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On the Lifts of $F(2K+S,S)-$Structure Satisfying $F^{2K+S}+F^S=0$, $(F\neq 0,\ K\geqslant 1,\ S\geqslant 1)$ on Cotangent and Tangent Bundle

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ABSTRACT

This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F(2K+S,S)-$structure Satisfying $F^{2K+S}+F^S=0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F(2K+S,S)-$structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the horizontal lifts of the structure. In the second part, all results obtained in the first section were obtained according to the complete and horizontal lifts of $F(2K+S,S)-$structure in tangent bundle $T(M^n)$.

Keywords: *Integrability conditions; Tachibana operators; lifts; Sasakian metric; tangent bundle; cotangent bundle.*

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1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [26]. There are a lot of structures on $n-$dimensional differentiable manifold M^n. Firstly, Ishıhara and Yano [12] have obtained the integrability conditions of a structure F satisfying $F^3+F=0$. Gouli-Andreou [3] has studied the integrabilty conditions of a structure F satisfying $F^5+F=0$. Later, R. Nivas and C.S. Prasad [16] studied on the form $F_a(5,1)-$structure. Also $F_\lambda(7,1)-$structure extended in M^n to $T^*(M^n)$ by L. S. Das, R. Nivas and V. N. Pathak [14]. In 1989, V. C. Gupta [11] studied on more generalized form $F(K,1)-$structure satisfying $F^K+F=0$, where K is a positive integer $\geqslant 2$. Later, L. Das studied on the structure $f(2K+4;2)$ and the structure satisfying $F^{K+1}-a^2F^{K-1}=0$ [9, 10]. In addition, manifolds with $F(2K+S,S)-$structure satisfying $F^{2K+S}+F^S=0$, $(F\neq 0$, fixed integer $K\geqslant 1$, fixed odd integer $S\geqslant 1)$ have been defined and studied by A. Singh [21] and the complete and horizontal lifts of $F(2K+S,S)-$structure extended in M^n to tangent bundle by A. Singh, R. K. Pandey and S. Khare [22].

This paper consists of two main sections. In the first part, we find integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F(2K+S,S)$ $-$structure satisfying $F^{2K+S}+F^S=0$, $(F\neq 0$, fixed integer $K\geqslant 1$, fixed odd integer $S\geqslant 1)$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F(2K+S,S)-$structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the horizontal lifts of the structure. In the second part, all results obtained in the first section were obtained according to the complete and horizontal lifts of $F(2K+S,S)-$structure in tangent bundle $T(M^n)$. Also the Riemannian manifolds and the tangent bundles studyed a lot of authors [1, 2, 4, 15, 17, 18, 19, 23, 24] too.

Let M^n be a differentiable manifold of class C^∞ and F be a non-null tensor field of type $(1,1)$ satisfying

$$F^{2K+S}+F^S=0, \tag{1.1}$$

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where K is a fixed integer greater than or equal to 1 and S is a fixed odd integer greater than or equal to 1. F is of constant rank r everywhere in M^n. We call such a structure an $F(2K+S,S)-$structure of rank $2r$.

Let the operators l and m be defined as

$$l=-F^{2K}, m=I+F^{2K}, \tag{1.2}$$

where I denotes the identity operator on M^n.

The operators l and m defined by (1.2) satisfy the following:

$$\begin{aligned} l^2 &= l, m^2=m,\ l+m=I, \\ lm &= ml=0, \\ Fl &= lF=F, Fm=mF=0, \end{aligned} \tag{1.3}$$

where I being the identity operator.

Consequently, if there is a tensor field $F\neq 0$ satisfying (1.1), then there exist on M^n two complementary distributions L and M. Corresponding to l and m respectively. Let the rank of F be constant and be equal to r ewerywhere, then the dimensions of L and M are r and $n-r$, respectively. We call such a structure a '$F(2K+S,S)-$structure of rank $2r$' and the manifold M^n with this structure a $F(2K+S,S)-$manifold,where dim $M^n=n$.

In the manifold M^n endowed with $F^{2K+S}+F^S=0$, ($F\neq 0$, fixed integer $K\geqslant 1$, fixed odd integer $S\geqslant 1$) structure, the $(1,1)$ tensor field ψ given by $\psi=l-m=-I-2F^{2K}$ gives an almost product structure.

1.1. Horizontal Lift of the Structure Satisfying $F^{2K+S}+F^S=0$, ($F\neq 0$, fixed integer $K\geqslant 1$, fixed odd integer $S\geqslant 1$) on Cotangent Bundle

Let F,G be two tensor field of type $(1,1)$ on the manifold M^n. If F^H denotes the horizontal lift of F, we have [14, 26]

$$F^HG^H+G^HF^H=(FG+GF)^H. \tag{1.4}$$

Taking F and G identical, we get

$$(F^H)^2=(F^2)^H,$$

Continuing the above process of replacing G in equation (1.4) by some higher powers of F, we obtain

$$\begin{aligned} (F^K)^H &= (F^H)^K \\ (F^S)^H &= (F^H)^S \\ (F^{2K+S})^H &= (F^H)^{2K+S} \end{aligned} \tag{1.5}$$

where $F\neq 0$, fixed integer $K\geqslant 1$, fixed odd integer $S\geqslant 1$. Also if G and H are tensors of the same type then

$$(G+H)^H=G^H+H^H \tag{1.6}$$

Taking horizontal lift on both sides of equation $F^{2K+S}+F^S=0$, we get

$$(F^{2K+S})^H+(F^S)^H=0. \tag{1.7}$$

In view of (1.5) and (1.6), we can write [14, 22]

$$(F^H)^{2K+S}+(F^H)^S=0. \tag{1.8}$$

Proposition 1.1. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle $T^*(M^n)$ of M^n satisfies the following*

$$\begin{aligned} i)\ [\omega^V,\theta^V] &= 0, \\ ii)\ \left[X^H,\omega^V\right] &= (\nabla_X\omega)^V, \\ iii)\ \left[X^H,Y^H\right] &= [X,Y]^H+\gamma R(X,Y)=[X,Y]^H+(pR(X,Y))^V \end{aligned} \tag{1.9}$$

for all $X,Y\in\Im_0^1(M^n)$ and $\omega,\theta\in\Im_1^0(M^n)$. (See [26] p. 238, p. 277 for more details).

2. Main Results

Definition 2.1. Let F be a tensor field of type $(1,1)$ admitting $F^{2K+S}+F^S=0$ structure in M^n. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^n is given by

$$N_F=[FX,FY]-F\left[X,FY\right]-F\left[FX,Y\right]+F^2\left[X,Y\right] \tag{2.1}$$

for any $X,Y\in\Im_0^1(M^n)$ [5, 19, 20]. The condition of $N_F(X,Y)=N(X,Y)=0$ is essential to integrability condition in these structures.

The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k\partial_k=(F_i^s\partial_s^kF_j^k-F_j^l\partial_lF_i^k-\partial_iF_j^lF_l^k+\partial_jF_i^sF_s^k)\partial_k \tag{2.2}$$

where $X=\partial_i$, $Y=\partial_j$, $F\in\Im_1^1(M^n)$.

2.1. The Nijenhuis Tensors of $(F^{2K+S})^H$ on Cotangent Bundle $T^*(M^n)$

Theorem 2.1. *The Nijenhuis tensors of $\left(F^{2K+S}\right)^H$ and F^S denote by $\tilde{N}$ and N, respectively. Thus, taking account of the definition of the Nijenhuis tensor, the formulas (1.9) stated in Proposition 1.1 and the structure $(F^{2K+S})^H+(F^S)^H=0$, we find the following results of computation.*

$$\begin{aligned}
i)\ \tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,Y^H\right) &= \{[F^SX,F^SY]-F^S[F^SX,Y]-F^S[X,F^SY]\\
&\quad+(F^S)^2[X,Y]\}^H+\gamma\{R(F^SX,F^SY)\\
&\quad-R(F^SX,Y)F^S-R(X,F^SY)F^S\\
&\quad+R(X,Y)(F^S)^2\}.\\
ii)\ \tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,\omega^V\right) &= \{\omega\circ(\nabla_{F^SX}F^S)-(\omega\circ(\nabla_XF^S)F^S\}^V,\\
iii)\ \tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(\omega^V,\theta^V\right) &= 0.
\end{aligned}$$

Proof. $i)$The Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(X^H,Y^H)$ of the horizontal lift $(F^{2K+S})^H$ vanishes if F^S is an almost complex structure i.e., $(F^S)^2=-I$ and $R(F^SX,F^SY)=R(X,Y)$.

$$\begin{aligned}
\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(X^H,Y^H) &= [(F^{2K+S})^HX^H,(F^{2K+S})^HY^H]\\
&\quad-(F^{2K+S})^H[(F^{2K+S})^HX^H,Y^H]\\
&\quad-(F^{2K+S})^H[X^H,(F^{2K+S})^HY^H]\\
&\quad+(F^{2K+S})^H(F^{2K+S})^H[X^H,Y^H]\\
&= [(F^S)^HX^H,(F^S)^HY^H]-(F^S)^H[(F^S)^HX^H,Y^H]\\
&\quad-(F^S)^H[X^H,(F^S)^HY^H]+((F^S)^H)^2[X^H,Y^H]\\
&= \{[F^SX,F^SY]-F^S[F^SX,Y]-F^S[X,F^SY]\\
&\quad+(F^S)^2[X,Y]\}^H+\gamma\{R(F^SX,F^SY)\\
&\quad-R(F^SX,Y)F^S-R(X,F^SY)F^S+R(X,Y)(F^S)^2\}.
\end{aligned}$$

$(F^{2K+S})^H$ is integrable if the curvature tensor R of ∇ satisfies $R(F^SX,F^SY)=R(X,Y)$ and F^S is an almost complex structure, then we get $R(F^SX,Y)=-R(X,F^SY)$. Hence using $(F^S)^2=-I$, we find $R(F^SX,F^SY)-R(F^SX,Y)F-R(X,F^SY)F+R(X,Y)(F^S)^2=0$. Therefore, it follows $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(X^H,Y^H)=0$.

$ii)$The Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(X^H,\omega^V)$ of the horizontal lift $(F^{2K+S})^H$ vanishes if $\nabla F^S=0$.

$$\begin{aligned}\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(X^H,\omega^V) &= [(F^{2K+S})^H X^H,(F^{2K+S})^H\omega^V]\\ &\quad -(F^{2K+S})^H[(F^{2K+S})^H X^H,\omega^V]\\ &\quad -(F^{2K+S})^H[X^H,(F^{2K+S})^H\omega^V]\\ &\quad +(F^{2K+S})^H(F^{2K+S})^H[X^H,\omega^V]\\ &= [(F^S X)^H,(\omega\circ F^S)^V]-(F^S)^H[(F^S X)^H,\omega^V]\\ &\quad -(F^S)^H[X^H,(\omega\circ F^S)^V]+((F^S)^H)^2(\nabla_X\omega)^V\\ &= \{\omega\circ(\nabla_{F^S X}F^S)-(\omega\circ(\nabla_X F^S)F^S\}^V,\end{aligned}$$

We now suppose $\nabla F^S=0$, then we see $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,\omega^V\right)=0$, where $F^S\in\Im_1^1(M^n)$, $X\in\Im_0^1(M^n)$, $\omega\in\Im_1^0(M^n)$.

$iii)$The Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(\omega^V,\theta^V)$ of the horizontal lift $(F^{2K+S})^H$ vanishes.

Because of $[\omega^V,\theta^V]=0$ for $\omega\circ F^S,\theta\circ F^S,\omega,\theta\in\Im_1^0(M^n)$ on $T^*(M^n)$, the Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}(\omega^V,\theta^V)$ of the horizontal lift $(F^{2K+S})^H$ vanishes. □

2.2. Tachibana Operators Applied to Vector and Covector Fields According to Lifts of $F^{2K+S}+F^S=0$ Structure on $T^*(M^n)$

Definition 2.2. Let $\varphi\in\Im_1^1(M^n)$, and $\Im(M^n)=\sum_{r,s=0}^{\infty}\Im_s^r(M^n)$ be a tensor algebra over R. A map $\phi_\varphi\mid_{r+s)0}$: $\overset{*}{\Im}(M^n)\to\Im(M^n)$ is called as Tachibana operatör or $\phi_\varphi$operator on M^n if

$a)$ ϕ_φ is linear with respect to constant coefficient,
$b)$ $\phi_\varphi:\overset{*}{\Im}(M^n)\to\Im_{s+1}^r(M^n)$ for all r and s,
$c)$ $\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M^n)$,
$d)$ $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M^n)$, where L_Y is the Lie derivation with respect to Y (see [6, 8, 13]),
$e)$

$$\begin{aligned}(\phi_{\varphi X}\eta)Y &= (d(\imath_Y\eta))(\varphi X)-(d(\imath_Y(\eta o\varphi)))X+\eta((L_Y\varphi)X)\\ &= \phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)+\eta((L_Y\varphi)X)\end{aligned}\tag{2.3}$$

for all $\eta\in\Im_1^0(M^n)$ and $X,Y\in\Im_0^1(M^n)$, where $\imath_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y$, $\overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C [5, 7, 19] (see [20] for applied to pure tensor field).

Remark 2.1. If $r=s=0$, then from $c),d)$ and $e)$ of Definition2.2 we have $\phi_{\varphi X}(\imath_Y\eta)=\phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)$ for $\imath_Y\eta\in\Im_0^0(M^n)$, which is not well-defined $\phi_\varphi-$operator. Different choices of Y and η leading to same function $f=\imath_Y\eta$ do get the same values. Consider $M^n=R^2$ with standard coordinates x,y. Let $\varphi=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Consider the function $f=1$. This may be written in many different ways as $\imath_Y\eta$. Indeed taking $\eta=dx$, we may choose $Y=\frac{\partial}{\partial_x}$ or $Y=\frac{\partial}{\partial_x}+x\frac{\partial}{\partial_y}$. Now the right-hand side of $\phi_{\varphi X}(\imath_Y\eta)=\phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)$ is $(\phi X)1-0=0$ in the first case, and $(\phi X)1-Xx=-Xx$ in the second case. For $X=\frac{\partial}{\partial_x}$, the latter expression is $-1\neq 0$. Therefore, we put $r+s>0$ [19].

Remark 2.2. From $d)$ of Definition2.2 we have

$$\phi_{\varphi X}Y=[\varphi X,Y]-\varphi[X,Y].\tag{2.4}$$

By virtue of

$$[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X\tag{2.5}$$

for any $f,g\in\Im_0^0(M^n)$, we see that $\phi_{\varphi X}Y$ is linear in X, but not Y [19].

Theorem 2.2. *Let $(F^{2K+S})^H$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator ϕ_φ applied to vector fields according to horizontal lifts of $F^{2K+S}+F^S=0$ structure defined by (1.7) on $T^*(M^n)$, then we get the following* *results.*

$$
\begin{aligned}
i)\ \phi_{(F^{2K+S})^H X^H} Y^H &= ((L_Y F^S)X)^H + (pR(Y, F^S X))^V \\
&\quad -((pR(Y,X) \circ F^S)^V,
\end{aligned}
$$

$$
ii)\ \phi_{(F^{2K+S})^H X^H} \omega^V = ((\nabla_X \omega) \circ F^S)^V - (\nabla_{(F^S X)}\omega)^V,
$$

$$
iii)\ \phi_{(F^{2K+S})^H \omega^V} X^H = (\omega \circ (\nabla_X F^S))^V,
$$

$$
iv)\ \phi_{(F^{2K+S})^H \omega^V} \theta^V = 0,
$$

where horizontal lifts $X^H, Y^H \in \Im_0^1(T^*(M^n))$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_1^0(M^n)$ are given, respectively.

Proof. $i)$

$$
\begin{aligned}
\phi_{(F^{2K+S})^H X^H} Y^H &= -(L_{Y^H}(F^{2K+S})^H)X^H \\
&= -L_{Y^H}(F^{2K+S})^H X^H + (F^{2K+S})^H L_{Y^H} X^H \\
&= L_{Y^H}(F^S)^H X^H - (F^S)^H([Y,X]^H + (pR(Y,X))^V) \\
&= ((L_Y F^S)X)^H + (pR(Y, F^S X))^V - ((pR(Y,X)) \circ F^S)^V
\end{aligned}
$$

$ii)$

$$
\begin{aligned}
\phi_{(F^{2K+S})^H X^H} \omega^V &= -(L_{\omega^V}(F^{2K+S})^H)X^H \\
&= -L_{\omega^V}(F^{2K+S})^H X^H + (F^{2K+S})^H L_{\omega^V} X^H \\
&= L_{\omega^V}(F^S X)^H + (F^S)^H(\nabla_X \omega)^V \\
&= -(\nabla_{(F^S X)}\omega)^V + ((\nabla_X \omega) \circ F^S)^V \\
&= ((\nabla_X \omega) \circ F^S)^V - (\nabla_{(F^S X)}\omega)^V
\end{aligned}
$$

$iii)$

$$
\begin{aligned}
\phi_{(F^{2K+S})^H \omega^V} X^H &= -(L_{X^H}(F^{2K+S})^H)\omega^V \\
&= -L_{X^H}(F^{2K+S})^H \omega^V + (F^{2K+S})^H L_{X^H} \omega^V \\
&= L_{X^H}(\omega \circ F^S)^V - (F^S)^H(\nabla_X \omega)^V \\
&= (\nabla_X(\omega \circ F^S))^V - ((\nabla_X \omega) \circ F^S)^V \\
&= (\omega \circ (\nabla_X F^S))^V
\end{aligned}
$$

$vi)$

$$
\begin{aligned}
\phi_{(F^{2K+S})^H \omega^V} \theta^V &= -(L_{\theta^V}(F^{2K+S})^H)\omega^V \\
&= -L_{\theta^V}(F^{2K+S})^H \omega^V + (F^{2K+S})^H (L_{\theta^V} \omega^V) \\
&= L_{\theta^V}(\omega \circ F^S)^V \\
&= 0
\end{aligned}
$$

□

2.3. *The Purity Conditions of Sasakian Metric with Respect to* $(F^{2K+S})^H$

Definition 2.3. A Sasakian metric ${}^S g$ is defined on $T^*(M^n)$ by the three equations

$$
{}^S g(\omega^V, \theta^V) = (g^{-1}(\omega, \theta))^V = g^{-1}(\omega, \theta) o \pi, \tag{2.6}
$$

$$
{}^S g(\omega^V, Y^H) = 0, \tag{2.7}
$$

$$
{}^S g(X^H, Y^H) = (g(X,Y))^V = g(X,Y) \circ \pi. \tag{2.8}
$$

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x(M^n)$ by

$$g^{-1}(\omega,\theta) = g^{ij}\omega_i\theta_j,$$

where $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$. Since any tensor field of type $(0,2)$ on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [26], p.280), it follows that ${}^Sg$ is completely determined by equations (2.6), (2.7) and (2.8).

Theorem 2.3. *Let $(T^*(M^n), {}^Sg)$ be the cotangent bundle equipped with Sasakian metric ${}^Sg$ and a tensor field $(F^{2K+S})^H$ of type $(1,1)$ defined by (1.7). Sasakian metric ${}^Sg$ is pure with respect to $(F^{2K+S})^H$ if $F^S = I$ (I = identity tensor field of type $(1,1)$).*

Proof. We put

$$S(\tilde{X},\tilde{Y}) = {}^Sg((F^{2K+S})^H\tilde{X},\tilde{Y}) - {}^Sg(\tilde{X},(F^{2K+S})^H\tilde{Y}).$$

If $S(\tilde{X},\tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^V, θ^V or X^H, Y^H, then $S = 0$. By virtue of $F^{2K+S} + F^S = 0$ and (2.6), (2.7), (2.8), we get

$i)$

$$\begin{aligned} S(\omega^V,\theta^V) &= {}^Sg((F^{2K+S})^H\omega^V,\theta^V) - {}^Sg(\omega^V,(F^{2K+S})^H\theta^V) \\ &= {}^Sg(-(F^S)^H\omega^V,\theta^V) - {}^Sg(\omega^V,-(F^S)^H\theta^V) \\ &= -({}^Sg((\omega\circ F^S)^V,\theta^V) - {}^Sg(\omega^V,(\theta\circ F^S)^V)). \end{aligned}$$

$ii)$

$$\begin{aligned} S(X^H,\theta^V) &= {}^Sg((F^{2K+S})^HX^H,\theta^V) - {}^Sg(X^H,(F^{2K+S})^H\theta^V) \\ &= {}^Sg(-(F^S)^HX^H,\theta^V) - {}^Sg(X^H,-(F^S)^H\theta^V) \\ &= -({}^Sg((F^SX)^H,\theta^V) - {}^Sg(X^H,(\omega\circ F^S)^V)) \\ &= 0. \end{aligned}$$

$iii)$

$$\begin{aligned} S(X^H,Y^H) &= {}^Sg((F^{2K+S})^HX^H,Y^H) - {}^Sg(X^H,(F^{2K+S})^HY^H) \\ &= {}^Sg(-(F^S)^HX^H,Y^H) - {}^Sg(X^H,-(F^S)^HY^H) \\ &= -({}^Sg((F^SX)^H,Y^H) - {}^Sg(X^H,(F^SY)^H)). \end{aligned}$$

Thus, $F^S = I$, then ${}^Sg$ is pure with respect to $(F^{2K+S})^H$. □

2.4. Complete Lift of $F(2K+S,S)-$Structure on Tangent Bundle $T(M^n)$

Let M^n be an $n-$dimensional differentiable manifold of class C^∞ and $T_P(M^n)$ the tangent space at a point p of M^n and

$$T(M^n) = \underset{p\in M^n}{U} T_P(M^n) \tag{2.9}$$

is the tangent bundle over the manifold M^n.

Let us denote by $T^r_s(M^n)$, the set of all tensor fields of class C^∞ and of type (r,s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift of F^C of an element of $T^1_1(M^n)$ with local components F^h_i has components of the form [25]

$$F^C = \begin{bmatrix} F^h_i & 0 \\ \delta^h_i & F^h_i \end{bmatrix}. \tag{2.10}$$

Now we obtain the following results on the complete lift of F satisfying $F^{2K+S} + F^S = 0$, ($F \neq 0$, fixed integer $K \geqslant 1$, fixed odd integer $S \geqslant 1$).

Let $F, G \in T^1_1(M^n)$. Then we have [25]

$$(FG)^C = F^CG^C. \tag{2.11}$$

Replacing G by F in (2.11) we obtain

$$(FF)^C = F^CF^C \text{ or } (F^2)^C = (F^C)^2. \tag{2.12}$$

Now putting $G = F^4$ in (2.11) since G is $(1,1)$ tensor field therefore F^4 is also $(1,1)$ so we obtain $(FF^4)^C = F^C(F^4)^C$ which in view of (2.12) becomes

$$(F^5)^C = (F^C)^5.$$

Continuing the above process of replacing G in equation (2.11) by some higher powers of F, we obtain

$$(F^K)^C = (F^C)^K,$$

where fixed integer $K \geqslant 1$. Also if G and H are tensors of the same type then

$$(G + H)^C = G^C + H^C \tag{2.13}$$

Taking complete lift on both sides of equation $F^{2K+S} + F^S = 0$, we get

$$(F^{2K+S} + F^S)^C = 0$$

Using (2.13) and $I^C = I$, we get

$$(F^{2K+S})^C + (F^S)^C = 0 \tag{2.14}$$

$$(F^C)^{2K+S} + (F^C)^S = 0.$$

Let F satisfying $(1,1)$ be an $F-$structure of rank r in M^n. Then the complete lifts $l^C = -(F^{2K})^C$ of l and $m^C = I + (F^{2K})^C$ of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions L^C and M^C determined by l^C and m^C, respectively.

Proposition 2.1. *The* $(1,1)$ *tensor field* $\tilde{\psi}$ *given by* $\tilde{\psi} = l^C - m^C = -2(F^{2K})^C - I$ *gives an almost product structure on* $T(M^n)$.

Proof. For $l^C = -(F^{2K})^C$,$m^C = I + (F^{2K})^C$ and $\tilde{\psi} = l^C - m^C = -2(F^{2K})^C - I$, we have

$$\begin{aligned}
\tilde{\psi}^2 &= 4(F^{4K})^C + 4(F^{2K})^C + I \\
&= 4(F^{2K})^C(F^{2K})^C + 4(F^{2K})^C + I \\
&= 4(-I^C)(F^{2K})^C + 4(F^{2K})^C + I \\
&= I,
\end{aligned}$$

where $\tilde{\psi} \in \Im_1^1(T(M^n))$, I = identity tensor field of type $(1,1)$. □

2.5. Horizontal Lift of $F(2K+S,S)-$Structure on Tangent Bundle $T(M^n)$

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are given by

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F_i^h & 0 \\ -\Gamma_t^h F_i^t + \Gamma_i^t F_t^h & F_i^h \end{pmatrix}. \tag{2.15}$$

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H = F^H G^H. \tag{2.16}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H. \tag{2.17}$$

Multiplying both sides by F^H and making use of the same (2.17) , we get

$$(F^H)^3 = (F^3)^H$$

Thus it follows that

$$(F^H)^4 = (F^4)^H, (F^H)^5 = (F^5)^H \tag{2.18}$$

and so on. Taking horizontal lift on both sides of equation $F^{2K+S} + F^S = 0$ we get

$$(F^{2K+S})^H + (F^S)^H = 0 \tag{2.19}$$

view of (2.18), we can write

$$(F^H)^{2K+S} + (F^H)^S = 0.$$

2.6. The Structure $(F^{2K+S})^C+(F^S)^C=0$ on Tangent Bundle $T(M^n)$

Definition 2.4. Let X and Y be any vector fields on a Riemannian manifold (M^n,g), we have [26]

$$\begin{aligned}\left[X^H,Y^H\right] &= [X,Y]^H-(R(X,Y)u)^V,\\ \left[X^H,Y^V\right] &= (\nabla_X Y)^V,\\ \left[X^V,Y^V\right] &= 0,\end{aligned}$$

where R is the Riemannian curvature tensor of g defined by

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}.$$

In particular, we have the vertical spray u^V and the horizontal spray u^H on $T(M^n)$ defined by

$$u^V=u^i(\partial_i)^V=u^i\partial_{\bar{i}},\ u^H=u^i(\partial_i)^H=u^i\delta_i,$$

where $\delta_i=\partial_i-u^j\Gamma^s_{ji}\partial_{\bar{s}}$. u^V is also called the canonical or Liouville vector field on $T(M^n)$.

Theorem 2.4. *The Nijenhuis tensors*

$$\tilde{N}_{(F^{2K+S})^C(F^{2K+S})^C}\left(X^C,Y^C\right),\tilde{N}_{(F^{2K+S})^C(F^{2K+S})^C}\left(X^C,Y^V\right),\tilde{N}_{(F^{2K+S})^C(F^{2K+S})^C}\left(X^V,Y^V\right)$$

of the complete lift $(F^{2K+S})^C$ vanishes if the Nijenhuis tensor of the F^S is zero.

Proof. In consequence of Definition 2.1 and the formulations in Definition 2.4, the Nijenhuis tensors of $\left(F^{2K+S}\right)^C$ are given by

$i)$

$$\begin{aligned}\tilde{N}_{(F^{2K+S})^C(F^{2K+S})^C}\left(X^C,Y^C\right) &= [\left(F^{2K+S}\right)^C X^C,\left(F^{2K+S}\right)^C Y^C]\\ &\quad-\left(F^{2K+S}\right)^C[\left(F^{2K+S}\right)^C X^C,Y^C]\\ &\quad-\left(F^{2K+S}\right)^C[X^C,\left(F^{2K+S}\right)^C Y^C]\\ &\quad+\left(F^{2K+S}\right)^C\left(F^{2K+S}\right)^C\left[X^C,Y^C\right]\\ &= [\left(F^SX\right)^C,\left(F^SY\right)^C]+(F^S)^C[\left(F^SX\right)^C,Y^C]\\ &\quad-(F^S)^C[X^C,\left(F^SY\right)^C]+(F^S)^C(F^S)^C\left[X^C,Y^C\right]\\ &= N_{F^S}(X,Y)^C\end{aligned}$$

$ii)$

$$\begin{aligned}\tilde{N}_{(F^{2K+S})^C(F^{2K+S})^C}\left(X^C,Y^V\right) &= [\left(F^{2K+S}\right)^C X^C,\left(F^{2K+S}\right)^C Y^V]\\ &\quad-\left(F^{2K+S}\right)^C[\left(F^{2K+S}\right)^C X^C,Y^V]\\ &\quad-\left(F^{2K+S}\right)^C[X^C,\left(F^{2K+S}\right)^C Y^V]\\ &\quad+\left(F^{2K+S}\right)^C\left(F^{2K+S}\right)^C\left[X^C,Y^V\right]\\ &= [\left(F^SX\right)^C,\left(F^SY\right)^V]-(F^S)^C[\left(F^SX\right)^C,Y^V]\\ &\quad-(F^S)^C[X^C,\left(F^SY\right)^V]+\left((F^S)^2\right)^C[X,Y]^V\\ &= N_{F^S}(X,Y)^V\end{aligned}$$

$iii)$ Because of $\left[X^V,Y^V\right]=0$ and $X,Y\in M$, easily we get

$$\tilde{N}_{(F^{2K+S})^C(F^{2K+S})^C}\left(X^V,Y^V\right)=0.$$

□

2.7. The Purity Conditions of Sasakian Metric with Respect to $(F^{2K+S})^{C}$ on $T(M^{n})$

Definition 2.5. The Sasaki metric ${}^{S}g$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^{n})$ which is derived from the given Riemannian metric on M^{n} as follows [19]:

$$\begin{aligned} {}^{S}g\left(X^{H},Y^{H}\right) &= g\left(X,Y\right), \\ {}^{S}g\left(X^{H},Y^{V}\right) &= {}^{S}g\left(X^{V},Y^{H}\right)=0, \\ {}^{S}g\left(X^{V},Y^{V}\right) &= g\left(X,Y\right) \end{aligned} \tag{2.20}$$

for all $X,Y\in\Im_{0}^{1}\left(M^{n}\right)$.

Theorem 2.5. *The Sasaki metric ${}^{S}g$ is pure with respect to $\left(F^{2K+S}\right)^{C}$ if $\nabla F^{S}=0$ and $F^{S}=I$, where I=ıdentity tensor field of type $(1,1)$.*

Proof. $S(\widetilde{X},\widetilde{Y})={}^{S}g((F^{2K+S})^{C}\widetilde{X},\widetilde{Y})-{}^{S}g(\widetilde{X},(F^{2K+S})^{C}\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^{V},Y^{V} or X^{H},Y^{H} then $S=0$.

$i)$

$$\begin{aligned} S\left(X^{V},Y^{V}\right) &= {}^{S}g(\left(F^{2K+S}\right)^{C}X^{V},Y^{V})-{}^{S}g(X^{V},\left(F^{2K+S}\right)^{C}Y^{V}) \\ &= -{}^{S}g(\left(F^{S}X\right)^{V},Y^{V})+{}^{S}g(X^{V},\left(F^{S}Y\right)^{V})\} \\ &= -\left(g\left(F^{S}X,Y\right)\right)^{V}+\left(g\left(X,F^{S}Y\right)\right)^{V} \end{aligned}$$

$ii)$

$$\begin{aligned} S\left(X^{V},Y^{H}\right) &= {}^{S}g(\left(F^{2K+S}\right)^{C}X^{V},Y^{H})-{}^{S}g(X^{V},\left(F^{2K+S}\right)^{C}Y^{H}) \\ &= {}^{S}g(X^{V},\left(F^{S}Y\right)^{H}+\left(\nabla_{\gamma}F^{S}\right)Y^{H}) \\ &= {}^{S}g\left(X^{V},\left(\nabla_{\gamma}F^{S}\right)Y^{H}\right) \\ &= {}^{S}g(X^{V},\left(\left(\left(\nabla F^{S}\right)u\right)Y\right)^{V}) \\ &= \left(g\left(X,\left(\left(\nabla F^{S}\right)u\right)Y\right)\right)^{V} \end{aligned}$$

$iii)$

$$\begin{aligned} S\left(X^{H},Y^{H}\right) &= {}^{S}g(\left(F^{2K+S}\right)^{C}X^{H},Y^{H})-{}^{S}g(X^{H},\left(F^{2K+S}\right)^{C}Y^{H}) \\ &= -{}^{S}g((F^{S})^{C}X^{H},Y^{H})+{}^{S}g(X^{H},(F^{S})^{C}Y^{H}) \\ &= -{}^{S}g(\left(F^{S}X\right)^{H}+\left(\nabla_{\gamma}F^{S}\right)X^{H},Y^{H}) \\ &\quad +{}^{S}g(X^{H},\left(F^{S}Y\right)^{H}+\left(\nabla_{\gamma}F^{S}\right)Y^{H}) \\ &= -g\left(\left(F^{S}X\right),Y\right)^{V}+g\left(X,\left(F^{S}Y\right)\right)^{V} \end{aligned}$$

□

Theorem 2.6. *Let ϕ_{φ} be the Tachibana operator and the structure $\left(F^{2K+S}\right)^{C}+(F^{S})^{C}=0$ defined by Definition 2.2 and (2.14), respectively. If $L_{Y}F^{S}=0$, then all results with respect to $\left(F^{2K+S}\right)^{C}$ is zero, where $X,Y\in\Im_{0}^{1}\left(M^{n}\right)$, the complete lifts $X^{C},Y^{C}\in\Im_{0}^{1}\left(T\left(M^{n}\right)\right)$ and the vertical lift $X^{V},Y^{V}\in\Im_{0}^{1}\left(T\left(M^{n}\right)\right)$.*

$$\begin{aligned} i)\ \phi_{(F^{2K+S})^{C}X^{C}}Y^{C} &= \left(\left(L_{Y}F^{S}\right)X\right)^{C} \\ ii)\ \phi_{(F^{2K+S})^{C}X^{C}}Y^{V} &= \left(\left(L_{Y}F^{S}\right)X\right)^{V} \\ iii)\ \phi_{(F^{2K+S})^{C}X^{V}}Y^{C} &= \left(\left(L_{Y}F^{S}\right)X\right)^{V} \\ iv)\ \phi_{(F^{2K+S})^{C}X^{V}}Y^{V} &= 0 \end{aligned}$$

Proof. $i)$

$$\begin{aligned}\phi_{(F^{2K+S})^C X^C} Y^C &= -(L_{Y^C}\left(F^{2K+S}\right)^C)X^C \\ &= L_{Y^C}\left(F^S X\right)^C - (F^S)^C L_{Y^C} X^C \\ &= \left(\left(L_Y F^S\right) X\right)^C\end{aligned}$$

$ii)$

$$\begin{aligned}\phi_{(F^{2K+S})^C X^C} Y^V &= -(L_{Y^V}\left(F^{2K+S}\right)^C)X^C \\ &= -L_{Y^V}\left(F^{2K+S}\right)^C X^C + \left(F^{2K+S}\right)^C L_{Y^V} X^C \\ &= L_{Y^V}\left(F^S X\right)^C - (F^S)^C L_{Y^V} X^C \\ &= \left(\left(L_Y F^S\right) X\right)^V\end{aligned}$$

$iii)$

$$\begin{aligned}\phi_{(F^{2K+S})^C X^V} Y^C &= -(L_{Y^C}\left(F^{2K+S}\right)^C)X^V \\ &= -L_{Y^C}\left(F^{2K+S}\right)^C X^V + \left(F^{2K+S}\right)^C L_{Y^C} X^V \\ &= L_{Y^C}\left(F^S X\right)^V - (F^S)^C L_{Y^C} X^V \\ &= \left(\left(L_Y F^S\right) X\right)^V\end{aligned}$$

$iv)$

$$\begin{aligned}\phi_{(F^{2K+S})^C X^V} Y^V &= -(L_{Y^V}\left(F^{2K+S}\right)^C)X^V \\ &= -L_{Y^V}\left(F^{2K+S}\right)^C X^V + \left(F^{2K+S}\right)^C L_{Y^V} X^V \\ &= 0\end{aligned}$$

□

Theorem 2.7. *If $L_Y F^S = 0$ for $Y \in M^n$, then its complete lift Y^C to the tangent bundle is an almost holomorfic vector field with respect to the structure $\left(F^{2K+S}\right)^C + (F^S)^C = 0$.*

Proof. $i)$

$$\begin{aligned}(L_{Y^C}\left(F^{2K+S}\right)^C)X^C &= L_{Y^C}\left(F^{2K+S}\right)^C X^C - \left(F^{2K+S}\right)^C L_{Y^C} X^C \\ &= -L_{Y^C}\left(F^S X\right)^C + (F^S)^C L_{Y^C} X^C \\ &= -\left(\left(L_Y F^S\right) X\right)^C\end{aligned}$$

$ii)$

$$\begin{aligned}(L_{Y^C}\left(F^{2K+S}\right)^C)X^V &= L_{Y^C}\left(F^{2K+S}\right)^C X^V - \left(F^{2K+S}\right)^C L_{Y^C} X^V \\ &= -L_{Y^C}\left(F^S X\right)^V + (F^S)^C L_{Y^C} X^V \\ &= -\left(\left(L_Y F^S\right) X\right)^V\end{aligned}$$

□

2.8. The Structure $\left(F^{2K+S}\right)^H + (F^S)^H = 0$ on Tangent Bundle $T\left(M^n\right)$

Theorem 2.8. *The Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H (F^{2K+S})^H}\left(X^H, Y^H\right)$ of the horizontal lift $\left(F^{2K+S}\right)^H$ vanishes if the Nijenhuis tensor of the F^S is zero and $\{-(\hat{R}\left(F^S X, F^S Y\right) u) + (F^S(\hat{R}\left(F^S X, Y\right) u)) + (F^S(\hat{R}\left(X, F^S Y\right) u) - ((F^S)^2(\hat{R}\left(X, Y\right) u))\}^V = 0$.*

Proof.

$$\begin{aligned}\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,Y^H\right) &= [\left(F^{2K+S}\right)^H X^H,\left(F^{2K+S}\right)^H Y^H] \\ &\quad -\left(F^{2K+S}\right)^H [\left(F^{2K+S}\right)^H X^H,Y^H] \\ &\quad -\left(F^{2K+S}\right)^H [X^H,\left(F^{2K+S}\right)^H Y^H] \\ &\quad +\left(F^{2K+S}\right)^H \left(F^{2K+S}\right)^H \left[X^H,Y^H\right] \\ &= [\left(F^S X\right)^H,\left(F^S Y\right)^H] - \left(F^S\right)^H [\left(F^S X\right)^H,Y^H] \\ &\quad -\left(F^S\right)^H [X^H,\left(F^S Y\right)^H] + \left(F^S\right)^H \left(F^S\right)^H \left[X^H,Y^H\right] \\ &= \left(N_{F^S}(X,Y)\right)^H - (\hat{R}\left(F^S X,F^S Y\right)u)^V \\ &\quad +(F^S(\hat{R}\left(F^S X,Y\right)u))^V + (F^S(\hat{R}\left(X,F^S Y\right)u))^V \\ &\quad -((F^S)^2(\hat{R}(X,Y)u))^V.\end{aligned}$$

□

If $N_{F^S}(X,Y)=0$ and $\{-(\hat{R}\left(F^S X,F^S Y\right)u)+(F^S(\hat{R}\left(F^S X,Y\right)u))+(F^S(\hat{R}\left(X,F^S Y\right)u)-((F^S)^2(\hat{R}(X,Y)u))\}^V=0$, then we get $N_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,Y^H\right)=0$, where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_X Y=\nabla_Y X+[X,Y]$ (see [26] p.88-89).

Theorem 2.9. *The Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,Y^V\right)$ of the horizontal lift $(F^{2K+S})^H$ vanishes if the Nijenhuis tensor of the F^S is zero and $\nabla F^S=0$.*

Proof.

$$\begin{aligned}\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^H,Y^V\right) &= [\left(F^{2K+S}\right)^H X^H,\left(F^{2K+S}\right)^H Y^V] \\ &\quad -\left(F^{2K+S}\right)^H [\left(F^{2K+S}\right)^H X^H,Y^V] \\ &\quad -\left(F^{2K+S}\right)^H [X^H,\left(F^{2K+S}\right)^H Y^V] \\ &\quad +\left(F^{2K+S}\right)^H \left(F^{2K+S}\right)^H \left[X^H,Y^V\right] \\ &= \left[F^S X+F^S Y\right]^V - \left(F^S\left[F^S X,Y\right]\right)^V \\ &\quad -\left(F^S\left[X,F^S Y\right]\right)^V + (\left(F^S\right)^2[X,Y])^V \\ &\quad +\left(\nabla_{F^S Y}F^S X\right)^V - \left(F^S\left(\nabla_Y F^S X\right)\right)^V \\ &\quad -\left(F^S\left(\nabla_{F^S Y}X\right)\right)^V + (\left(F^S\right)^2\nabla_Y X)^V \\ &= \left(N_{F^S}(X,Y)\right)^V + (\left(\nabla_{F^S Y}F^S\right)X)^V \\ &\quad -\left(F^S\left(\left(\nabla_Y F^S\right)X\right)\right)^V.\end{aligned}$$

□

Theorem 2.10. *The Nijenhuis tensor $\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^V,Y^V\right)$ of the horizontal lift $(F^{2K+S})^H$ vanishes.*

Proof. Because of $\left[X^V,Y^V\right]=0$ for $X,Y\in M^n$, easily we get

$$\tilde{N}_{(F^{2K+S})^H(F^{2K+S})^H}\left(X^V,Y^V\right)=0.$$

□

Theorem 2.11. *The Sasakian metric ${}^S g$ is pure with respect to $\left(F^{2K+S}\right)^H$ if $F^S=I$, where I =ıdentity tensor field of type $(1,1)$.*

Proof. $S(\widetilde{X},\widetilde{Y})={}^S g(\left(F^{2K+S}\right)^H\widetilde{X},\widetilde{Y})-{}^S g(\widetilde{X},\left(F^{2K+S}\right)^H\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.

$i)$

$$\begin{aligned} S\left(X^{V},Y^{V}\right) &= {}^{S}g((F^{2K+S})^{H}X^{V},Y^{V})-{}^{S}g(X^{V},(F^{2K+S})^{H}Y^{V}) \\ &= -{}^{S}g((F^{S}X)^{V},Y^{V})+{}^{S}g(X^{V},(F^{S}Y)^{V}) \\ &= -\left(g\left(F^{S}X,Y\right)\right)^{V}+\left(g\left(X,F^{S}Y\right)\right)^{V}\} \end{aligned}$$

$ii)$

$$\begin{aligned} S\left(X^{V},Y^{H}\right) &= {}^{S}g((F^{2K+S})^{H}X^{V},Y^{H})-{}^{S}g(X^{V},(F^{2K+S})^{H}Y^{H}) \\ &= {}^{S}g(X^{V},(F^{S}Y)^{H}) \\ &= 0 \end{aligned}$$

$iii)$

$$\begin{aligned} S\left(X^{H},Y^{H}\right) &= {}^{S}g((F^{2K+S})^{H}X^{H},Y^{H})-{}^{S}g(X^{H},(F^{2K+S})^{H}Y^{H}) \\ &= -({}^{S}g\left(F^{S}X\right)^{H},Y^{H})+{}^{S}g(X^{H},(F^{S}Y)^{H}) \\ &= -\left(g\left(F^{S}X\right),Y\right)^{V}+(g(X,(F^{S}Y)^{H}))^{V} \end{aligned}$$

□

Theorem 2.12. *Let ϕ_{φ} be the Tachibana operator and the structure $\left(F^{2K+S}\right)^{H}+(F^{S})^{H}=0$ defined by Definition 2.2 and (2.19), respectively. if $L_{Y}F^{S}=0$ and $F^{S}=I$, then all results with respect to $\left(F^{2K+S}\right)^{H}$ is zero, where $X,Y\in\Im_{0}^{1}(M^{n})$, the horizontal lifts $X^{H},Y^{H}\in\Im_{0}^{1}(T(M^{n}))$ and the vertical lift $X^{V},Y^{V}\in\Im_{0}^{1}(T(M^{n}))$.*

$$\begin{aligned} i)\ \phi_{(F^{2K+S})^{H}X^{H}}Y^{H} &= -\left(\left(L_{Y}F^{S}\right)X\right)^{H}+(\hat{R}\left(Y,F^{S}X\right)u)^{V} \\ &\quad -(F^{S}(\hat{R}\left(Y,X\right)u))^{V}, \\ ii)\ \phi_{(F^{2K+S})^{H}X^{H}}Y^{V} &= \left(\left(L_{Y}F^{S}\right)X\right)^{V}-\left(\left(\nabla_{Y}F^{S}\right)X\right)^{V}, \\ iii)\ \phi_{(F^{2K+S})^{H}X^{V}}Y^{H} &= \left(\left(L_{Y}F^{S}\right)X\right)^{V}+\left(\nabla_{F^{S}X}Y\right)^{V}-\left(F^{S}\left(\nabla_{X}Y\right)\right)^{V}, \\ iv)\ \phi_{(F^{2K+S})^{H}X^{V}}Y^{V} &= 0. \end{aligned}$$

Proof. $i)$

$$\begin{aligned} \phi_{(F^{2K+S})^{H}X^{H}}Y^{H} &= -(L_{Y^{H}}\left(F^{2K+S}\right)^{H})X^{H} \\ &= -L_{Y^{C}}\left(F^{2K+S}\right)^{H}X^{H}+\left(F^{2K+S}\right)^{H}L_{Y^{H}}X^{H} \\ &= \left[Y,F^{S}X\right]^{H}-\gamma\hat{R}\left[Y,F^{S}X\right] \\ &\quad -\left(F^{S}\left[Y,X\right]\right)^{H}+(F^{S})^{H}(\hat{R}\left(Y,X\right)u)^{V} \\ &= -\left(\left(L_{Y}F^{S}\right)X\right)^{H}+(\hat{R}\left(Y,F^{S}X\right)u)^{V} \\ &\quad -(F^{S}(\hat{R}\left(Y,X\right)u))^{V} \end{aligned}$$

$ii)$

$$\begin{aligned} \phi_{(F^{2K+S})^{H}X^{H}}Y^{V} &= -(L_{Y^{V}}\left(F^{2K+S}\right)^{H})X^{H} \\ &= -L_{Y^{V}}\left(F^{2K+S}X\right)^{H}+\left(F^{2K+S}\right)^{H}L_{Y^{V}}X^{H} \\ &= \left[Y,F^{S}X\right]^{V}-\left(\nabla_{Y}F^{S}X\right)^{V} \\ &\quad -\left(F^{S}\left[Y,X\right]\right)^{V}+\left(F^{S}\left(\nabla_{Y}X\right)\right)^{V} \\ &= \left(\left(L_{Y}F^{S}\right)X\right)^{V}-\left(\left(\nabla_{Y}F^{S}\right)X\right)^{V} \end{aligned}$$

$iii)$

$$
\begin{aligned}
\phi_{(F^{2K+S})^H X^V} Y^H &= -(L_{Y^H}\left(F^{2K+S}\right)^H)X^V \\
&= -L_{Y^H}\left(F^{2K+S}X\right)^V + \left(F^{2K+S}\right)^H L_{Y^H}X^V \\
&= -\left[Y, F^S X\right]^V + \left(\nabla_{F^S X}Y\right)^V \\
&\quad -\left(F^S\left[Y,X\right]\right)^H - \left(F^S\left(\nabla_X Y\right)\right)^V \\
&= \left(\left(L_Y F^S\right)X\right)^V + \left(\nabla_{F^S X}Y\right)^V - \left(F^S\left(\nabla_X Y\right)\right)^V
\end{aligned}
$$

$iv)$

$$
\begin{aligned}
\phi_{(F^{2K+S})^H X^V} Y^V &= -(L_{Y^V}\left(F^{2K+S}\right)^H)X^V \\
&= L_{Y^V}\left(F^S X\right)^V - (F^S)^H L_{Y^V}X^V \\
&= 0
\end{aligned}
$$

□

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