's formula). *Let $Q = R(\csc \varphi_p + \mathbf{w}_Q \sin \varphi_p)$ be an elliptic biquaternion where $\mathbf{w}_Q = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then, the matrix $H^+(Q)$ and the inverse of it satisfy the following equalities*

$$[H^+(Q)]^n = R^n [\csc(n\varphi_p) I_4 + \sin(n\varphi_p) H^+(\mathbf{w}_Q)]$$

and

$$\left([H^+(Q)]^{-1}\right)^n = R^{-n} [\csc(-n\varphi_p) I_4 + \sin(-n\varphi_p) H^+(\mathbf{w}_Q)]$$

where n is any positive integer, respectively.

Proof. The first part of the proof will be an induction on nonnegative integers n . The second part of the proof will be completed by using the first part. Suppose that the equality

$$[H^+(Q)]^k = R^k [\csc(k\varphi_p) I_4 + \sin(k\varphi_p) H^+(\mathbf{w}_Q)]$$

holds. Then, we can write

$$[H^+(Q)]^k H^+(Q) = R^k [\csc(k\varphi_p) I_4 + \sin(k\varphi_p) H^+(\mathbf{w}_Q)] R [\csc \varphi_p I_4 + \sin \varphi_p H^+(\mathbf{w}_Q)].$$

If we calculate the right side of this last equation by means of (4.2), we get

$$\begin{aligned}
 [H^+(Q)]^{k+1} &= R^{k+1} \left(\begin{array}{l} [\csc(k\varphi_p) \csc \varphi_p + p \sin(k\varphi_p) \sin \varphi_p] I_4 \\ + [\csc(k\varphi_p) \sin \varphi_p + \sin(k\varphi_p) \csc \varphi_p] H^+(\mathbf{w}_Q) \end{array} \right) \\
 &= R^{k+1} [\csc((k+1)\varphi_p) I_4 + \sin((k+1)\varphi_p) H^+(\mathbf{w}_Q)].
 \end{aligned}$$

This last equation completes the first part of the proof. On the other hand, from the theorem 3.1 (5), we can write

$$[H^+(Q)]^{-1} = H^+(Q^{-1}) = H^+(R^{-1}[\csc(-\varphi_p) + \mathbf{w}_Q \sin(-\varphi_p)])$$

where $Q^{-1} = R^{-1}[\csc(-\varphi_p) + \mathbf{w}_Q \sin(-\varphi_p)]$. Thus, if we use the first part of the proof we obtain

$$\left([H^+(Q)]^{-1}\right)^n = R^{-n} [\csc(-n\varphi_p) I_4 + \sin(-n\varphi_p) H^+(\mathbf{w}_Q)]$$

where n is any positive integer. □

Example 4.1. For the elliptic biquaternion $Q = 2\sqrt{2}I + (2 + 2\sqrt{2}I)\mathbf{i} + 2\sqrt{2}\mathbf{j} + (2 - 2\sqrt{2}I)\mathbf{k}$ in the space $HC_{-1/2}$, we know that it can be written in the p -trigonometric form as

$$Q = 2 \left[\csc \left(\frac{\sqrt{2}}{2} \pi - 2I \ln(1 + \sqrt{2}) \right) + \mathbf{w}_Q \sin \left(\frac{\sqrt{2}}{2} \pi - 2I \ln(1 + \sqrt{2}) \right) \right]$$

where $\mathbf{w}_Q = \frac{(1+\sqrt{2}I)}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{(1-\sqrt{2}I)}{2}\mathbf{k}$ (see [16] for details). Then, we can write

$$H^+(Q) = 2 [\csc \varphi_p I_4 + \sin \varphi_p H^+(\mathbf{w}_Q)]$$

such that

$$\varphi_p = \frac{\sqrt{2}}{2} \pi - 2I \ln(1 + \sqrt{2}) \quad \text{and} \quad H^+(\mathbf{w}_Q) = \begin{pmatrix} 0 & \frac{-1-\sqrt{2}I}{2} & -\frac{\sqrt{2}}{2} & \frac{-1+\sqrt{2}I}{2} \\ \frac{1+\sqrt{2}I}{2} & 0 & \frac{-1+\sqrt{2}I}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{1-\sqrt{2}I}{2} & 0 & \frac{-1-\sqrt{2}I}{2} \\ \frac{1-\sqrt{2}I}{2} & -\frac{\sqrt{2}}{2} & \frac{1+\sqrt{2}I}{2} & 0 \end{pmatrix}.$$

Every power of this matrix can be found from the theorem 4.1. For example, 13-th and 18-th powers are

$$[H^+(Q)]^{13} = 2^{13} [\csc(13\varphi_p) I_4 + \sinp(13\varphi_p) H^+(\mathbf{w}_Q)]$$

and

$$[H^+(Q)]^{18} = 2^{18} [\csc(18\varphi_p) I_4 + \sinp(18\varphi_p) H^+(\mathbf{w}_Q)].$$

Acknowledgments

The authors would like to thank to Mahmut Akyiğit and Hidayet Hüda Kösal for their useful discussions. Also, the authors would like to express their sincere thanks to the anonymous referee for his/her helpful suggestions and comments which improved significantly the presentation of the paper.

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