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Constructions of Type III^+ Helicoidal Surfaces in Minkowski Space with Desity

Önder Gökmen Yıldız* and Mahmut Akyiğit

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ABSTRACT

In this paper, we construct a helicoidal surface of type III^+ with prescribed weighted mean curvature and weighted Gaussian curvature in the Minkowski $3-$space R_1^3 with a positive density function. We get a result for minimal case. Also we give examples of helicoidal surface with prescribed weighted mean curvature and Gaussian curvature.

Keywords: *Minkowski space; manifold with density; weighted curvature; helicoidal.*

AMS Subject Classification (2010): *Primary: 53A10 ; Secondary: 53C50.*

1. Introduction

It is well known that a helicoidal surface is a generalization of a rotation surface. There are many studies about these surfaces under some given certain conditions [1, 7, 9, 16, 19]. Recently, the popular question is whether a helicoidal surface can be constructed when its curvatures are prescribed. Several researchers worked on this problem and obtained useful results. Firstly, helicoidal surfaces with prescribed mean and Gaussian curvature in $\mathbb{R}^3$ have been studied by Baikoussis et. al [2]. Then, Beneki et. al [3] and Ji et. al [10] have studied the similar work in $\mathbb{R}_1^3$. This problem is extended to manifolds with density. Dae Won Yoon et. al have studied the helicoidal surfaces with prescribed weighted mean and weighted Gaussian curvature in $\mathbb{R}^3$ with density [22]. Furthermore, Yıldız et. al have constructed the type I^+ helicoidal surfaces with prescribed weighted curvatures in $\mathbb{R}_1^3$ with density [20].

A manifold with a positive density function ψ used to weight the volume and the hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the new, weighted volume and area are given by $dV = \psi dV_0$ and $dA = \psi dA_0$, respectively. One of the most important examples of manifolds with density, with applications to probability and statistics, is Gauss space with density $\psi = e^{a\left(-x^2-y^2-z^2\right)}$ for $a \in \mathbb{R}, (x, y, z) \in \mathbb{R}^3$ [15]. For more details on manifolds with density, see [8, 12, 13, 14, 15, 17, 18].

In the Minkowski $3-$space with density e^{φ}, the weighted mean curvature is given with

$$H_{\varphi} = H - \frac{1}{2}\langle N, \bigtriangledown\varphi\rangle$$

where H is the mean curvature of the surface, N is the unit normal vector of the surface and $\bigtriangledown\varphi$ is the gradient vector of φ [17]. If $H_{\varphi} = 0$ then the surface is called weighted minimal surface. The weighted Gaussian curvature with density e^{φ} is

$$G_{\varphi} = G - \triangle\varphi$$

where G is the Gaussian curvature of the surface and $\triangle$ is the Laplacian operator [5].

In this paper, we study helicoidal surfaces in the Minkowski $3-$space R_1^3 with density e^{φ}, where $\varphi = -x_2^2 - x_3^2$. Firstly, we consider helicoidal surfaces of type III^+, defined in [3]. Then, we construct a helicoidal surface of type III^+ with prescribed weighted mean and weighted Gaussian curvature. We give the classification of weighted minimal helicoidal surfaces. Finally, we give examples to illustrate our result.

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*** Corresponding author***

2. Preliminaries

The Minkowski $3-$space $\mathbb{R}_1^3$ is the real vector space $\mathbb{R}^3$ provided with the standard flat metric given by

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$$

where $(x_1, x_{2,} x_3)$ is a rectangular coordinate system of $\mathbb{R}_1^3$.

For a given plane curve and an axis in the plane in $\mathbb{R}_1^3$, a helicoidal surface can be constructed by the plane curve under helicoidal motions $g_t : \mathbb{R}_1^3 \to \mathbb{R}_1^3, t \in \mathbb{R}$ around the axis. So, a helicoidal surface is non-degenerate and invariant under $g_t, t \in \mathbb{R}$ for which one parameter subgroup of rigid motions is in $\mathbb{R}_1^3$. There exist four kinds of helicoidal surfaces in $\mathbb{R}_1^3$ which are defined by Beneki et. al [3] and these are called type I, type II, type III , type IV. In this study, type III^+ is considered which has the timelike axis of revolution and the profile curve in x_1x_2-plane. In addition, the helicoidal surface is called type III^+ since the discriminant of the first fundamental form $u^2\left(1-g'^2\right)-c^2$ is positive [3].

Let γ be a C^2-curve on x_1x_2-plane of type $\gamma(u) = (g(u), u, 0)$ where $u \in I$ for an open interval $I \subset \mathbb{R} - \{0\}$. By using helicoidal motion on γ, we can obtain the helicoidal as

$$X(u,v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{bmatrix} \begin{bmatrix} g(u) \\ u \\ 0 \end{bmatrix} + \begin{bmatrix} cv \\ 0 \\ 0 \end{bmatrix} \tag{2.1}$$

with x_1-axis and a pitch $c \in \mathbb{R}$. So the parametric equation can be given in the form

$$X(u,v) = (g(u) + cv, u\cos v, u\sin v). \tag{2.2}$$

It is straightforward to see that the mean curvature H, the Gaussian curvature G and the unit normal of surface N are

$$\begin{aligned} H &= \frac{\left(1-g'^2\right)u^2g' + \left(u^2-c^2\right)ug'' - 2c^2g'}{2\left[u^2\left(1-g'^2\right)-c^2\right]^{3/2}}, \\ G &= \frac{u^3g'g''-c^2}{\left[u^2\left(1-g'^2\right)-c^2\right]^2}, \\ N &= \frac{1}{\sqrt{u^2\left(1-g'^2\right)-c^2}}\left(-u, c\sin v - ug'\cos v, -c\cos v - ug'\sin v\right), \end{aligned}$$

where $u^2\left(1-g'^2\right)-c^2 > 0$ [3]. We assume that M is the surface in $\mathbb{R}_1^3$ with density e^φ, where $\varphi = -x_2^2 - x_3^2$. By considering density function, we can calculate the weighted mean curvature H_φ and the weighted Gaussian curvature G_φ as

$$H_\varphi = \frac{\left(u^2-c^2\right)ug'' - \left(u^2-2u^4\right)g'^3 + \left(u^2+2c^2-2u^4+2c^2u^2\right)g'}{2(u^2\left(1-g'^2\right)-c^2)^{3/2}} \tag{2.3}$$

and

$$G_\varphi = \frac{u^3g'g''-c^2}{2\left(u^2\left(1-g'^2\right)-c^2\right)^2} + 4. \tag{2.4}$$

3. Helicoidal surfaces with prescribed mean or Gaussian curvature

Theorem 3.1. *Let $\gamma(u)$ be a profile curve of the helicoidal surface given with $X(u,v) = (g(u)+cv, u\cos v, u\sin v)$ in $\mathbb{R}_1^3$ with density $e^{-x_2^2-x_3^2}$ and $H_\varphi(u)$ be the weighted mean curvature. Then, there exists a two-parameter family of helicoidal surface given by the curves*

$$\gamma(u, H_\varphi(u), c, c_1, c_2) = \left(\mp\int \frac{e^{u^2}\sqrt{u^2-c^2}\left(2\int ue^{-u^2}H_\varphi du + c_1\right)}{u\sqrt{u^2+e^{2u^2}\left(2\int ue^{-u^2}H_\varphi du + c_1\right)^2}}du + c_2, u, 0\right).$$

Conversely, for a given smooth function $H_\varphi(u)$, one can obtain the two-parameter family of curves $\gamma(u, H_\varphi(u), c, c_1, c_2)$ being the two-parameter family of helicoidal surfaces, accepting $H_\varphi(u)$ as the weighted mean curvature c as a pitch.

Proof. Let's solve the equation (2.3) which is a second-order nonlinear ordinary differential equation. If we apply $A=\frac{g'(u)}{\sqrt{(u^2(1-g'^2)-c^2)}}$ into the equation, then we get

$$H_\varphi=\frac{u}{2}A'+\left(1-u^2\right)A. \tag{3.1}$$

The equation (3.1) becomes a first-order linear ordinary differential equation with respect to A and we rewrite the equation as follows

$$A'+\left(\frac{2}{u}-2u\right)A=\frac{2}{u}H_\varphi. \tag{3.2}$$

Then the general solution of (3.2) is

$$A=\frac{e^{u^2}}{u^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right) \tag{3.3}$$

where $c_1\in\mathbb{R}$. By using $A=\frac{g'(u)}{\sqrt{(u^2(1-g'^2)-c^2)}}$ and the equation (3.3), we obtain

$$\left[u^2+e^{2u^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right)^2\right]g'^2(u)=\frac{\left(u^2-c^2\right)}{u^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right)^2. \tag{3.4}$$

From the above equation, we get

$$g(u)=\mp\int\frac{e^{u^2}\sqrt{u^2-c^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right)}{u\sqrt{u^2+e^{2u^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right)^2}}du+c_2 \tag{3.5}$$

where $c_2\in\mathbb{R}$.

On the contrary, for a given smooth function $H_\varphi(u)$, it is clear that there exists a two-parameter family of the curves as

$$\gamma\left(u,H_\varphi(u),c,c_1,c_2\right)=\left(\mp\int\frac{e^{u^2}\sqrt{u^2-c^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right)}{u\sqrt{u^2+e^{2u^2}\left(2\int ue^{-u^2}H_\varphi du+c_1\right)^2}}du+c_2,u,0\right).$$

□

The following corollary is an immediate consequence of the Theorem 3.1 and the definition of a minimal surfaces.

Corollary 3.1. *Let M be a minimal helicoidal surface in $\mathbb{R}^3_1$ with density $e^{-x_2^2-x_3^2}$. Then M is an open part of either a helicoid or a surface parametrized by*

$$X(u,v)=\left(\mp\int\frac{c_1e^{u^2}\sqrt{u^2-c^2}}{u\sqrt{u^2+c_1^2e^{2u^2}}}du+c_2+cv,u\cos v,u\sin v\right)$$

where $c_1,c_2\in\mathbb{R}$.

Example 3.1. Consider a helicoidal surface with the weighted mean curvature

$$H_\varphi(u)=-\frac{\sqrt{3}u}{4}$$

and the pitch $c=1$ in $\mathbb{R}^3_1$ with density $e^{-x_2^2-x_3^2}$. By considering the equation (3.5), we get $\gamma(u)$. So we obtain the parametrization of the surface as follows

$$X(u,v)=\left(\frac{\sqrt{3}\left(\sqrt{-1+u^2}+\arctan\left(\frac{1}{\sqrt{-1+u^2}}\right)\right)}{2}+v,u\cos v,u\sin v\right)$$

and the figure of the domain

$$\begin{cases} 1 < u < 5 \\ -5 < v < 5 \end{cases}$$

is given in Figure 1.

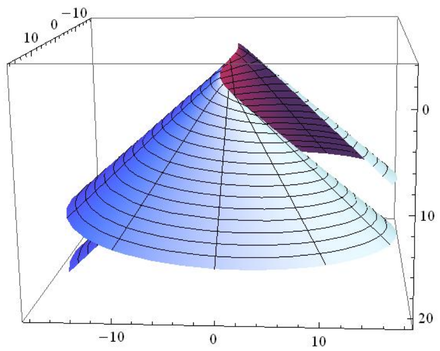

Figure 1. The helicoidal surface with the weighted mean curvature.

Theorem 3.2. *Let $\gamma(u)$ be a profile curve of the helicoidal surface given with $X(u,v) = (g(u) + cv, u\cos v, u\sin v)$ in $\mathbb{R}^3_1$ with density $e^{-x_2^2-x_3^2}$ and $G_\varphi(u)$ be the weighted Gaussian curvature at $(g(u), u, 0)$. Then, there exists two-parameter family of the helicoidal surface given by the curves*

$$\gamma(u, G_\varphi(u), c, c_1, c_2) = \left(\mp\int \frac{1}{u}\left[\frac{(u^2-c^2)\left(4u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 + 4u^2 - 2\int uG_\varphi du + c_1}\right]^{\frac{1}{2}} du + c_2, u, 0\right)$$

where, c_1 and c_2 are constants. Conversely, for a given smooth function $G_\varphi(u)$, one can obtain the two-parameter family of curves $\gamma(u, G_\varphi(u), c, c_1, c_2)$ being the two-parameter family of helicoidal surfaces, accepting $G_\varphi(u)$ as the weighted Gaussian curvature c as a pitch.

Proof. Let's solve the equation (2.4), which is a second-order nonlinear ordinary differential equation. If we apply

$$B = \frac{-u^2 g'^2 - c^2}{(u^2(1-g'^2) - c^2)} \tag{3.6}$$

into the equation (3.6), then we obtain

$$G_\varphi = -\frac{1}{2u}B' + 4$$

that is,

$$B' = -2uG_\varphi + 8u. \tag{3.7}$$

The general solution of the equation (3.7) becomes

$$B = 4u^2 - 2\int uG_\varphi du + c_1 \tag{3.8}$$

where $c_1 \in \mathbb{R}$. Combining the equation (3.6) and the equation (3.8), we get

$$u^2\left(-1 + 4u^2 - 2\int uG_\varphi du + c_1\right)g'^2(u) = (u^2 - c^2)\left(4u^2 - 2\int uG_\varphi du + c_1\right) + c^2. \tag{3.9}$$

It follows that

$$g(u) = \mp\int \frac{1}{u}\left[\frac{(u^2-c^2)\left(4u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 + 4u^2 - 2\int uG_\varphi du + c_1}\right]^{\frac{1}{2}} du + c_2 \tag{3.10}$$

where $c_2 \in \mathbb{R}$.

Conversely, for a given $c \in \mathbb{R}$ and a smooth function $G_{\varphi}(u)$ defined on an open interval $I \subset \mathbb{R}^{+}$ and an arbitrary $u_0 \in I$, there exists an open sub-interval $I' \subset I$ containing u_0 and an open interval $J \subset \mathbb{R}$ containing

$$\hat{c}_1 = \left(1 + 2\int uG_{\varphi}du\right)(u_0)$$

such that

$$F(u, c_1) = -1 + 4u^2 - 2\int uG_{\varphi}du > 0$$

is defined on $I' \times J$ and it is easily seen F is positive. Thus, two-parameter family of the curves can be given as

$$\gamma(u, G_{\varphi}(u), c, c_1, c_2) = \left(\mp\int \frac{1}{u}\left[\frac{(u^2 - c^2)(4u^2 - 2\int uG_{\varphi}du + c_1) + c^2}{-1 + 4u^2 - 2\int uG_{\varphi}du + c_1}\right]^{\frac{1}{2}} du + c_2, u, 0\right)$$

where $(u, c_1) \in I' \times J; c_2 \in \mathbb{R}, c \in \mathbb{R}$ and G_{φ} is smooth function. □

Example 3.2. Consider a helicoidal surface with the weighted Gaussian curvature

$$G_{\varphi}(u) = \frac{-3 + 2u^2}{3}$$

in R^3_1 with density $e^{-x_2^2 - x_3^2}$. By using the equation (3.10), we obtain

$$g(u) = \sqrt{-1 + 2u^2} + \arctan\left(\frac{1}{\sqrt{-1 + 2u^2}}\right)$$

for $c = 1, c_1 = 0, c_2 = 0$ and the parametrization of the surface as follows

$$X(u, v) = (g(u) + v, u\cos v, u\sin v).$$

The figure of the surface of the domain

$$\begin{cases} 2 < u < 5 \\ -10 < v < 10 \end{cases}$$

is given in Figure 3.

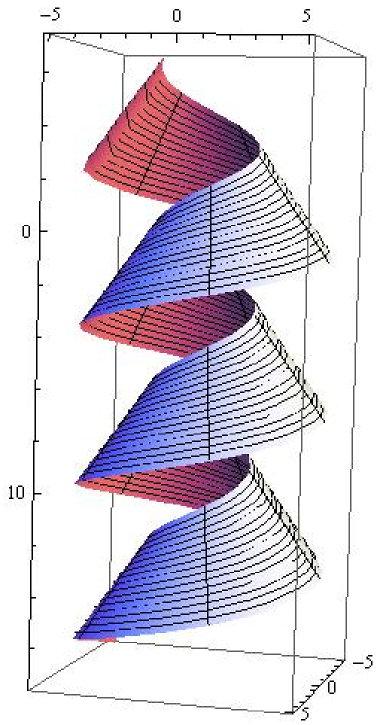

Figure 2. The helicoidal surface with the weighted Gaussian curvature.

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Affiliations

ÖNDER GÖKMEN YILDIZ
ADDRESS: Bilecik Şeyh Edebali University, Dept. of Mathematics,
11210, Bilecik-Turkey
E-MAIL: ogokmen.yildiz@bilecik.edu.tr
ORCID ID: orcid.org/0000-0002-2760-1223

MAHMUT AKYIĞIT
ADDRESS: Sakarya University, Dept. of Mathematics,
54050, Sakarya-Turkey
E-MAIL: makyigit@sakarya.edu.tr
ORCID ID: orcid.org/0000-0002-8398-365X