

# Estimates of B.-Y. Chen's $\hat{\delta}$ -Invariant in Terms of Casorati Curvature and Mean Curvature for Strictly Convex Euclidean Hypersurfaces

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## ABSTRACT

B.-Y. Chen's  $\hat{\delta}$ -invariants can be estimated in function of other curvature terms through an algebraic process using the AM-GM and AM-QM inequalities. This procedure works on strictly convex smooth hypersurfaces lying in an Euclidean ambient space, and the estimates relate some  $\hat{\delta}$ -invariants to Germain's mean curvature and Casorati curvature. As a consequence, we obtain a new string of inequalities in the geometry of strictly convex smooth hypersurfaces.

*Keywords:* principal curvatures; Casorati curvature; mean curvature; smooth hypersurfaces; AM QM inequality.

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## 1. Introduction

Felice Casorati introduced in 1890 what is today called the Casorati curvature [3], an invariant that, in L. Verstraelen's words [30], "in best accordance with our visual intuition and common sense, relates to the form proper of surfaces". The study of Casorati's curvature inspired recent new works, e.g. [1, 13, 14, 29], and invite further comparison with the classical curvature invariants corresponding to Gaussian curvature [17] and Sophie Germain's mean curvature [18], which were studied in the classical literature [4, 5, 16].

The Casorati curvature of a submanifold  $M^n$  of a Riemannian manifold  $\tilde{M}^{n+m}$  usually denoted by  $\mathcal{C}$ , is an extrinsic invariant defined in some references (e.g. [19, 21]) as the normalized square of the length of the second fundamental form of the submanifold. We prefer to work instead with the square of the length of the second fundamental form of the submanifold [1], as we will see below, thinking of a better compatibility with the Hilbert-Schmidt norm, while still preserving all the geometric information encoded in the square of the length of the second fundamental form of the submanifold as a curvature invariant. The Casorati curvature, which is of interest in computer vision [20], was preferred by Casorati over the traditional curvature because, as he wrote, it corresponds better with the common intuition of curvature.

By the other hand, B.-Y. Chen introduced in the early 1990s a class of fundamental curvature invariants [6, 7, 10] that have been extended and investigated, among other contexts, to the Kählerian context [9, 10, 15]. These inspiring works have motivated and inspired several other constructions e.g., among many other papers, important works as [12] or [22] as well as many other investigations e.g. [2, 11, 23, 24, 25, 26, 27, 28, 31], etc. For a complete overview of the developments, the best reference is [10].

### 1.1. Notations in the Geometry of Smooth Hypersurfaces

To recall a few concepts in the differential geometry of smooth hypersurfaces, we start by considering  $\sigma : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ , a smooth hypersurface given by the smooth map  $\sigma$ . Let  $p$  be a point on the hypersurface. Denote  $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$ , for all  $k$  from 1 to  $n$ . Consider  $\{\sigma_1(p), \sigma_2(p), \dots, \sigma_n(p), N(p)\}$ , the Gauss frame of the hypersurface, where  $N$  denotes the normal vector field. We denote by  $g_{ij}(p)$  the coefficients of the first fundamental form and

by  $h_{ij}(p)$  the coefficients of the second fundamental form. Then

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \quad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The Weingarten map  $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \rightarrow T_{\sigma(p)}\sigma$  is linear. Denote by  $(h_j^i(p))_{1 \leq i, j \leq n}$  the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p)) = h_i^k(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. Weingarten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h_j^i(p) - \lambda(p)\delta_j^i) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots  $k_1(p), k_2(p), \dots, k_n(p)$  of this algebraic equation. The mean curvature at the point  $p$  is

$$H(p) = \frac{1}{n}[k_1(p) + \dots + k_n(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = k_1(p)k_2(p)\dots k_n(p).$$

The Casorati curvature is

$$C(p) = k_1^2(p) + \dots + k_n^2(p).$$

If all the principal curvature of a smooth regular hypersurface are  $\geq 0$ , then the hypersurface is convex. If all the principal curvature of a smooth regular hypersurface are  $> 0$ , then the hypersurface is strictly convex.

### 1.2. Notations in the Geometry of Smooth Submanifolds

Let  $M^n$  be a Riemannian  $n$ -manifold. For any orthonormal basis  $e_1, \dots, e_n$  of the tangent space  $T_pM$ , the scalar curvature is defined to be  $scal(p) = \sum_{i < j} sec(e_i \wedge e_j)$ . For any  $r$ -dimensional subspace of  $T_pM$  denoted  $L$  with orthonormal basis  $e_1, \dots, e_r$  one may define

$$scal(L) = \sum_{1 \leq i < j \leq r} sec(e_i \wedge e_j). \tag{1.1}$$

In [8], Chen considered the finite set  $S(n)$  of  $k$ -tuples  $(n_1, \dots, n_k)$  with  $k \geq 0$  which satisfy the conditions:  $n_1 < n, n_i \geq 2$ , and  $n_1 + \dots + n_k \leq n$ . For each  $(n_1, \dots, n_k) \in S(n)$  he introduced the following Riemannian invariants:

$$\delta(n_1, \dots, n_k)(p) = scal(p) - \inf \{ scal(L_1) + \dots + scal(L_k) \}(p), \tag{1.2}$$

where the infimum is taken for all possible choices of orthogonal subspaces  $L_1, \dots, L_k$ , satisfying  $n_j = \dim L_j$ , ( $j = 1, \dots, k$ ). Note that the Chen invariants with  $k = 0$  is nothing but the scalar curvature. Similarly, B.-Y.Chen's  $\hat{\delta}$ -invariants can be defined as (see equation (13.2) in [10]).

$$\hat{\delta}(n_1, \dots, n_k)(p) = scal(p) - \sup \{ scal(L_1) + \dots + scal(L_k) \}(p). \tag{1.3}$$

As in [8], one may denote put

$$c(n_1, \dots, n_k) = \frac{n^2(n+k-1 - \sum_{j=1}^k n_j)}{2(n+k - \sum_{j=1}^k n_j)},$$

$$b(n_1, \dots, n_k) = \frac{1}{2} \left\{ (n(n-1) - \sum_{j=1}^k n_j(n_j-1)) \right\}.$$

The original Chen's fundamental inequalities obtained in [8] were stated as follows:

**Theorem 1.1.** For any  $n$ -dimensional submanifold  $M$  of a Riemannian space form  $R^{n+m}(\varepsilon)$  of constant sectional curvature  $\varepsilon$  and for any  $k$ -tuple  $(n_1, \dots, n_k) \in S(n)$ , we have

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)|H|^2 + b(n_1, \dots, n_k)\varepsilon. \tag{1.4}$$

The equality case of the inequality above holds at a point  $p \in M$  if and only if there exists an orthonormal basis  $e_1, \dots, e_{n+m}$  at  $p$  such that the shape operators of  $M$  in  $R^{n+m}(\varepsilon)$  at  $p$  take the following forms:  $S_r = \text{diag}(A_1^r, \dots, A_k^r, \mu_r, \dots, \mu_r)$  for  $r = n + 1, \dots, m$ , where each  $A_j^r$  is a symmetric  $n_j \times n_j$  submatrix such that

$$\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = \mu_r.$$

Based on this fundamental theorem, in the present work we investigate the following.

**Question.** Are there any geometric contexts in which there exist any relations between mean curvature and B.-Y. Chen's  $\hat{\delta}$ -invariants?

Note that the most general form of Chen's fundamental inequalities is Theorem 13.3 in [10]. The form stated above as Theorem 1.1 allows us to see the case when the submanifold  $M$  is isometrically immersed into an Euclidean space, i.e  $\varepsilon = 0$ . In that case,  $\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)|H|^2$ . We compare this inequality with the relations (3.1) and (3.7) we prove in the present work.

## 2. The Case of a Four-Dimensional Strictly Convex Smooth Hypersurface

**Theorem 2.1.** Let  $\sigma : U \subset \mathbb{R}^4 \rightarrow \mathbb{R}^5$  be a strictly convex smooth hypersurface in the Euclidean ambient space. Let  $p$  be a point on the hypersurface. Denote by  $a, b, c, d > 0$  the principal curvatures at  $p$ , and by  $\mathcal{C}(p) = a^2 + b^2 + c^2 + d^2$  the Casorati curvature at  $p$ . Then

$$H \geq \frac{1}{2\sqrt{\mathcal{C}}} \hat{\delta}(2, 2). \tag{2.1}$$

Equality holds if and only if  $p$  is an umbilical point.

*Proof:* Denote by  $e_1, e_2, e_3, e_4$  the principal vectors at  $p$ , corresponding to principal curvatures  $a, b, c, d$ , respectively. The plane spanned by  $\{e_1, e_3\}$  is  $L_1$  and lies perpendicularly to the plane spanned by  $\{e_2, e_4\}$ , which is denoted  $L_2$ . We need this fundamental fact in order to estimate below  $\hat{\delta}(2, 2)$ . By AM-HM inequality, we have

$$\frac{1}{a+c} + \frac{1}{b+d} \geq \frac{4}{a+b+c+d}. \tag{2.2}$$

By AM-QM inequality we have

$$\frac{4}{a+b+c+d} \geq \sqrt{\frac{4}{a^2+b^2+c^2+d^2}} = \frac{2}{\sqrt{\mathcal{C}}}. \tag{2.3}$$

The left hand side in (2.2) has as denominator the product  $(a+c)(b+d)$ . The geometric interpretation of this product is given by the following estimate:

$$ab + ad + bc + cd = ab + ad + bc + cd + ac + bd - ac - bd = \text{scal}(p) - ac - bd.$$

On the other hand, the product  $ac$  represents the sectional curvature in the direction of a 2-planar section  $L_1$ , and  $bd$  is the sectional curvature in the direction of a 2-planar section  $L_2$ , and since  $a, b, c, d$  are principal curvatures of a smooth hypersurface,  $L_1 \perp L_2$ . Furthermore,  $ac + bd \leq \sup\{\text{sec}(L_1) + \text{sec}(L_2)\}$ , hence

$$\text{scal}(p) - ac - bd \geq \hat{\delta}(2, 2)(p). \tag{2.4}$$

By combining (2.2), (2.3), (2.4) we obtain the claimed inequality.

In the step where we used AM-QM, the equality holds iff  $a = b = c = d$ , which means the point  $p$  is umbilical.  $\square$

Note that one may write the above inequality in the spirit of the original's Chen's fundamental inequality [7, 8] as a relationship between extrinsic and intrinsic quantities as

$$2H\sqrt{C} \geq \hat{\delta}(2, 2).$$

A natural quest is to see whether the procedure employed in the above proof can be further extended for more general cases. If such, the section hereby concluded serves mainly for expository reasons.

### 3. The Case of a Generic Strictly Convex Smooth Hypersurface

**Theorem 3.1.** *Let  $\sigma : U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n+1}$  be a strictly convex smooth hypersurface in the Euclidean ambient space. Let  $p$  be a point on the hypersurface. Then the mean curvature  $H$ , the Casorati curvature  $C$  and Chen's  $\hat{\delta}(n, n)$ -invariant satisfy the inequality*

$$H \geq \frac{\sqrt{2}}{n\sqrt{n}} \cdot \frac{1}{\sqrt{C}} \cdot \hat{\delta}(n, n). \tag{3.1}$$

The equality holds if and only if the point  $p$  is an umbilical point.

*Proof:* Consider two mutually orthogonal spaces  $L_1$  and  $L_2$  such that  $\dim L_1 = \dim L_2 = n$ ,  $L_1 \perp L_2$ , and  $L_1 \oplus L_2 = T_p\sigma$ . Denote the principal curvatures at  $p$  by  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0$ . By applying the AM-HM inequality, we have

$$\frac{\frac{1}{a_1+a_2+\dots+a_n} + \frac{1}{b_1+b_2+\dots+b_n}}{2} \geq \frac{2}{a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n}, \tag{3.2}$$

which turns into

$$\frac{a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n}{(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)} \geq \frac{4}{2nH}. \tag{3.3}$$

This last relation yields

$$nH^2 \geq \frac{(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)}{n}. \tag{3.4}$$

Now we apply the AM-QM inequality to the positive numbers  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0$ . We obtain

$$\frac{a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n}{2n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2 + b_1^2 + b_2^2 + \dots + b_n^2}{2n}},$$

which is equivalent to

$$\frac{1}{H} \geq \sqrt{\frac{2n}{C}}. \tag{3.5}$$

By combining (3.4) with (3.5) we obtain

$$H \geq \frac{\sqrt{2}}{n\sqrt{n}} \cdot \frac{1}{\sqrt{C}} \cdot (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n) \geq \frac{\sqrt{2}}{n\sqrt{n}} \cdot \frac{1}{\sqrt{C}} \cdot \hat{\delta}(n, n), \tag{3.6}$$

where we have used the fact that

$$(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n) \geq \hat{\delta}(n, n),$$

through an argument similar to the one described in (2.4).

The equality case is determined by the AM-QM inequality, which will insure the umbilicity of point  $p$ .  $\square$

**Theorem 3.2.** *Let  $\sigma : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$  be a convex smooth hypersurface in the Euclidean ambient space. Let  $p$  be a point on the hypersurface. Then for any couple of natural numbers  $n_1, n_2$  such that  $n_1 + n_2 = n$ , the mean curvature  $H$ , the Casorati curvature  $C$  and Chen's  $\hat{\delta}(n_1, n_2)$ -invariant satisfy the inequality*

$$H \geq \frac{4}{n\sqrt{n}} \cdot \frac{1}{\sqrt{C}} \cdot \hat{\delta}(n_1, n_2). \tag{3.7}$$

The equality holds if and only if the point  $p$  is an umbilical point.

*Proof:* Consider two mutually orthogonal spaces  $L_1$  and  $L_2$  such that  $\dim L_1 = n_1$ ,  $\dim L_2 = n_2$ ,  $L_1 \perp L_2$ , and  $L_1 \oplus L_2 = T_p \sigma$ . Denote the principal curvatures at  $p$  by  $a_1, a_2, \dots, a_{n_1}, b_1, b_2, \dots, b_{n_2} > 0$ . As in the previous argument when we obtained (3.5), we derive

$$\frac{1}{H} \geq \sqrt{\frac{n}{C}}. \quad (3.8)$$

By applying the AM-HM inequality we get

$$\frac{(a_1 + a_2 + \dots + a_{n_1}) + (b_1 + b_2 + \dots + b_{n_2})}{2} \geq \frac{2}{\frac{1}{a_1 + a_2 + \dots + a_{n_1}} + \frac{1}{b_1 + b_2 + \dots + b_{n_2}}}.$$

By interpreting this inequality we derive immediately

$$\frac{nH}{2} \geq \frac{2(a_1 + a_2 + \dots + a_{n_1}) \cdot (b_1 + b_2 + \dots + b_{n_2})}{nH}. \quad (3.9)$$

By combining (3.8) with (3.9) we have

$$H \geq \frac{4}{n\sqrt{n}} \cdot \frac{1}{\sqrt{C}} \cdot (a_1 + a_2 + \dots + a_{n_1}) \cdot (b_1 + b_2 + \dots + b_{n_2}) \geq \frac{4}{n\sqrt{n}} \cdot \frac{1}{\sqrt{C}} \cdot \hat{\delta}(n_1, n_2).$$

The AM-HM inequality fully controls the equality case as stated.  $\square$

#### 4. A Global Consequence

An important theorem in the geometry of submanifolds of finite type (see [10], p.173) states the following.

**Theorem 4.1.** *Let  $\sigma : N \rightarrow \mathbb{R}^m$  be an isometric immersion of a compact Riemannian  $n$ -manifold into the Euclidean space of dimension  $m$ . Then*

$$\int_N \|H\|^k dV \leq \left(\frac{\lambda_q}{n}\right)^{k/2} \cdot \text{vol}(N), \quad (4.1)$$

for  $k = 1, 2, 3, 4$ , where  $q$  is the upper order of the submanifold with finite type  $N$ . The equality sign in (4.1) holds for some  $k$  if and only if  $N$  is of 1-type.

By combining Theorem 3.2 with Theorem 4.1 we obtain the following.

**Theorem 4.2.** *Let  $\sigma : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$  be an isometric immersion of a convex compact Riemannian  $n$ -manifold into the Euclidean space of dimension  $m$ . Denote by  $N$  the image of  $\sigma$ . Consider any couple of natural numbers  $n_1, n_2$  such that  $n_1 + n_2 = n$ . Denote the mean curvature by  $H$ , the Casorati curvature by  $C$ . Then the Chen's  $\hat{\delta}(n_1, n_2)$ -invariant provides the following lower bound inequality for the volume of  $N$ :*

$$\frac{4^k}{n^{3k/2}} \cdot \int_N \frac{1}{(C)^{k/2}} \cdot [\hat{\delta}(n_1, n_2)]^k dV \leq \left(\frac{\lambda_q}{n}\right)^{k/2} \cdot \text{vol}(N), \quad (4.2)$$

for  $k = 1, 2, 3, 4$ , where  $q$  is the upper order of the submanifold with finite type  $N$ . The equality sign in (4.1) holds for some  $k$  if and only if  $N$  is of 1-type and if the hypersurface is totally umbilical.  $\square$

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