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# A Generalized Wintgen Inequality for Legendrian Submanifolds in Almost Kenmotsu Statistical Manifolds

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**ABSTRACT**

**Main interest of the present paper is to obtain the generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu statistical manifolds.**

***Keywords:*** *Statistical Manifolds; Cosymplectic Manifolds; Kenmotsu Manifolds; Wintgen inequality; Legendrian submanifold.*

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## 1. Introduction

One of the most fundamental problems in a Riemannian submanifold theory is to establish a simple sharp relationship between intrinsic and extrinsic invariants. The main extrinsic invariants are the extrinsic normal curvature, the squared mean curvature and the main intrinsic invariants include the Ricci curvature and the scalar curvature. In 1979, Wintgen [27] obtained a basic inequality involving Gauss curvature $K$, normal curvature $K^{\perp}$ and the squared mean curvature $\|H\|^2$ of an oriented surface $M^2$ in $E^4$ , that is ,

$$K \leq \|H\|^2 - \left|K^{\perp}\right| \tag{1.1}$$

with the equality holding if and only if the ellipse of curvature of $M^2$ in $E^4$ is a circle. The inequality (1.1), now called Wintgen inequality, attracted the attention of several authors.

Over time P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [11] gave a conjecture for Wintgen inequality in an $n$-dimensional Riemannian submanifold $M^n$ of a real space form $R^{n+p}(c)$, namely,

$$\rho \leq \|H\|^2 - \rho^{\perp} + c, \tag{1.2}$$

where

$$\rho = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \langle R(e_i, e_j)e_j, e_i\rangle \,, \tag{1.3}$$

is the normalized scalar curvature of $M^n$

$$\rho^{\perp} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n} \sum_{1\leq \alpha<\beta\leq m} \langle R^{\perp}(e_i, e_j)u_\alpha, u_\beta\rangle^2} \,, \tag{1.4}$$

where $\{e_1, ..., e_n\}$ and $\{u_1, ..., u_p\}$ respectively orthonormal frames of tangent bundle $TM$ and normal bundle $T^{\perp}M$ and they also proved that this conjecture holds for codimension $p = 2$. This type of inequality later came to be known as the DDVV conjecture. A special version of the DDVV conjecture,

$$\rho \leq \|H\|^2 + c, \tag{1.5}$$

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was proved by B.Y. Chen in [9]. F. Dillen, J. Fastenakels and J. Van der Veken [12] proved that DDVV conjecture is equivalent to an algebraic conjecture. Recently DDVV-conjecture was proven by Z. Lu [19] and by Ge and Z. Tang [15] indepently.

In recent years, I. Mihai [20] proved DDVV conjecture for Lagrangian submanifolds in complex space forms and obtained Wintgen inequality for Legendrian submanifolds in Sasakian space forms (see [21]). On the other hand, the product spaces $S^n(c)\times\mathbb{R}$ and $\mathbb{R}\times H^n(c)$ are studied to obtain generalized Wintgen inequality by Q. Chen and Q. Cui [10]. Then J. Roth [24] extended DDVV inequality to submanifolds of warped product manifolds $\mathbb{R}\times_f M^n(c)$. Furuhata et al. [14] studied on statistical warped product and Kenmotsu statistical manifolds.

Nowadays, Wintgen inequality of statistical submanifolds in statistical manifolds of constant curvature has been studied in [2], [3] and [4]. The generalized Wintgen inequality for statistical submanifolds of statistical warped product manifolds was proved in [22]. Furthermore, in [5], the generalized Wintgen inequality for statistical submanifolds in statistical manifolds of quasi-constant curvature was obtained. Motivated by the studies in [22], we consider generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu statistical manifolds in this article.

## 2. Preliminaries

An almost Hermitian manifold $(N^{2n},g,J)$ is a smooth manifold endowed with an almost complex structure $J$ and a Riemannian metric $g$ compatible in the sense

$$J^2X=-X,\ g(JX,Y)=-g(X,JY)$$

for any $X,Y\in\Gamma(TN)$. The fundamental 2-form $\Omega$ of an almost Hermitian manifold is defined by

$$\Omega(X,Y)=g(JX,Y)$$

for any vector fields $X,Y$ on $N$. For an almost Hermitian manifold $(N^{2n},g,J)$ with Riemannian connection $\nabla$, the fundamental 2-form $\Omega$ and the Nijenhuis torsion of $J$, $N_J$ satisfy

$$2g((\nabla_X J)Y,Z)=g(JX,N_J(Y,Z)+3d\Omega(X,JY,JZ)-3d\Omega(X,Y,Z) \tag{2.1}$$

where $N_J(X,Y)=[X,Y]-[JX,JY]+J[X,JY]+J[JX,Y]$ (see [28]). An almost Hermitian manifold is said to be an almost Kaehler manifold if its fundamental form $\Omega$ is closed, that is, $d\Omega=0$. If $d\Omega=0$ and $N_J=0$, the structure is called Kaehler. Thus by (2.1), an almost Hermitian manifold $(N,J,g)$ is Kaehler if and only if its almost complex structure $J$ is parallel with respect to the Levi-Civita connection $\nabla^0$, that is, $\nabla^0 J=0$ ([28]).

It is known that a Kaehler manifold $N^{2n}$ is of constant holomorphic sectional curvature $c$ if and only if

$$R(X,Y)Z=\frac{c}{4}(g(X,Z)Y-g(Y,Z)X+g(JX,Z)Y-g(JY,Z)JX+2g(JX,Y)JZ), \tag{2.2}$$

and is denoted by $N^{2n}(c)$ (see [28]).

Let $M$ be a $(2n+1)$-dimensional differentiable manifold and $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field and $\eta$ is a one-form on $M$. If $\phi^2=-Id+\eta\otimes\xi,\quad \eta(\xi)=1$ then $(\phi,\xi,\eta)$ is called an almost contact structure on $M$. The manifold $M$ is said to be an almost contact manifold if it is endowed with an almost contact structure [6].

Let $M$ be an almost contact manifold. $M$ will be called an almost contact metric manifold if it is additionally endowed with a Riemannian metric $g$ , i.e.

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.3}$$

For such manifold, we have

$$\eta(X)=g(X,\xi),\ \phi(\xi)=0,\ \eta\circ\phi=0. \tag{2.4}$$

Furthermore, a 2-form $\Phi$ is defined by

$$\Phi(X,Y)=g(\phi X,Y), \tag{2.5}$$

and usually is called fundamental form.

On an almost contact manifold, the $(1,2)$-tensor field $N^{(1)}$ is defined by

$$N^{(1)}(X,Y)=[\phi,\phi]\,(X,Y)-2d\eta(X,Y)\xi,$$

where $[\phi,\phi]$ is the Nijenhuis torsion of $\phi$

$$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost contact manifold (structure) is said to be normal [6]. The normality condition says that the almost complex structure $J$ defined on $M\times\mathbb{R}$

$$J(X,\lambda\frac{d}{dt})=(\phi X+\lambda\xi,\eta(X)\frac{d}{dt}),$$

is integrable.

An almost contact metric manifold $M^{2n+1}$, with a structure $(\phi,\xi,\eta,g)$ is said to be an almost cosymplectic manifold, if

$$d\eta=0,\quad d\Phi=0. \tag{2.6}$$

If additionally normality conditon is fulfilled, then manifold is called cosymplectic.

On the other hand, Kenmotsu studied in [16] another class of almost contact metric manifolds, defined by the following conditions on the associated almost contact structure

$$d\eta=0,\quad d\Phi=2\eta\wedge\Phi. \tag{2.7}$$

A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

## 3. Statistical Manifolds

Let $(M,g)$ be a Riemannian manifold and $\nabla$ an affine connection on $M$. An affine connection $\nabla^*$ is said to be dual connection of $\nabla$ if

$$Zg(X,Y)=g(\nabla_ZX,Y)+g(X,\nabla^*_ZY) \tag{3.1}$$

for any $X,Y,Z\in\Gamma(M)$.The notion of "conjugate connection" is given an excellent survey by Simon [25]. In the triple $(g,\nabla,\nabla^*)$ is called a dualistic structure on $M$. It appears that $(\nabla^*)^*=\nabla$. The manifold structure of statistical distributions was first introduced by Amari [1] and used by Lauritzen [17].

A statistical manifold $(M,\nabla,g)$ is a Riemannian manifold $(M,g)$ endowed torsion free connection $\nabla$ such that the Codazzi equation

$$(\nabla_Xg)(Y,Z)=(\nabla_Yg)(X,Z) \tag{3.2}$$

holds for any $X,Y,Z\in\Gamma(TM)$ (see [1]). If $(M,\nabla,g)$ is a statistical manifold, so is $(M,\nabla^*,g)$. For a statistical manifold $(M,g,\nabla,\nabla^*)$ the difference $(1,2)$ tensor $\mathcal{K}$ of a torsion free affine connection $\nabla$ and Levi-Civita connection $\nabla^0$ is defined as

$$\mathcal{K}_XY=\mathcal{K}(X,Y)=\nabla_XY-\nabla^0_XY. \tag{3.3}$$

Because of $\nabla$ and $\nabla^0$ are torsion free, we have

$$\mathcal{K}_XY=\mathcal{K}_YX,\quad g(\mathcal{K}_XY,Z)=g(Y,\mathcal{K}_XZ) \tag{3.4}$$

for any $X,Y,Z\in\Gamma(TM)$. By (3.1) and (3.3), one can obtain

$$\mathcal{K}_XY=\nabla^0_XY-\nabla^*_XY. \tag{3.5}$$

Using (3.3) and (3.5), we find

$$2\mathcal{K}_XY=\nabla_XY-\nabla^*_XY. \tag{3.6}$$

By (3.3), we have

$$g(\nabla_XY,Z)=g(\mathcal{K}_XY,Z)+g(\nabla^0_XY,Z). \tag{3.7}$$

It can be also shown that any torsion-free affine connection $\nabla$ has a dual connection given by

$$\nabla^0=\frac{1}{2}(\nabla+\nabla^*), \tag{3.8}$$

where $\nabla^0$ is Levi-Civita connection of the Riemannian manifold $(M,g)$. If $\nabla=\nabla^*$ then $(M,\nabla,g)$ is called trivial statistical manifold.

Denote by $R$ and $R^*$ the curvature tensors on $M$ with respect to the affine connection $\nabla$ and its conjugate $\nabla^*$, respectively. Then the relation between $R$ and $R^*$ can be given as following

$$g(R(X,Y)Z,W) = -g(Z,R^*(X,Y)W) \tag{3.9}$$

for any $X,Y,Z,W \in \Gamma(TM)$.

By using (3.3) and (3.5), we have

$$R(X,Y)Z + R^*(X,Y)Z = 2R^0(X,Y)Z + 2[\mathcal{K},\mathcal{K}](X,Y)Z,$$

where $[\mathcal{K},\mathcal{K}](X,Y)Z = [\mathcal{K}_X,\mathcal{K}_Y]Z = \mathcal{K}_X\mathcal{K}_Y Z - \mathcal{K}_Y\mathcal{K}_X Z$ (see [23]).

In [29], L.Todjihounde gave a method how to establish a dualistic structure on the warped product manifold. If we adapt this method for $I \times_f N$, we have the following result.

**Proposition 3.1** ( [29]). *Let $(\mathbb{R}, dt, \nabla)$ be a trivial statistical manifold and $(N, g_N, {}^N\nabla, {}^N\nabla^*)$ be a statistical manifold. If the connections $\bar{\nabla}$ and $\bar{\nabla}^*$ satisfy the following relations on $\mathbb{R} \times N$*

*(a)* $\bar{\nabla}_{\bar{\partial}_t}\bar{\partial}_t = \nabla_{\partial_t}\partial_t = 0$,

*(b)* $\bar{\nabla}_{\bar{\partial}_t}\bar{X} = \bar{\nabla}_{\bar{X}}\bar{\partial}_t = \frac{f'(t)}{f(t)}X$,

*(c)* $\bar{\nabla}_{\bar{X}}\bar{Y} = {}^N\nabla_X Y - \frac{<X,Y>}{f}f'(t)\partial_t$,

*and*

*(i)* $\bar{\nabla}^*_{\bar{\partial}_t}\bar{\partial}_t = \nabla^*_{\partial_t}\partial_t = 0$,

*(ii)* $\bar{\nabla}^*_{\bar{\partial}_t}\bar{X} = \bar{\nabla}^*_{\bar{X}}\bar{\partial}_t = \frac{f'(t)}{f(t)}X$,

*(iii)* $\bar{\nabla}^*_{\bar{X}}\bar{Y} = {}^N\nabla^*_X Y - \frac{<X,Y>}{f}f'(t)\partial_t$,

*then $(\mathbb{R} \times_f N, <,>, \bar{\nabla}, \bar{\nabla}^*)$ is a statistical manifold, where $\bar{X}, \bar{Y}$ are vertical lifts of $X, Y \in \Gamma(TN)$ and $\bar{\partial}_t = \frac{\partial}{\partial t}$ is horizontal lift of $\partial_t$ and the notation is simplified by writing $f$ for $f \circ \pi$ and $grad f$ for $grad(f \circ \pi)$.*

Assuming $(\mathbb{R}, dt, \nabla)$ is trivial statistical manifold and denoting $R$ and $R^*$ are curvature tensors respect to the dualistic structure $(<,>, \bar{\nabla}, \bar{\nabla}^*)$ on $\mathbb{R} \times N$ then we can give the following lemma by using Proposition 3.1. In practise, $(-)$ is ommited from lifts.

**Lemma 3.1** ( [29]). *Let $(\tilde{M} = \mathbb{R} \times_f N, <,>, \bar{\nabla}, \bar{\nabla}^*)$ be a statistical warped product. If $U, V, W \in \Gamma(N)$, then:*

$(a)$ $R(V,\partial_t)\partial_t = -\frac{f''(t)}{f(t)}V$,

$(b)$ $R(V,U)\partial_t = 0$,

$(c)$ $R(\partial_t,V)W = -\frac{f''(t)}{f(t)} < V,W > \partial_t$,

$(d)$ $R(V,W)U = R^N(V,W)U - \frac{(f'(t))^2}{(f(t))^2}[< W,U > V - < V,U > W]$,

*and*

$(a^*)$ $R^*(V,\partial_t)\partial_t = -\frac{f''(t)}{f(t)}V$,

$(b^*)$ $R^*(V,U)\partial_t = 0$,

$(c^*)$ $R^*(\partial_t,V)W = -\frac{f''(t)}{f(t)} < V,W > \partial_t$,

$(d^*)$ $R^*(V,W)U = R^{*N}(V,W)U - \frac{(f'(t))^2}{(f(t))^2}[< W,U > V - < V,U > W]$

*where $R^{*N}$ and $R^N$ are curvature tensors of $N$ with respect to the connections ${}^N\nabla$ and ${}^N\nabla^*$ .*

### *3.1. Statistical submanifolds*

In this section, we will give some basic notations, formulas, definitions taken from reference [26].

Let $(M^n, g)$ be a statistical submanifold of $(\tilde{M}^{n+d}, <,>)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{3.10}$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \quad \tilde{\nabla}^*_X \xi = -A^*_\xi X + D^*_X \xi, \tag{3.11}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, respectively. Furthermore, the followings hold :

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),$$

$$< h(X,Y), \xi >= g(A^*_\xi X, Y), \quad < h^*(X,Y), \xi >= g(A_\xi X, Y)$$

and

$$X < \xi, \eta >=< \quad D_X\xi, \eta > + < \xi, D^*_X\eta >$$

for $X, Y, Z \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^\perp M)$.

The mean curvature vector fields of $M$ are defined with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \quad \text{and} \quad H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i)$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame of the tangent bundle $TM$ of $M$. By (3.10) and (3.11), we have $2h^0 = h + h^*$ and $2H^0 = H + H^*$, where $h^0$ and $H^0$ are second fundamental form and mean curvature with respect to Levi-Civita connection $\tilde{\nabla}^0$.

**Proposition 3.2** ([26]). *Let $(M^m, g, \nabla, \nabla^*)$ be a statistical submanifold of $(\tilde{M}^{m+n}, <,>, \tilde{\nabla}, \tilde{\nabla}^*)$. Denote $\tilde{R}$ and $\tilde{R}^*$ the curvature tensors on $\tilde{M}^{m+n}$ with respect to connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$. Then*

$$< \tilde{R}(X,Y)Z, W >= g_M(R(X,Y)Z, W) + < h(X,Z), h^*(Y,W) > - < h^*(X,W), h(Y,Z) >, \tag{3.12}$$

$$< \tilde{R}^*(X,Y)Z, W >= g_M(R^*(X,Y)Z, W) + < h^*(X,Z), h(Y,W) > - < h(X,W), h^*(Y,Z) >, \tag{3.13}$$

$$< (R^\perp(X,Y)\xi, \eta >=< \tilde{R}(X,Y)\xi, \eta > + g_M([A^*_\xi, A_\eta]X, Y), \tag{3.14}$$

$$< (R^{*\perp}(X,Y)\xi, \eta >=< \tilde{R}^*(X,Y)\xi, \eta > + g_M([A_\xi, A^*_\eta]X, Y), \tag{3.15}$$

*where $R^\perp$ and $R^{*\perp}$ are curvature tensors with respect to $D$ and $D^*$* and

$$\begin{aligned} [A_\xi, A^*_\eta] &= A_\xi A^*_\eta - A^*_\eta A_\xi, \\ [A^*_\xi, A_\eta] &= A^*_\xi A_\eta - A_\eta A^*_\xi \end{aligned}$$

*for $X, Y, Z, W \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^\perp M)$.*

## 4. Almost Kenmotsu statistical manifolds

**Definition 4.1** ([13]). Let $(M, g, \nabla)$ be a statistical manifold with almost complex structure $J \in \Gamma(TM^{(1,1)})$. Denote by $\Omega$ the fundamental form with respect to $J$ and $g$, that is, $\Omega(X,Y) = g(X, JY)$. The triplet $(\nabla, g, J)$ is called a holomorphic statistical structure on M if $\Omega$ is a $\nabla$-parallel 2-form.

**Definition 4.2** ([30]). Let $(N^{2n}, g, \nabla, \nabla^*)$ be a statistical manifold. If $(N^{2n}, g, J)$ is an almost Hermitian manifold then $(N^{2n}, g, J, \nabla, \nabla^*)$ is called almost Hermitian statistical manifold. If $(N^{2n}, g, J)$ is an (almost) Kaehler manifold then $(N^{2n}, g, J, \nabla, \nabla^*)$ is called (almost) Kaehler statistical manifold.

**Lemma 4.1** ([30]). *For an almost Hermitian statistical manifold we have*

$$(\nabla_X\Omega)(Y,Z) = g((\nabla_X J)Y, Z) - 2g(\mathcal{K}_X JY, Z), \tag{4.1}$$

*and*

$$(\nabla^*_X\Omega)(Y,Z) = g((\nabla^*_X J)Y, Z) + 2g(\mathcal{K}_X JY, Z) \tag{4.2}$$

*for any $X, Y, Z \in \Gamma(TM)$.*

**Corollary 4.1** ([30]). *For an almost Hermitian statistical manifold we have*

$$(\nabla_X\Omega)(Y,Z) = (\nabla^0_X\Omega)(Y,Z) - g(\mathcal{K}_X JY + J\mathcal{K}_X Y, Z) \tag{4.3}$$

*and*

$$(\nabla^*_X\Omega)(Y,Z) = (\nabla^0_X\Omega)(Y,Z) + g(\mathcal{K}_X JY + J\mathcal{K}_X Y, Z) \tag{4.4}$$

*for any $X, Y, Z \in \Gamma(TM)$.*

By Lemma 4.1 and Corollary 4.1, we have following.

**Proposition 4.1** ([13],[30].). *Let $(M, g, \nabla, J)$ be a holomorphic statistical manifold and $\mathcal{K}_X JY + J\mathcal{K}_X Y = 0$ for any $X, Y \in \Gamma(TM)$. Then following staments are equivalent.*

- *$(M, g, \nabla, J)$ is a holomorphic statistical manifold,*
- *$(M, g, \nabla^*, J)$ is a holomorphic statistical manifold,*
- *$(M, g, \nabla^0, J)$ is a Kaehler manifold.*

**Definition 4.3.** Let $(M^{2n+1}, g, \nabla, \nabla^*)$ be a statistical manifold. If $M^{2n+1}$ is an almost contact metric manifold then $M^{2n+1}$ is called almost contact metric statistical manifold.

**Corollary 4.2** ([30]). *For an almost contact metric statistical manifold we have*

$$(\nabla_X \Phi)(Y, Z) = (\nabla^0_X \Phi)(Y, Z) - g(\mathcal{K}_X \phi Y + \phi \mathcal{K}_X Y, Z) \tag{4.5}$$

*and*

$$(\nabla^*_X \Phi)(Y, Z) = (\nabla^0_X \Phi)(Y, Z) + g(\mathcal{K}_X \phi Y + \phi \mathcal{K}_X Y, Z) \tag{4.6}$$

*for any $X, Y, Z \in \Gamma(TM)$.*

**Proposition 4.2** ([30]). *Let $(M^n, g, \nabla, \nabla^*)$ be a statistical manifold and $\psi$ be a skew symmetric $(1, 1)$ tensor field on $M$. Then we have*

$$g(\mathcal{K}_X \psi Y + \psi \mathcal{K}_X Y, Z) + g(\mathcal{K}_Z \psi X + \psi \mathcal{K}_Z X, Y) + g(\mathcal{K}_Y \psi Z + \psi \mathcal{K}_Y Z, X) = 0 \tag{4.7}$$

*for any $X, Y, Z \in \Gamma(TM)$.*

If we resort to the relation (4.5) and (4.7), we have

$$\begin{aligned}
(\nabla_X \Phi)(Y, Z) + (\nabla_Z \Phi)(X, Y) + (\nabla_Y \Phi)(Z, X) &= (\nabla^0_X \Phi)(Y, Z) + (\nabla^0_Z \Phi)(X, Y) \\
&\quad + (\nabla^0_Y \Phi)(Z, X) \\
&\quad - 2(g(\mathcal{K}_X \phi Y + \phi \mathcal{K}_X Y, Z) \\
&\quad + g(\mathcal{K}_Z \phi X + \phi \mathcal{K}_Z X, Y) \\
&\quad + g(\mathcal{K}_Y \phi Z + \phi \mathcal{K}_Y Z, X)) \\
&= (\nabla^0_X \Phi)(Y, Z) + (\nabla^0_Z \Phi)(X, Y) \\
&\quad + (\nabla^0_Y \Phi)(Z, X),
\end{aligned}$$

where $U, V, W \in \Gamma(TM)$.

This relation shows clearly that

$$\begin{aligned}
3d\Phi(X, Y, Z) &= (\nabla^0_X \Phi)(Y, Z) + (\nabla^0_Z \Phi)(X, Y) + (\nabla^0_Y \Phi)(Z, X) \\
&= (\nabla_X \Phi)(Y, Z) + (\nabla_Z \Phi)(X, Y) + (\nabla_Y \Phi)(Z, X).
\end{aligned} \tag{4.8}$$

Let $(N, \nabla, g, J)$ be an almost Hermitian statistical manifold and $(\mathbb{R}, dt, ^{\mathbb{R}}\nabla)$ be trivial statistical manifold. Let us consider the warped product $\tilde{M} = \mathbb{R} \times_f N$, with warping function $f > 0$, endowed with the Riemannian metric

$$<, >= dt^2 + f^2 g.$$

Denoting by $\xi = \frac{\partial}{\partial t}$ the structure vector field on $\tilde{M}$, one can define arbitrary any vector field on $\tilde{M}$ by $\tilde{X} = \eta(\tilde{X})\xi + X$, where $X$ is any vector field on $N$ and $dt = \eta$. By the help of tensor field $J$, a new tensor field $\phi$ of type $(1, 1)$ on $\tilde{M}$ can be given by

$$\phi \tilde{X} = JX, \quad X \in \Gamma(TN), \tag{4.9}$$

for $\tilde{X} \in \Gamma(T\tilde{M})$. So we get $\phi\xi = 0$, $\eta \circ \phi = 0$, $\phi^2 \tilde{X} = -\tilde{X} + \eta(\tilde{X})\xi$ and $< \phi\tilde{X}, \tilde{Y} >= - < \tilde{X}, \phi\tilde{Y} >$ for $\tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M})$. Furthermore, we have $< \phi\tilde{X}, \phi\tilde{Y} >=< \tilde{X}, \tilde{Y} > -\eta(\tilde{X})\eta(\tilde{Y})$. Thus $(\tilde{M}, <, >, \phi, \xi, \eta)$ is an almost contact metric manifold. By Proposition 3.1 and similar argument as in [8] we have

$$(\tilde{\nabla}_{\tilde{X}} \phi)\tilde{Y} = (\nabla_X J)Y - \frac{f'(t)}{f(t)} < \tilde{X}, \phi\tilde{Y} > \xi - \frac{f'(t)}{f(t)} \eta(\tilde{Y})\phi\tilde{X}. \tag{4.10}$$

Using Proposition 3.1, we get

$$\tilde{\mathcal{K}}_X Y = \mathcal{K}_X Y, \tilde{\mathcal{K}}_X \xi = \tilde{\mathcal{K}}_\xi X = 0, \tilde{\mathcal{K}}_\xi \xi = 0,$$

where $\mathcal{K}_X Y = \nabla_X Y - \nabla^0_X Y$ and $\tilde{\mathcal{K}}_X Y = \tilde{\nabla}_X Y - \tilde{\nabla}^0_X Y$.

By (4.10) and (4.8), it is readily found that the relation

$$\begin{aligned}(\tilde{\nabla}_{\tilde{X}}\Phi)(\tilde{Y},\tilde{Z}) &= f^2(\nabla_X\Omega)(Y,Z) - \frac{f'(t)}{f(t)} < \tilde{X}, \phi\tilde{Y} > \eta(\tilde{Z}) \\ &\quad -\frac{f'(t)}{f(t)}\eta(\tilde{Y})\Phi(\tilde{X},\tilde{Z}).\end{aligned}$$

We thus conclude that

$$d\Phi = f^2 d\Omega + 2(-\frac{f'(t)}{f(t)})\eta \wedge \Phi \tag{4.11}$$

and Proposition 3.1 leading to the following theorem.

**Theorem 4.1.** *Let $(\mathbb{R}, dt, ^{\mathbb{R}}\nabla)$ be a trivial statistical manifold. Then the warped product $\tilde{M} = \mathbb{R} \times_f N$ is an almost $(-\frac{f'(t)}{f(t)})$−Kenmotsu statistical manifold if and only if $(N, \nabla, g, J)$ is an almost Kaehler statistical manifold. Moreover $\tilde{\mathcal{K}}_X Y = \mathcal{K}_X Y$, $\tilde{\mathcal{K}}_X \xi = \tilde{\mathcal{K}}_\xi X = 0$, $\tilde{\mathcal{K}}_\xi \xi = 0$, where $\mathcal{K} = \nabla - \nabla^g$, and $\tilde{\mathcal{K}} = \tilde{\nabla} - \tilde{\nabla}^{<,>}$.*

Chosing $f = const \neq 0$, we have following corollary.

**Corollary 4.3.** *Let $(\mathbb{R}, dt, ^{\mathbb{R}}\nabla)$ be a trivial statistical manifold. Then the product manifold $\tilde{M} = \mathbb{R} \times N$ is an almost cosymplectic statistical manifold if and only if $(N, \nabla, g, J)$ is an almost Kaehler statistical manifold.*

Using same methods as in [18], we get following proposition.

**Proposition 4.3.** *Let $\tilde{M} = I \times_f N(c)$ be a statistical warped product manifold and $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in \Gamma(\tilde{M})$, where $I \subset \mathbb{R}$ is trivial statistical manifold and $N(c)$ is statistical complex space form. Then the curvature tensors $\tilde{R}$ and $\tilde{R}^*$ are given by*

$$\begin{aligned}\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) &= \tilde{R}^*(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) \\ &= [\frac{c}{4f^2} - \frac{(f')^2}{f^2}][< \tilde{Y},\tilde{Z} >< \tilde{X},\tilde{W} > - < \tilde{X},\tilde{Z} >< \tilde{Y},\tilde{W} >] \\ +[\frac{c}{4f^2} - \frac{(f')^2}{f^2} + \frac{f''}{f}][ &< \tilde{X},\tilde{Z} >< \tilde{Y},\partial_t >< \tilde{W},\partial_t > \\ - &< \tilde{Y},\tilde{Z} >< \tilde{X},\partial_t >< \tilde{W},\partial_t > + < \tilde{Y},\tilde{W} >< \tilde{X},\partial_t >< \tilde{Z},\partial_t > \\ - &< \tilde{X},\tilde{W} >< \tilde{Y},\partial_t >< \tilde{Z},\partial_t >] \\ +\frac{c}{4f^2}[ &< \tilde{X},\phi\tilde{Z} >< \phi\tilde{Y},\tilde{W} > - < \tilde{Y},\phi\tilde{Z},>< \phi\tilde{X},\tilde{W} > \\ +2 &< \tilde{X},\phi\tilde{Y} >< \phi\tilde{Z},\tilde{W} >]\end{aligned}$$

*and $[\mathcal{K}, \mathcal{K}] = 0$.*

*Remark* 4.1. In [14] Furuhata et al. introduced Kenmotsu statistical manifolds. They proved that if $M$ has a holomorphic statistical structure, $(N = \mathbb{R} \times_{e^t} M, <,>, \phi, \xi)$ is Kenmotsu manifold satisfying property $\tilde{\mathcal{K}}_X Y = \mathcal{K}_X Y$, $\tilde{\mathcal{K}}_X \xi = \tilde{\mathcal{K}}_\xi X = 0$, $\tilde{\mathcal{K}}_\xi \xi = \lambda\xi$ then $N$ has a holomorphic statistical structure, where $\lambda \in C^\infty(N)$.

We now give a new example of a statistical warped product manifold.

**Example 4.1** ([22])**.** We consider $(\mathbb{R}^2, \tilde{g} = dx^2 + dy^2)$ Euclidean space and define the affine connection by

$$\begin{aligned}\tilde{\nabla}^2_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= \frac{\partial}{\partial y}, \ \tilde{\nabla}^2_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = 0, \\ \tilde{\nabla}^2_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} &= \tilde{\nabla}^2_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}.\end{aligned} \tag{4.12}$$

Then its conjugate $\tilde{\nabla}^{2*}$ is given as follows;

$$\begin{aligned} \tilde{\nabla}^{2*}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= -\frac{\partial}{\partial y},\ \tilde{\nabla}^{2*}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=0, \\ \tilde{\nabla}^{2*}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} &= \tilde{\nabla}^{2*}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}=-\frac{\partial}{\partial x}. \end{aligned} \tag{4.13}$$

Thus we can verify that $(\mathbb{R}^2, \tilde{\nabla}^2, \tilde{g})$ is a statistical manifold of constant curvature $-1$. An affine connection and its conjugate connection are defined on $(\mathbb{R}, dt^2)$ Euclidean space as following

$$\tilde{\nabla}^{1}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=0, \tilde{\nabla}^{1*}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=-0.$$

On the other hand, $(\mathbb{R}\times_{e^t}\mathbb{R}^2, <,>= dt^2+e^{2t}(dx^2+dy^2))$ is a warped product model of hyperbolic space $(\tilde{H}^3 = \{(x,y,z)\in\mathbb{R}^3 \mid z>0\}, \tilde{g}_{\tilde{H}^3}=\frac{dx^2+dy^2+dz^2}{z^2})$ and it has natural Kenmotsu structure. We also have $(\mathbb{R}\times_{e^t}\mathbb{R}^2, <,>)$ is a statistic manifold with following affine connection $\bar{\nabla}$ ;

$$\begin{aligned} \bar{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} &= 0,\ \bar{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x}=\frac{\partial}{\partial x}, \bar{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial y}=\frac{\partial}{\partial y}, \\ \bar{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial t} &= \frac{\partial}{\partial x},\ \bar{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=\frac{\partial}{\partial y}-e^{2t}\frac{\partial}{\partial t}, \bar{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=\frac{\partial}{\partial x}, \\ \bar{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial t} &= \frac{\partial}{\partial y},\ \bar{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}=\frac{\partial}{\partial x}, \bar{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=-e^{2t}\frac{\partial}{\partial t}. \end{aligned}$$

## 5. Generalized Wintgen Inequality for $(-\frac{f'(t)}{f(t)})-$Kenmotsu statistical manifold

Let $\bar{M}^m$ be a complex $m$-dimensional (real 2m dimensional) almost Hermitian manifold with Hermitian metric $g_{\bar{M}}$ and almost complex structure $J$ and $N^n$ be a Riemannian manifold with Riemannian metric $g_N$. If $J(T_pN)\subset T_p^{\perp}N$, at any point $p\in N$, then is called totally real submanifold. In particular, a toatally real submanifold of maximum dimension is called a Lagrangian submanifold.

Let $M^n$ be a submanifold of $\tilde{M}^{2m+1}$. $\phi$ maps any tangent space of $M^n$ into the normal space, that is, $\phi(T_pM^n)\subset T_p^{\perp}\tilde{M}^{2m+1}$, for every $p\in M^n$, then $M^n$ is called anti invarant submanifold. If $\dim(\tilde{M}) = 2\dim(M)+1$ and $\xi_p$ is orthogonal to $T_pM$ for all $p\in M^n$ then $M^n$ is called Legendre submanifold.

I. Mihai, [20],[21] obtained the DDVV inequality, also known as generalized Wintgen inequality for Lagrangian submanifold of a complex space form $\bar{M}^m(4c)$ and Legendrian submanifolds in Sasakian space forms,

$$(\rho^{\perp})^2 \le (\|H\|^2-\rho+c)^2+\frac{4}{n(n-1)}(\rho-c)+\frac{2c^2}{n(n-1)},$$

$$(\rho^{\perp})^2 \le (\|H\|^2-\rho+c)^2+\frac{4}{n(n-1)}(\rho-\frac{c+3}{4})\frac{c-1}{4}+\frac{(c-1)^2}{8n(n-1)},$$

respectively.

In [7], the following theorem is proved.

**Theorem 5.1** ([7]). *Let $M^n$ be a Lagrangian submanifold of a holomorphic statistical space form $\bar{M}^m(c)$. Then*

$$(\rho^{\perp})^2 \ge \frac{c}{n(n-1)}(\rho-\frac{c}{4})+\frac{c}{(n-1)^2}[g(H^*,H)-\|H\|\,\|H^*\|].$$

Now, we will prove Generalized Wintgen Inequality for almost $(-\frac{f'(t)}{f(t)})-$Kenmotsu statistical manifold.

**Theorem 5.2.** *Let $(\mathbb{R}, dt, {}^{\mathbb{R}}\nabla)$ be a trivial statistical manifold and $N(c)$ be a holomorphic statistical space form. If $M^n$ is a Legendrian submanifold of the statistical warped product manifold $\tilde{M}=\mathbb{R}\times_f N(c)$, then we have*

$$\begin{aligned} \rho^{\perp\nabla,\nabla^*} &\le 2\rho^{\nabla,\nabla^*}-8\rho^0+\frac{1}{4f^2}(2f\mid c\mid -c+4(f')^2) \\ &+4 \quad \| \quad H^0\|^2+\|H\|^2+\|H^*\|^2\,. \end{aligned}$$

*Proof.* Let $M^n$ be an $n$-dimensional Legendrian real submanifold of a $2n+1$-dimensional almost $(-\frac{f'(t)}{f(t)})-$Kenmotsu statistical manifold $\tilde{M} = \mathbb{R} \times_f N(c)$ and $\{e_1, e_2, ..., e_n\}$ an orthonormal frame on $M^n$ and $\{e_{n+1} = \phi e_1, e_{n+2} = \phi e_2, ..., e_{2n} = \phi e_n, e_{2n+1} = \xi\}$ an orthonormal frame in normal bundle $T^\perp M^n$, respectively. By Proposition 4.3 and (3.12), we have

$$\begin{aligned} g_M(R(X,Y)Z,W) &= < \tilde{R}(X,Y)Z,W > + < h^*(X,W), h(Y,Z) > \\ &- < h(X,Z), h^*(Y,W) > \\ &= [\frac{c}{4f^2} - \frac{(f')^2}{f^2}][< Y,Z >< X,W > - < X,Z >< Y,W >] \\ &+ < h^*(X,W), h(Y,Z) > - < h(X,Z), h^*(Y,W) > . \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} g_M(R^*(X,Y)Z,W) &= [\frac{c}{4f^2} - \frac{(f')^2}{f^2}][< Y,Z >< X,W > - < X,Z >< Y,W >] \\ &+ < h(X,W), h^*(Y,Z) > - < h^*(X,Z), h(Y,W) > . \end{aligned} \tag{5.2}$$

*for* $X, Y, Z, W \in \Gamma(TM)$. Setting $X = e_i = W$, $Y = e_j = Z$ in (5.1) and (5.2), we have

$$\begin{aligned} g_M(R(e_i,e_j)e_i,e_j) &= (\frac{c}{4f^2} - \frac{(f')^2}{f^2})(< e_j,e_j >< e_i,e_i > - < e_i,e_j >< e_i,e_j >) \\ &+ < h^*(e_i,e_i), h(e_j,e_j) > - < h(e_i,e_j), h^*(e_i,e_j) > \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} g_M(R^*(e_i,e_j)e_i,e_j) &= (\frac{c}{4f^2} - \frac{(f')^2}{f^2})(< e_j,e_j >< e_i,e_i > - < e_i,e_j >< e_i,e_j >) \\ &+ < h(e_i,e_i), h^*(e_j,e_j) > - < h^*(e_i,e_j), h(e_i,e_j) > . \end{aligned} \tag{5.4}$$

Using (5.1) in (3.14), we have

$$\begin{aligned} &< (R^\perp(X,Y)U,V >= \frac{c}{4f^2}(- < \phi X,U >< \phi Y,V > \\ &+ < \phi X,V >< \phi Y,U >) + g_M([A^*_U, A_V]X,Y), \end{aligned} \tag{5.5}$$

If we make use of the equality (5.2) in (3.15), we obtain

$$\begin{aligned} &< (R^{*\perp}(X,Y)U,V >= \frac{c}{4f^2}(- < \phi X,U >< \phi Y,V > \\ &+ < \phi X,V >< \phi Y,U >) + g_M([A_\xi, A^*_\eta]X,Y). \end{aligned} \tag{5.6}$$

Since $< \tilde{R}(X,Y)Z,W >$ is not skew-symmetric relative to $Z$ and $W$. Then the sectional curvature on $\tilde{M}$ can not be defined. But $< R(X,Y)Z,W > + < R^*(X,Y)Z,W) >$ is skew-symmetric relative to $Z$ and $W$. So the sectional curvature $K^{\nabla,\nabla^*}$ is defined by

$$K^{\nabla,\nabla^*}(X \wedge Y) = \frac{1}{2}[< R(X,Y)Y,X > + < R^*(X,Y)Y,X) >],$$

for any orthonormal vectors $X, Y, \in T_pM$, $p \in M$, (see [3]).

In [3], the normalized scalar curvature $\rho^{\nabla,\nabla^*}$and the normalized normal scalar curvature $\rho^{\perp\nabla,\nabla^*}$are respectively defined by

$$\begin{aligned} \rho^{\nabla,\nabla^*} &= \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} K^{\nabla,\nabla^*}(e_i \wedge e_j) \\ &= \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} (< R(e_i,e_j)e_j,e_i > + < R^*(e_i,e_j)e_j,e_i >) \end{aligned}$$

and

$$\rho^{\perp\nabla,\nabla^*}=\frac{1}{n(n-1)}\left\{\sum_{n+1\leq\alpha<\beta\leq 2n+1}\sum_{1\leq i<j\leq n}(<R^{\perp}(e_i,e_j)e_\alpha,e_\beta>+<R^{*\perp}(e_i,e_j)e_\alpha,e_\beta>)^2\right\}^{1/2},$$

where $\{e_1,...,e_n\}$ and $\{e_{n+1}=\phi e_1,...,e_{2n}=\phi e_n,e_{2n+1}=\xi\}$ are respectively orthonormal basis of $T_pM$ and $T_p^{\perp}M$ for $p\in M$. Due to the equations (5.2) and (5.3), we obtain

$$\begin{aligned}
\rho^{\nabla,\nabla^*} &= \frac{1}{2n(n-1)}\sum_{i\neq j}[(\frac{c}{4f^2}-\frac{(f')^2}{f^2})+<h^*(e_i,e_i),h(e_j,e_j)> \\
&- <h(e_i,e_j),h^*(e_i,e_j)>+(\frac{c}{4f^2}-\frac{(f')^2}{f^2}) \\
&+ <h(e_i,e_i),h^*(e_j,e_j)>-<h^*(e_i,e_j),h(e_i,e_j)>] \\
&= (\frac{c}{4f^2}-\frac{(f')^2}{f^2})+\frac{1}{2n(n-1)}\sum_{i\neq j}[<h^*(e_i,e_i),h(e_j,e_j)> \\
&+ <h(e_i,e_i),h^*(e_j,e_j)>-2<h(e_i,e_j),h^*(e_i,e_j)>] \\
&= (\frac{c}{4f^2}-\frac{(f')^2}{f^2}) \\
+\frac{1}{2n(n-1)}\sum_{i\neq j}[ &< h(e_i,e_i)+h^*(e_i,e_i),h^*(e_j,e_j)+h(e_j,e_j)> \\
&- <h(e_i,e_i),h(e_j,e_j)>-<h^*(e_j,e_j),h^*(e_j,e_j)> \\
-( &< h(e_i,e_j)+h^*(e_i,e_j),h(e_i,e_j)+h^*(e_i,e_j> \\
&- <h(e_i,e_j),h(e_i,e_j)>-<h^*(e_i,e_j),h^*(e_i,e_j)>)].
\end{aligned}$$

Because of $2h^0=h+h^*$ and $2H^0=H+H^*$, we thus get

$$\begin{aligned}
\rho^{\nabla,\nabla^*} &= (\frac{c}{4f^2}-\frac{(f')^2}{f^2})+\frac{1}{2n(n-1)}\sum_{i\neq j}[4<h^0(e_i,e_i),h^0(e_j,e_j))> \\
&- <h(e_i,e_i),h(e_j,e_j)>-<h^*(e_j,e_j),h^*(e_j,e_j)> \\
-(4 &< h^0(e_i,e_j),h^0(e_i,e_j)>-<h(e_i,e_j),h(e_i,e_j)> \\
&- <h^*(e_i,e_j),h^*(e_i,e_j)>)].
\end{aligned}$$

$$\begin{aligned}
\rho^{\nabla,\nabla^*} &= (\frac{c}{4f^2}-\frac{(f')^2}{f^2})+\frac{1}{2n(n-1)}[4n^2\parallel H^0\parallel^2-n^2\parallel H\parallel^2-n^2\parallel H^*\parallel^2 \\
-(4 &\parallel h^0\parallel^2-\parallel h\parallel^2-\parallel h^*\parallel^2].
\end{aligned}$$

Denote $\tau^0=h^0-H^0g,\tau=h-Hg$ and $\tau^*=h^*-H^*g$ the traceless part of second fundamental forms. Then we find $\parallel\tau^0\parallel^2=\parallel h^0\parallel^2-n^2\parallel H^0\parallel^2$, $\parallel\tau\parallel^2=\parallel h\parallel^2-n^2\parallel H\parallel^2$ and $\parallel\tau^*\parallel^2=\parallel h^*\parallel^2-n^2\parallel H^*\parallel^2$. Thus, we get

$$\begin{aligned}
\rho^{\nabla,\nabla^*} &= (\frac{c}{4f^2}-\frac{(f')^2}{f^2})+\frac{1}{2n(n-1)}[4n^2\parallel H^0\parallel^2-n^2\parallel H\parallel^2-n^2\parallel H^*\parallel^2 \\
-(4 &\parallel \tau^0\parallel^2+4n\parallel H^0\parallel^2-\parallel\tau\parallel^2-n\parallel H\parallel^2-\parallel\tau^*\parallel^2-n\parallel H^*\parallel^2).
\end{aligned}$$

This relation gives rise to

$$\begin{aligned}
\rho^{\nabla,\nabla^*} &= \frac{c}{4f^2}-\frac{(f')^2}{f^2} \\
+2 &\parallel H^0\parallel^2-\frac{2}{n(n-1)}\parallel\tau^0\parallel^2 \\
-\frac{1}{2} &\parallel H\parallel^2+\frac{1}{2n(n-1)}\parallel\tau\parallel^2 \\
-\frac{1}{2} &\parallel H^*\parallel^2+\frac{1}{2n(n-1)}\parallel\tau^*\parallel^2.
\end{aligned}\tag{5.7}$$

From (5.5) and (5.6), the normalized normal scalar curvature satisfies

$$\rho^{\perp\nabla,\nabla^*}=\frac{1}{n(n-1)}\left\{\sum_{1\leq r<s\leq n+1}\sum_{1\leq i<j\leq n}\left[\begin{array}{c} g([A^*_{e_{n+r}},A_{e_{n+s}}]e_i,e_j)+g([A_{e_{n+r}},A^*_{e_{n+s}}]e_i,e_j)\\ +\frac{2c}{4f^2}(-<\phi e_i,e_{n+r}><\phi e_j,e_{n+s}>\\ +<\phi e_i,e_{n+s}><\phi e_j,e_{n+r}>)\end{array}\right]^2\right\}^{1/2} \tag{5.8}$$

By Proposition 3.1 and the equations (3.10), (3.11), we have

$$A_\xi X=A^*_\xi X=-\frac{f'(t)}{f(t)}X\,. \tag{5.9}$$

Hence we have

$$\begin{aligned} g([A^*_\xi,A_{e_{n+s}}]e_i,e_j) &= g(A^*_\xi A_{e_{n+s}}e_i,e_j)-g(A_{e_{n+s}}A^*_\xi e_i,e_j)\\ &\overset{(5.9)}{=} -\frac{f'(t)}{f(t)}g(A_{e_{n+s}}e_i,e_j)+\frac{f'(t)}{f(t)}g(A_{e_{n+s}}e_i,e_j)\\ &= 0 \end{aligned} \tag{5.10}$$

and by using similar calculation we obtain

$$([A_{e_{n+r}},A^*_{e_{n+s}}]e_i,e_j)=0. \tag{5.11}$$

On the other hand, we recall

$$<\phi X,\xi>=0. \tag{5.12}$$

Using the equations (5.10), (5.11) and (5.12) in (5.8) we find that

$$\rho^{\perp\nabla,\nabla^*}=\frac{1}{n(n-1)}\left\{\sum_{1\leq r<s\leq n}\sum_{1\leq i<j\leq n}\left[\begin{array}{c} g([A^*_{e_{n+r}},A_{e_{n+s}}]e_i,e_j)\\ +g([A_{e_{n+r}},A^*_{e_{n+s}}]e_i,e_j)\\ -\frac{2c}{4f^2}(\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr})\end{array}\right]^2\right\}^{1/2} \tag{5.13}$$

which is equivalent to

$$\rho^{\perp\nabla,\nabla^*}=\frac{1}{n(n-1)}\left\{\sum_{1\leq r<s\leq n}\sum_{1\leq i<j\leq n}\left[\begin{array}{c} 4g([A^0_{e_{n+r}},A^0_{e_{n+s}}]e_i,e_j)+g([A_{e_{n+r}},A_{e_{n+s}}]e_i,e_j)\\ +g([A^*_{e_{n+r}},A^*_{n+s_s}]e_i,e_j)-\frac{2c}{4f^2}(\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr})\end{array}\right]^2\right\}^{1/2}, \tag{5.14}$$

where $2A^0_{\xi_r}=A_{\xi_r}+A^*_{\xi_r}$. By the Cauchy–Schwarz inequality, we have the algebraic inequality

$$(\lambda+\mu+\nu+w)^2\leq 4(\lambda^2+\mu^2+\nu^2+w^2),\forall\lambda,\mu,\nu\in\mathbb{R}. \tag{5.15}$$

We obtain from (5.15) that

$$\begin{aligned} \rho^{\perp\nabla,\nabla^*} &\leq \frac{2}{n(n-1)}\left\{\begin{array}{c}\sum\limits_{1\leq r<s\leq n}(\sum\limits_{1\leq i<j\leq n}(16g([A^0_{e_{n+r}},A^0_{e_{n+s}}]e_i,e_j)^2\\ +g([A_{e_{n+r}},A_{e_{n+s}}]e_i,e_j)^2\\ +g([A^*_{e_{n+r}},A^*_{n+s_s}]e_i,e_j)^2+\frac{c^2}{4f^2}(\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr})^2\end{array}\right\}^{1/2}\\ &\leq \frac{2}{n(n-1)}\left\{\begin{array}{c}\frac{c^2}{4f^2}n^2(n-1)^2+\frac{1}{4}\sum\limits_{r,s=1}^{n}(\sum\limits_{i,j=1}^{m}16g([A^0_{\xi_r},A^0_{\xi_s}]e_i,e_j)^2\\ +g([A_{\xi_r},A_{\xi_s}]e_i,e_j)^2+g([A^*_{\xi_r},A^*_{\xi_s}]e_i,e_j))^2\end{array}\right\}^{1/2}\\ &\leq \frac{2}{n(n-1)}\left\{\begin{array}{c}\frac{c^2}{4f^2}n^2(n-1)^2\\ +\frac{1}{4}\sum\limits_{r,s=1}^{n}(16\|[A^0_{\xi_r},A^0_{\xi_s}\|^2+\|[A_{\xi_r},A_{\xi_s}]\|^2+\|[A^*_{\xi_r},A^*_{\xi_s}]\|^2)\end{array}\right\}^{1/2}. \end{aligned}$$

Now we define sets $\{S^0_1,...,S^0_n\}$, $\{S_1,...,S_n\}$, $\{S^*_1,...,S^*_n\}$ of symmetric with trace zero operators on $T_pM$ by

$$\begin{aligned} <\ S^0_\alpha X,Y>&=<\tau^0(X,Y),\xi_\alpha>,\\ <\ S_\alpha X,Y>&=<\tau(X,Y),\xi_\alpha>,\\ <\ S^*_\alpha X,Y>&=<\tau^*(X,Y),\xi_\alpha> \end{aligned}$$

for all $X, Y, \in T_pM$, $p \in M$. Clearly, we obtain

$$\begin{aligned} S^0_\alpha &= A^0_{\xi_\alpha} - < H^0, \xi_\alpha > I, \\ S_\alpha &= A_{\xi_\alpha} - < H, \xi_\alpha > I, \\ S^*_\alpha &= A^*_{\xi_\alpha} - < H^*, \xi_\alpha > I \end{aligned}$$

and

$$\begin{aligned} [S^0_\alpha, S^0_\beta] &= [A^0_{\xi_\alpha}, A^0_{\xi_\beta}], \\ [S_\alpha, S_\beta] &= [A_{\xi_\alpha}, A_{\xi_\beta}], \\ [S^*_\alpha, S^*_\beta] &= [A^*_{\xi_\alpha}, A^*_{\xi_\beta}]. \end{aligned}$$

Therefore, it is clear that

$$\rho^{\perp\nabla,\nabla^*} \leq \frac{2}{n(n-1)} \left\{ \frac{c^2}{4f^2} n^2(n-1)^2 + \frac{1}{4} \sum_{r,s=1}^{n} (16\|[S^0_r, S^0_s]\|^2 + \|[S_r, S_s]\|^2 + \|[S^*_r, S^*_s]\|^2) \right\}^{1/2}. \tag{5.16}$$

In [19], Lu proved following theorem.

**Theorem 5.3** ( [19]). *For every set $\{B_1, ..., B_n\}$ of symmetric $(n \times n)$-matrices with trace zero the following inequality holds:*

$$\sum_{\alpha,\beta=1}^{n} \|[B_\alpha, B_\beta]\|^2 \leq (\sum_{\alpha=1}^{n} \|B_\alpha\|^2)^2.$$

By Theorem 5.3, (5.16) can be written as

$$\begin{aligned} \rho^{\perp\nabla,\nabla^*} &\leq \frac{|\,c\,|}{2f} + \frac{4}{n(n-1)} \sum_{r=1}^{n} \|S^0_r\|^2 + \frac{1}{n(n-1)} \sum_{r=1}^{n} \|[S_r\|^2 + \frac{1}{n(n-1)} \sum_{r=1}^{n} \|[S^*_r\|^2 \\ &\leq \frac{|\,c\,|}{2f} + \frac{4}{n(n-1)} \parallel \tau^0 \parallel^2 + \frac{1}{n(n-1)} \parallel \tau \parallel^2 + \frac{1}{n(n-1)} \parallel \tau^* \parallel^2 . \end{aligned} \tag{5.17}$$

Using (5.7) in (5.17), we get

$$\begin{aligned} \rho^{\perp\nabla,\nabla^*} &\leq \frac{|\,c\,|}{2f} + \frac{8}{n(n-1)} \parallel \tau^0 \parallel^2 + 2\rho^{\nabla,\nabla^*} - \frac{2c}{4f^2} + \frac{2(f')^2}{f^2} \\ -4 &\parallel H^0 \parallel^2 + \parallel H \parallel^2 + \parallel H^* \parallel^2 . \end{aligned} \tag{5.18}$$

On the other hand normalized scalar curvature $\rho^0$ of $M^m$ with respect to Levi-civita connection $\nabla^0$ can be obtained as

$$\rho^0 = (\frac{c}{4f^2} - \frac{(f')^2}{f^2}) + \frac{1}{n(n-1)} [n^2 \parallel H^0 \parallel^2 - \parallel h^0 \parallel^2] \tag{5.19}$$

(see [24]).

Now, if we set $\parallel \tau^0 \parallel^2 = \parallel h^0 \parallel^2 - n \parallel H^0 \parallel^2$ in (5.19), then we get

$$\rho^0 = (\frac{c}{4f^2} - \frac{(f')^2}{f^2}) + \parallel H^0 \parallel^2 - \frac{1}{n(n-1)} \parallel \tau^0 \parallel^2 . \tag{5.20}$$

In view of the equations (5.18) and (5.19), we have

$$\begin{aligned} \rho^{\perp\nabla,\nabla^*} &\leq 2\rho^{\nabla,\nabla^*} - 8\rho^0 + \frac{1}{4f^2}(2f \mid c \mid -c + 4(f')^2) \\ +4 &\parallel H^0 \parallel^2 + \parallel H \parallel^2 + \parallel H^* \parallel^2 \end{aligned}$$

which completes the proof. □

**Corollary 5.1.** *Let* $(\mathbb{R}, dt, ^{\mathbb{R}}\nabla)$ *be a trivial statistical manifold and* $N(c=0) = \mathbb{C}^n$ *be a holomorphic statistical space form. If* $M^n$ *be a Legendrian submanifold of the statistical Kenmotsu manifold* $\mathbb{R} \times_{e^t} \mathbb{C}^n$*, then we get*

$$\rho^{\perp\nabla,\nabla^*} \leq 2\rho^{\nabla,\nabla^*} - 8\rho^0 + 4 \parallel H^0 \parallel^2 + \parallel H \parallel^2 + \parallel H^* \parallel^2 + 1.$$

*In this case* $\mathbb{R} \times_{e^t} \mathbb{C}^n$ *is locally isometric to the hyperbolic space* $H^{2n+1}(-1)$*.*

**Corollary 5.2.** *Let* $(\mathbb{R}, dt, ^{\mathbb{R}}\nabla)$ *be a trivial statistical manifold and* $N(c)$ *be a holomorphic statistical space form. If* $M^n$ *be a Legendrian submanifold of the statistical cosymplectic manifold* $\mathbb{R} \times N(c)$*, then we have*

$$\rho^{\perp\nabla,\nabla^*} \leq 2\rho^{\nabla,\nabla^*} - 8\rho^0 + 4 \parallel H^0 \parallel^2 + \parallel H \parallel^2 + \parallel H^* \parallel^2 + \frac{1}{4}(2 \mid c \mid -c).$$

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