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Improved exponential type estimators of finite population mean under complete and partial auxiliary information

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Abstract

This paper proposes some improved exponential type estimators of finite population mean under simple random sampling and double sampling. Expressions for biases and mean squared errors of the proposed estimators are derived up to the first order of approximation. Theoretical and numerical comparisons are made to investigate the performances of the estimators. The proposed estimators always perform better than the difference estimator of the population mean. They also perform better than the estimators suggested by Gupta and Shabbir [3] and Grover and Kaur [2].

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1. Introduction

The auxiliary information is frequently used to increase precision of the population estimates by taking advantage of the correlation between the study variable and the auxiliary variable. Several authors including Kadilar and Cingi [4], Kadilar and Cingi [5], Kadilar and Cingi [6], Kadilar and Cingi [7] and Gupta and Shabbir [3] have proposed different estimators by utilizing information on the auxiliary variable for estimation of the population mean.

In this paper, we propose some improved exponential type estimators for estimating finite population mean using complete and partial auxiliary information. Explicit expressions for biases and mean squared errors (*MSE*s) of the proposed estimators are derived up to the first order of approximation. An empirical study is conducted to assess the

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performance of the proposed estimators. It is observed that the proposed estimators are more precise than the existing estimators of the finite population mean.

Consider a finite population comprises of N units. We draw a sample of size n from this population by using simple random sampling without replacement (SRSWOR). Let y and x be the study and the auxiliary variables of the characteristics y_i and x_i, respectively, for the ith unit. Let $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the sample means corresponding to the population means $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$, respectively. Let $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n} (y_i - \bar{y})^2$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$ be the sample variances corresponding to the population variances $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N} \left(y_i - \bar{Y}\right)^2$ and $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N} \left(x_i - \bar{X}\right)^2$, respectively. Let ρ be the correlation coefficient between y and x. Let $C_y = \frac{S_y}{\bar{Y}}$ and $C_x = \frac{S_x}{\bar{X}}$ be the coefficients of variation of y and x, respectively.

The rest of the paper is organized as follows: Section 2 includes the estimators adopted by several authors when using complete auxiliary information. In Section 3, the proposed estimators based on complete information are discussed in detail. Theoretical comparisons of the proposed estimators with the existing estimators are given in Section 4. Section 5 contains some suggested estimators when partial auxiliary information is available. The work on the proposed estimators is extended to two-phase sampling in Section 6. Section 7 contains theoretical comparisons of the suggested estimators and existing estimators. For numerical comparisons of estimators, we consider three real data sets in Section 8, and concluding remarks are given in Section 9.

2. Estimators based on complete auxiliary information

In the following subsequent sections, we discuss the properties of the difference, difference-ratio-type and exponential-type estimators of finite population mean suggested by several authors.

2.1. Usual difference estimator of population mean. The unbiased difference estimator of population mean is

(2.1) $$\hat{\bar{Y}}_D = \bar{y} + k\left(\bar{X} - \bar{x}\right),$$

where k is an unknown constant.
The minimum variance of $\hat{\bar{Y}}_D$, at optimum value of k, i.e., $k_{(opt)} = \frac{\bar{Y}\rho C_y}{\bar{X}C_x}$, is given by

(2.2) $$Var_{\min}\left(\hat{\bar{Y}}_D\right) \cong \bar{Y}^2\lambda\left(1-\rho^2\right)C_y^2,$$

where $\lambda = \frac{1-f}{n}$ and $f = \frac{n}{N}$.

2.2. Gupta and Shabbir [3] family of estimators. Gupta and Shabbir [3] introduced the following family of estimators for estimating finite population mean:

(2.3) $$\hat{\bar{Y}}_{GS} = \left\{s_1\bar{y} + s_2\left(\bar{X} - \bar{x}\right)\right\}\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right),$$

where s_1 and s_2 are two unknown constants. Here a and b are the known population parameters which may be coefficient of skewness (β_{1x}), coefficient of kurtosis (β_{2x}), coefficient of variation (CV) and correlation coefficient (ρ).

Expressions for *Bias* and *MSE* of $\hat{\hat{Y}}_{GS}$, to first order of approximation, are given by

$$\text{(2.4)}\qquad Bias\left(\hat{\hat{Y}}_{GS}\right) \cong -\bar{Y} + \bar{Y}\left\{1 + \lambda\tau C_x\left(\tau C_x - \rho C_y\right)\right\} s_1 + \bar{X}\lambda\tau C_x^2 s_2$$

and

$$\begin{aligned} MSE\left(\hat{\hat{Y}}_{GS}\right) &\cong \bar{Y}^2 + \bar{Y}^2\left\{1 + \lambda\left(3\tau^2 C_x^2 - 4\rho\tau C_x C_y + C_y^2\right)\right\} s_1^2 + \bar{X}\lambda C_x^2 s_2\left(-2\bar{Y}\tau + \bar{X}s_2\right) \\ \text{(2.5)}\qquad & -2\bar{Y}s_1\left[\bar{Y} + \lambda C_x\left\{\tau C_x\left(\bar{Y}\tau - 2\bar{X}s_2\right) + \rho C_y\left(-\bar{Y}\tau + \bar{X}s_2\right)\right\}\right], \end{aligned}$$

where $\tau = \frac{a\bar{X}}{a\bar{X}+b}$.

The optimum values of s_1 and s_2, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{GS}$, are given by $s_{1(opt)} = \frac{-1+\lambda\tau^2C_x^2}{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2}$ and $s_{2(opt)} = \frac{\bar{Y}\left[-\rho C_y+\tau C_x\left\{1-\lambda\tau^2C_x^2+\lambda\rho\tau C_xC_y+\lambda\left(-1+\rho^2\right)C_y^2\right\}\right]}{\bar{X}C_x\left\{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2\right\}}$.

The minimum *MSE* of $\hat{\hat{Y}}_{GS}$, at optimum values of s_1 and s_2, is given by

$$\text{(2.6)}\qquad MSE_{\min}\left(\hat{\hat{Y}}_{GS}\right) \cong \frac{\bar{Y}^2\lambda\left(1-\rho^2\right)\left(-1+\lambda\tau^2C_x^2\right)C_y^2}{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2}.$$

Gupta and Shabbir [3] estimator $\hat{\hat{Y}}_{GS}$ will perform better than the difference estimator $\hat{\hat{Y}}_D$, if

$$\frac{\bar{Y}^2\lambda^2\left(-1+\rho^2\right)^2C_y^4}{1-\lambda\tau^2C_x^2+\lambda\left(1-\rho^2\right)C_y^2} > 0.$$

2.3. Grover and Kaur [2] estimator. Grover and Kaur [2] proposed the following estimator of finite population mean:

$$\text{(2.7)}\qquad \hat{\hat{Y}}_{GK} = \left\{t_1\bar{y} + t_2\left(\bar{X} - \bar{x}\right)\right\}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right),$$

where t_1 and t_2 are two unknown constants, whose values are to be determined later on. Expressions for *Bias* and *MSE* of $\hat{\hat{Y}}_{GK}$, to first order of approximation, are given by

$$\text{(2.8)}\qquad Bias\left(\hat{\hat{Y}}_{GK}\right) \cong \frac{1}{8}\left[-8\bar{Y} + \bar{Y}\left\{8 + \lambda C_x\left(3C_x - 4\rho C_y\right)\right\}t_1 + 4\bar{X}\lambda C_x^2 t_2\right]$$

and

$$\begin{aligned} MSE\left(\hat{\hat{Y}}_{GK}\right) &\cong \bar{Y}^2 + \bar{Y}^2\left\{1 + \lambda\left(C_x^2 - 2\rho C_xC_y + C_y^2\right)\right\}t_1^2 + \bar{X}\lambda C_x^2 t_2\left(-\bar{Y} + \bar{X}t_2\right) \\ \text{(2.9)}\qquad & +\frac{1}{4}\bar{Y}t_1\left[-8\bar{Y} + \lambda C_x\left\{4\rho C_y\left(\bar{Y} - 2\bar{X}t_2\right) + C_x\left(-3\bar{Y} + 8\bar{X}t_2\right)\right\}\right]. \end{aligned}$$

The optimum values of t_1 and t_2, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{GK}$, are given by $t_{1(opt)} = \frac{-8+\lambda C_x^2}{-8+8\lambda\left(-1+\rho^2\right)C_y^2}$ and $t_{2(opt)} = \frac{\bar{Y}\left[-8\rho C_y+C_x\left\{4-\lambda C_x^2+\lambda\rho C_xC_y+4\lambda\left(-1+\rho^2\right)C_y^2\right\}\right]}{8\bar{X}C_x\left\{-1+\lambda\left(-1+\rho^2\right)C_y^2\right\}}$.

The minimum *MSE* of $\hat{\hat{Y}}_{GK}$, at optimum values of t_1 and t_2, is given by

$$\text{(2.10)}\qquad MSE_{\min}\left(\hat{\hat{Y}}_{GK}\right) \cong \frac{\bar{Y}^2\lambda\left\{\lambda C_x^4 - 16\left(-1+\rho^2\right)\left(-4+\lambda C_x^2\right)C_y^2\right\}}{64\left\{-1+\lambda\left(-1+\rho^2\right)C_y^2\right\}}.$$

Grover and Kaur [2] estimator $\hat{\hat{Y}}_{GK}$ will perform better than the difference estimator $\hat{\hat{Y}}_D$, if

$$\frac{\bar{Y}^2\lambda^2\left\{C_x^2 - 8\left(-1+\rho^2\right)C_y^2\right\}^2}{64\left\{1+\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

Gupta and Shabbir [3] estimator $\hat{\bar{Y}}_{GS}$ will perform better than the Grover and Kaur [2] estimator $\hat{\bar{Y}}_{GK}$, if

$$\bar{Y}^2\lambda\left(\frac{\left(-1+\rho^2\right)\left(-1+\lambda\tau^2C_x^2\right)C_y^2}{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2}+\frac{\lambda C_x^4-16\left(-1+\rho^2\right)\left(-4+\lambda C_x^2\right)C_y^2}{64\left\{-1+\lambda\left(-1+\rho^2\right)C_y^2\right\}}\right)>0.$$

3. Proposed estimators

In this section, we propose some improved exponential type estimators for estimating finite population mean when complete auxiliary information is available.

3.1. First proposed estimator. On the lines of Singh and Espejo [8], the average ratio-product estimator is given by

$$\hat{\bar{Y}}_{SE}=\frac{1}{2}\bar{y}\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right). \quad (3.1)$$

By replacing $\hat{\bar{Y}}_{SE}$ in place of $\bar{y}$ in (2.7), the proposed estimator becomes

$$\hat{\bar{Y}}_{P1}=\left\{u_1\hat{\bar{Y}}_{SE}+u_2\left(\bar{X}-\bar{x}\right)\right\}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \quad (3.2)$$

where u_1 and u_2 are two unknown constants, whose values are determined for optimality. Expressions for *Bias* and *MSE* of $\hat{\bar{Y}}_{P1}$, to first order of approximation, are given by

$$Bias\left(\hat{\bar{Y}}_{P1}\right)\cong\frac{1}{8}\left[-8\bar{Y}+\bar{Y}\left\{8+\lambda C_x\left(7C_x-4\rho C_y\right)\right\}u_1+4\bar{X}\lambda C_x^2u_2\right] \quad (3.3)$$

and

$$MSE\left(\hat{\bar{Y}}_{P1}\right)\cong\bar{Y}^2+\bar{Y}^2\left\{1+\lambda\left(2C_x^2-2\rho C_xC_y+C_y^2\right)\right\}u_1^2+\bar{X}\lambda C_x^2u_2\left(-\bar{Y}+\bar{X}u_2\right)$$

$$+\frac{1}{4}\bar{Y}u_1\left[-8\bar{Y}+\lambda C_x\left\{4\rho C_y\left(\bar{Y}-2\bar{X}u_2\right)+C_x\left(-7\bar{Y}+8\bar{X}u_2\right)\right\}\right]. \quad (3.4)$$

The optimum values of u_1 and u_2, obtained by minimizing the *MSE* of $\hat{\bar{Y}}_{P1}$, are given by $u_{1(opt)}=\frac{8+3\lambda C_x^2}{8\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}}$ and $u_{2(opt)}=\frac{\bar{Y}\left[8\rho C_y+C_x\left\{-4+\lambda\left(C_x^2+3\rho C_xC_y-4\left(-1+\rho^2\right)C_y^2\right)\right\}\right]}{8\bar{X}C_x\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}}$.
The minimum *MSE* of $\hat{\bar{Y}}_{P1}$, at optimum values of u_1 and u_2, is given by

$$MSE_{\min}\left(\hat{\bar{Y}}_{P1}\right)\cong\frac{\bar{Y}^2\lambda\left\{-25\lambda C_x^4+16\left(-1+\rho^2\right)\left(-4+\lambda C_x^2\right)C_y^2\right\}}{64\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}}. \quad (3.5)$$

3.2. Second proposed estimator. On the line of Bahl and Tuteja [1], we can define the average exponential ratio-product type estimator, given by

$$\hat{\bar{Y}}_{BTW}=\frac{1}{2}\bar{y}\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right)\right\}, \quad (3.6)$$

By replacing $\hat{\bar{Y}}_{BTW}$ in place of $\bar{y}$ in (2.7), the proposed estimator becomes

$$\hat{\bar{Y}}_{P2}=\left\{v_1\hat{\bar{Y}}_{BTW}+v_2\left(\bar{X}-\bar{x}\right)\right\}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \quad (3.7)$$

where v_1 and v_2 are two unknown constants.
Expressions for *Bias* and *MSE* of $\hat{\bar{Y}}_{P2}$, to first order of approximation, are given by

$$Bias\left(\hat{\bar{Y}}_{P2}\right)\cong\frac{1}{2}\left[-2\bar{Y}+\bar{Y}\left\{2+\lambda C_x\left(C_x-\rho C_y\right)\right\}v_1+\bar{X}\lambda C_x^2v_2\right] \quad (3.8)$$

and

$$MSE\left(\hat{\hat{Y}}_{P2}\right) \cong \bar{Y}^2 + \frac{1}{4}\bar{Y}^2\left(4+5\lambda C_x^2 - 8\lambda\rho C_x C_y + 4\lambda C_y^2\right)v_1^2 + \bar{X}\lambda C_x^2 v_2\left(-\bar{Y}+\bar{X}v_2\right)$$
$$+\bar{Y}v_1\left\{-2\bar{Y} - \lambda C_x\left(C_x - \rho C_y\right)\left(\bar{Y} - 2\bar{X}v_2\right)\right\}. \qquad (3.9)$$

The optimum values of v_1 and v_2, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{P2}$, are given by $v_{1(opt)} = \frac{4}{4+\lambda C_x^2 - 4\lambda\left(-1+\rho^2\right)C_y^2}$ and $v_{2(opt)} = \frac{\bar{Y}}{2\bar{X}}\left(1+\frac{-8C_x+8\rho C_y}{C_x\left\{4+\lambda C_x^2-4\lambda\left(-1+\rho^2\right)C_y^2\right\}}\right)$.
The minimum *MSE* of $\hat{\hat{Y}}_{P2}$, at optimum values of v_1 and v_2, is given by

$$MSE_{\min}\left(\hat{\hat{Y}}_{P2}\right) \cong \frac{\bar{Y}^2\lambda\left\{-\lambda C_x^4 + 4\left(-1+\rho^2\right)\left(-4+\lambda C_x^2\right)C_y^2\right\}}{4\left\{4+\lambda C_x^2 - 4\lambda\left(-1+\rho^2\right)C_y^2\right\}}. \qquad (3.10)$$

3.3. Third proposed estimator. Replacing $\hat{\hat{Y}}_{SE}$ from (3.1) in place of $\bar{y}$ given in (3.6), the estimator becomes

$$\hat{\hat{Y}}_{BTSEW} = \bar{y}\frac{1}{4}\left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}}\right)\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right)\right\}. \qquad (3.11)$$

Also replacing $\hat{\hat{Y}}_{BTSEW}$ in place of $\bar{y}$ in (2.7), the proposed estimator turns out to be

$$\hat{\hat{Y}}_{P3} = \left[w_1\hat{\hat{Y}}_{BTSEW} + w_2\left(\bar{X}-\bar{x}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \qquad (3.12)$$

where w_1 and w_2 are two unknown constants.
Expressions for *Bias* and *MSE* of $\hat{\hat{Y}}_{P3}$, to first order of approximation, are given by

$$Bias\left(\hat{\hat{Y}}_{P3}\right) \cong \frac{1}{2}\left[-2\bar{Y} + \bar{Y}\left\{2+\lambda C_x\left(2C_x - \rho C_y\right)\right\}w_1 + \bar{X}\lambda C_x^2 w_2\right] \qquad (3.13)$$

and

$$MSE\left(\hat{\hat{Y}}_{P3}\right) \cong \bar{Y}^2 + \frac{1}{4}\bar{Y}^2\left(4+9\lambda C_x^2 - 8\lambda\rho C_x C_y + 4\lambda C_y^2\right)w_1^2 + \bar{X}\lambda C_x^2 w_2\left(-\bar{Y}+\bar{X}w_2\right)$$
$$+\bar{Y}w_1\left[-2\bar{Y} + \lambda C_x\left\{\rho C_y\left(\bar{Y}-2\bar{X}w_2\right) - 2C_x\left(\bar{Y}-\bar{X}w_2\right)\right\}\right]. \qquad (3.14)$$

The optimum values of w_1 and w_2, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{P3}$, are given by $w_{1(opt)} = \frac{4+2\lambda C_x^2}{4+5\lambda C_x^2 - 4\lambda\left(-1+\rho^2\right)C_y^2}$ and $w_{2(opt)} = \frac{\bar{Y}\left[8\rho C_y + C_x\left\{-4+\lambda\left(C_x^2+4\rho C_x C_y - 4\left(-1+\rho^2\right)C_y^2\right)\right\}\right]}{2\bar{X}C_x\left\{4+5\lambda C_x^2 - 4\lambda\left(-1+\rho^2\right)C_y^2\right\}}$.
The minimum *MSE* of $\hat{\hat{Y}}_{P3}$, at optimum values of w_1 and w_2, is given by

$$MSE_{\min}\left(\hat{\hat{Y}}_{P3}\right) \cong \frac{\bar{Y}^2\lambda\left\{-9\lambda C_x^4 + 4\left(-1+\rho^2\right)\left(-4+\lambda C_x^2\right)C_y^2\right\}}{4\left\{4+5\lambda C_x^2 - 4\lambda\left(-1+\rho^2\right)C_y^2\right\}}. \qquad (3.15)$$

Remarks: Expressions given in (3.5), (3.10) and (3.15) contain unknown population parameters, which can be estimated either from the sample values or through repeated survey or by experience gathered in due course of time.

4. Efficiency comparisons under simple random sampling

In this section, we compare the proposed estimators with the existing estimators.

(**a**) Comparison with difference type estimator

(i) From (2.2) and (3.5), $MSE_{\min}\left(\hat{\hat{Y}}_{P1}\right) < Var_{\min}\left(\hat{\hat{Y}}_{D}\right)$, if

$$\frac{\bar{Y}^2\lambda^2\left\{5C_x^2 - 8\left(-1+\rho^2\right)C_y^2\right\}^2}{64\left\{1+\lambda C_x^2 + \lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

(ii) From (2.2) and (3.10), $MSE_{\min}\left(\hat{\hat{Y}}_{P2}\right) < Var_{\min}\left(\hat{\hat{Y}}_{D}\right)$, if

$$\frac{\bar{Y}^2\lambda^2\left\{C_x^2-4\left(-1+\rho^2\right)C_y^2\right\}^2}{4\left\{4+\lambda C_x^2+4\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

(iii) From (2.2) and (3.15), $MSE_{\min}\left(\hat{\hat{Y}}_{P3}\right) < Var_{\min}\left(\hat{\hat{Y}}_{D}\right)$, if

$$\frac{\bar{Y}^2\lambda^2\left\{3C_x^2-4\left(-1+\rho^2\right)C_y^2\right\}^2}{4\left\{4+5\lambda C_x^2+4\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

Note: Conditions (i)-(iii) are always true.

(b) Comparison with Gupta and Shabbir [3] **estimator**

(iv) From (2.6) and (3.5), $MSE_{\min}\left(\hat{\hat{Y}}_{P1}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{GS}\right)$, if

$$\frac{\bar{Y}^2\lambda^2C_x^2\left[25C_x^2\left(-1+\lambda\tau^2C_x^2\right)-5\left(-1+\rho^2\right)\left\{-16+\lambda\left(-5+16\tau^2\right)C_x^2\right\}C_y^2+16\lambda\left(-1+\rho^2\right)^2\left(-1+4\tau^2\right)C_y^4\right]}{64\left\{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2\right\}\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

(v) From (2.6) and (3.10), $MSE_{\min}\left(\hat{\hat{Y}}_{P2}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{GS}\right)$, if

$$\frac{1}{4}\bar{Y}^2\left\{\lambda\left(1-4\tau^2\right)C_x^2+\frac{16}{4+\lambda C_x^2-4\lambda\left(-1+\rho^2\right)C_y^2}+\frac{4\left(-1+\lambda\tau^2C_x^2\right)^2}{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2}\right\} > 0.$$

(vi) From (2.6) and (3.15), $MSE_{\min}\left(\hat{\hat{Y}}_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{GS}\right)$, if

$$\frac{\bar{Y}^2\lambda^2C_x^2\left[9C_x^2\left(-1+\lambda\tau^2C_x^2\right)-3\left(-1+\rho^2\right)\left\{-8+\lambda\left(-3+8\tau^2\right)C_x^2\right\}C_y^2+4\lambda\left(-1+\rho^2\right)^2\left(-1+4\tau^2\right)C_y^4\right]}{4\left\{4+5\lambda C_x^2-4\lambda\left(-1+\rho^2\right)C_y^2\right\}\left\{-1+\lambda\tau^2C_x^2+\lambda\left(-1+\rho^2\right)C_y^2\right\}} > 0.$$

Note: The proposed estimators $\hat{\hat{Y}}_{Pi}(i=1,2,3)$ perform better than the Gupta and Shabbir [3] if conditions (iv)-(vi) are satisfied.

(c) Comparison with Grover and Kaur [2] **estimator**

(vii) From (2.10) and (3.5), $MSE_{\min}\left(\hat{\hat{Y}}_{P1}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{GK}\right)$, if

$$\frac{\bar{Y}^2\lambda^2C_x^2\left\{C_x^2\left(-24+\lambda C_x^2\right)+8\left(-1+\rho^2\right)\left(8+\lambda C_x^2\right)C_y^2\right\}}{64\left\{-1+\lambda\left(-1+\rho^2\right)C_y^2\right\}\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

(viii) From (2.10) and (3.10), $MSE_{\min}\left(\hat{\hat{Y}}_{P2}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{GK}\right)$, if

$$\bar{Y}^2\left(\frac{4}{4+\lambda C_x^2-4\lambda\left(-1+\rho^2\right)C_y^2}+\frac{\left(-8+\lambda C_x^2\right)^2}{-64+64\lambda\left(-1+\rho^2\right)C_y^2}\right) > 0.$$

(ix) From (2.10) and (3.15), $MSE_{\min}\left(\hat{\hat{Y}}_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{GK}\right)$, if

$$\frac{5\bar{Y}^2\lambda^2C_x^2\left\{C_x^2\left(-28+\lambda C_x^2\right)+4\left(-1+\rho^2\right)\left(16+3\lambda C_x^2\right)C_y^2\right\}}{64\left\{4+5\lambda C_x^2-4\lambda\left(-1+\rho^2\right)C_y^2\right\}\left\{-1+\lambda\left(-1+\rho^2\right)C_y^2\right\}} > 0.$$

Note: The proposed estimators $\hat{\hat{Y}}_{Pi}(i=1,2,3)$ perform better than the Grover and Kaur (2011) if conditions (vii)-(ix) are satisfied.

(d) Comparisons among proposed estimators

(x) From (3.5) and (3.10), $MSE_{\min}\left(\hat{\hat{Y}}_{P2}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{p1}\right)$, if

$$\bar{Y}^2\left(\frac{4}{4+\lambda C_x^2+4\lambda\left(1-\rho^2\right)C_y^2}-\frac{\left(8+3\lambda C_x^2\right)^2}{64\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}}\right) > 0.$$

(xi) From (3.5) and (3.15), $MSE_{\min}\left(\hat{\hat{Y}}_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{p1}\right)$, if

$$\frac{\bar{Y}^2\lambda^2C_x^2\left\{C_x^2\left(44+19\lambda C_x^2\right)+4\left(1-\rho^2\right)\left(16+7\lambda C_x^2\right)C_y^2\right\}}{64\left\{4+5\lambda C_x^2+4\lambda\left(1-\rho^2\right)C_y^2\right\}\left\{1+\lambda C_x^2+\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

(xii) From (3.10) and (3.15), $MSE_{\min}\left(\hat{\hat{Y}}_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}_{p2}\right)$, if

$$\frac{\bar{Y}^2\lambda^2C_x^2\left\{C_x^2\left(8+\lambda C_x^2\right)+4\left(1-\rho^2\right)\left(4+\lambda C_x^2\right)C_y^2\right\}}{\left\{4+\lambda C_x^2+4\lambda\left(1-\rho^2\right)C_y^2\right\}\left\{4+5\lambda C_x^2+4\lambda\left(1-\rho^2\right)C_y^2\right\}} > 0.$$

Note: Conditions (xi) and (xii) are always true.

5. Estimators under two-phase sampling (partial information)

When the population mean of the auxiliary variable, x, is unknown, it is customary to apply the two-phase sampling procedure. The two-phase sampling scheme is explained as follows

(i) In first-phase, a sample of size $(n_1 < N)$ is selected from the population using SRSWOR to estimate $\bar{X}$.

(ii) In second-phase, a sample of size $(n < n_1)$ is selected to observe both y and x.

Let $\bar{x}_1$ be the sample mean based on first-phase sample of size n_1, and let $\bar{y}$ and $\bar{x}$ be the sample means based on second-phase sample of size n. Let $(\bar{x}_1, \bar{x})$ and $\bar{y}$ are the unbiased estimators of $\bar{X}$ and $\bar{Y}$, respectively. Now we discuss different estimators of finite population mean based on two-phase sampling.

5.1. Unbiased difference estimator. The unbiased difference estimator of population mean under two-phase sampling is

$$\hat{\hat{Y}}_D^* = \bar{y} + k^*\left(\bar{x}_1 - \bar{x}\right), \tag{5.1}$$

where k^* is an unknown constant.
The expression for variance of $\hat{\hat{Y}}_D^*$, at optimum value of k^*, i.e., $k^*_{(opt)} = \frac{\bar{Y}\rho C_y}{\bar{X}C_x}$ is given by

$$Var_{\min}\left(\hat{\hat{Y}}_D^*\right) \cong \bar{Y}^2\left(\lambda - \lambda\rho^2 + \lambda_1\rho^2\right)C_y^2, \tag{5.2}$$

where $\lambda_1 = \frac{1}{n} - \frac{1}{n_1}$.

5.2. Gupta and Shabbir [3] family of estimators. Under two-phase sampling, Gupta and Shabbir [3] family of estimators for estimating finite population mean, is given by

$$\hat{\hat{Y}}_{GS}^* = \left\{s_1^*\bar{y} + s_2^*\left(\bar{x}_1 - \bar{x}\right)\right\}\left(\frac{a\bar{x}_1 + b}{a\bar{x} + b}\right), \tag{5.3}$$

where s_1^* and s_2^* are two unknown constants.
The expressions for *Bias* and *MSE* of $\hat{\hat{Y}}_{GS}^*$, to first order of approximation, are given by

$$Bias\left(\hat{\hat{Y}}_{GS}^*\right) \cong -\bar{Y} + \bar{Y}\left\{1 + \left(\lambda - \lambda_1\right)\tau C_x\left(\tau C_x - \rho C_y\right)\right\}s_1^* + \bar{X}\lambda\tau C_x^2 s_2^* \tag{5.4}$$

and

$$\begin{aligned} MSE\left(\hat{\hat{Y}}_{GS}^*\right) &\cong \bar{Y}^2 + \bar{Y}^2\left\{1 + 3\left(\lambda - \lambda_1\right)\tau^2C_x^2 + 4\left(-\lambda + \lambda_1\right)\rho\tau C_xC_y + \lambda C_y^2\right\}s_1^{*2} \\ &+ \bar{X}\left(\lambda - \lambda_1\right)C_x^2 s_2^*\left(-2\bar{Y}\tau + \bar{X}s_2^*\right) - 2\bar{Y}s_1^*\left[\bar{Y} + \left(\lambda - \lambda_1\right)C_x\left\{\tau C_x\left(\bar{Y}\tau - 2\bar{X}s_2^*\right) + \rho C_y\left(-\bar{Y}\tau + \bar{X}s_2^*\right)\right\}\right], \end{aligned} \tag{5.5}$$

where τ is defined earlier.
The optimum values of s_1^* and s_2^*, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{GS}^*$, are given by

$s_{1(opt)}^* = \frac{-1+(\lambda-\lambda_1)\tau^2 C_x^2}{-1+(\lambda-\lambda_1)\tau^2 C_x^2+\{-\lambda+(\lambda-\lambda_1)\}C_y^2}$ and
$s_{2(opt)}^* = \frac{\bar{Y}\left[-\rho C_y+\tau C_x\left\{1+(-\lambda+\lambda_1)\tau^2 C_x^2+(\lambda-\lambda_1)\rho\tau C_x C_y+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2\right\}\right]}{\bar{X}C_x\left\{-1+(\lambda-\lambda_1)\tau^2 C_x^2+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2\right\}}$.
The minimum *MSE* of $\hat{\bar{Y}}_{GS}^*$, at optimum values of s_1^* and s_2^*, is given by

$$MSE_{\min}\left(\hat{\bar{Y}}_{GS}^*\right) \cong \frac{\bar{Y}^2\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{1-(\lambda-\lambda_1)\tau^2 C_x^2\right\}C_y^2}{-1+(\lambda-\lambda_1)\tau^2 C_x^2+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}. \tag{5.6}$$

Gupta and Shabbir [3] estimator $\hat{\bar{Y}}_{GS}^*$ will perform better than the difference estimator $\hat{\bar{Y}}_D^*$, if

$$\frac{\bar{Y}^2\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)^2 C_y^4}{1-(\lambda-\lambda_1)\tau^2 C_x^2-\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2} > 0.$$

5.3. Grover and Kaur [2] **estimator.** Grover and Kaur [2] estimator under double sampling for estimation of the population mean is given by

$$\hat{\bar{Y}}_{GK}^* = \left\{t_1^*\bar{y}+t_2^*(\bar{x}_1-\bar{x})\right\}\exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right), \tag{5.7}$$

where t_1^* and t_2^* are two unknown constants.
The expressions for *Bias* and *MSE* of $\hat{\bar{Y}}_{GK}^*$, to first order of approximation, are given by

$$Bias\left(\hat{\bar{Y}}_{GK}^*\right) \cong \frac{1}{8}\left[-8\bar{Y}+\bar{Y}\left\{8+(\lambda-\lambda_1)C_x(3C_x-4\rho C_y)\right\}t_1^*+4\bar{X}(\lambda-\lambda_1)C_x^2 t_2^*\right] \tag{5.8}$$

and

$$\begin{aligned} MSE\left(\hat{\bar{Y}}_{GK}^*\right) &\cong \bar{Y}^2+\bar{Y}^2\left\{1+(\lambda-\lambda_1)C_x^2+2(-\lambda+\lambda_1)\rho C_x C_y+\lambda C_y^2\right\}t_1^{*2} \\ &+\bar{X}(\lambda-\lambda_1)C_x^2 t_2^*\left(-\bar{Y}+\bar{X}t_2^*\right)+\frac{1}{4}\bar{Y}t_1^*\left[-8\bar{Y}+(\lambda-\lambda_1)C_x\left\{4\rho C_y\left(\bar{Y}-2\bar{X}t_2^*\right)+C_x\left(-3\bar{Y}+8\bar{X}t_2^*\right)\right\}\right]. \end{aligned} \tag{5.9}$$

The optimum values of t_1^* and t_2^*, obtained by minimizing the *MSE* of $\hat{\bar{Y}}_{GK}^*$, are given by
$t_{1(opt)}^* = \frac{-8+(\lambda-\lambda_1)C_x^2}{-8+8\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}$ and $t_{2(opt)}^* = \frac{\bar{Y}\left[8\rho C_y+C_x\left\{-4+(\lambda-\lambda_1)C_x^2+(-\lambda+\lambda_1)\rho C_x C_y+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}\right]}{8\bar{X}C_x\left\{1+\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}$.
The minimum *MSE* of $\hat{\bar{Y}}_{GK}^*$, at optimum values of t_1^* and t_2^*, is given by

$$MSE_{\min}\left(\hat{\bar{Y}}_{GK}^*\right) \cong \frac{\bar{Y}^2\left\{(\lambda-\lambda_1)^2 C_x^4-16\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left(-4+(\lambda-\lambda_1)C_x^2\right)C_y^2\right\}}{-64+64\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}. \tag{5.10}$$

Grover and Kaur [2] estimator $\hat{\bar{Y}}_{GK}^*$ will perform better than the difference estimator $\hat{\bar{Y}}_D^*$, if

$$\frac{\bar{Y}^2\left\{(\lambda-\lambda_1)C_x^2+8\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}^2}{64\left\{1+\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}} > 0.$$

Gupta and Shabbir [3] estimator $\hat{\bar{Y}}_{GS}^*$ will perform better than the Grover and Kaur [2] estimator $\hat{\bar{Y}}_{GK}^*$, if

$$\bar{Y}^2\left[\frac{\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left(-1+(\lambda-\lambda_1)\tau^2 C_x^2\right)C_y^2}{-1+(\lambda-\lambda_1)\tau^2 C_x^2+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2} + \frac{(\lambda-\lambda_1)^2 C_x^4-16\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{-4+(\lambda-\lambda_1)C_x^2\right\}C_y^2}{64\left\{-1+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2\right\}}\right] > 0.$$

6. Proposed estimators under two-phase sampling

In this section, we derive the mathematical expressions of the biases and MSEs of the proposed estimators of finite population mean when partial auxiliary information is available.

6.1. First proposed estimator. Similar to (3.2), the proposed estimator under double sampling is given by

$$\hat{\hat{Y}}_{P1}^{*} = \left\{ u_1^{*} \frac{1}{2} \bar{y} \left(\frac{\bar{x}_1}{\bar{x}} + \frac{\bar{x}}{\bar{x}_1} \right) + u_2^{*} \left(\bar{x}_1 - \bar{x} \right) \right\} \exp\left(\frac{\bar{x}_1 - \bar{x}}{\bar{x}_1 + \bar{x}} \right), \quad (6.1)$$

where u_1^{*} and u_2^{*} are two unknown constants.
The expressions for *Bias* and *MSE* of $\hat{\hat{Y}}_{P1}^{*}$, to first order of approximation, are given by

(6.2)

$$Bias\left(\hat{\hat{Y}}_{P1}^{*}\right) \cong \frac{1}{8}\left[-8\bar{Y} + \bar{Y}\left\{8 + (\lambda - \lambda_1) C_x \left(7C_x - 4\rho C_y\right)\right\} u_1^{*} + 4\bar{X}\left(\lambda - \lambda_1\right) C_x^2 u_2^{*}\right]$$

and

$$MSE\left(\hat{\hat{Y}}_{P1}^{*}\right) \cong \bar{Y}^2 + \bar{Y}^2\left\{1 + 2\left(\lambda - \lambda_1\right) C_x^2 + 2\left(-\lambda + \lambda_1\right)\rho C_x C_y + \lambda C_y^2\right\} u_1^{*2} + \bar{X}\left(\lambda - \lambda_1\right) C_x^2 u_2^{*}\left(-\bar{Y} + \bar{X} u_2^{*}\right)$$
$$+ \frac{1}{4}\bar{Y} u_1^{*}\left[-8\bar{Y} + \left(\lambda - \lambda_1\right) C_x\left\{4\rho C_y\left(\bar{Y} - 2\bar{X} u_2^{*}\right) + C_x\left(-7\bar{Y} + 8\bar{X} u_2^{*}\right)\right\}\right]. \quad (6.3)$$

The optimum values of u_1^{*} and u_2^{*}, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{P1}^{*}$, are given by
$u_{1(opt)}^{*} = \frac{8+3(\lambda-\lambda_1)C_x^2}{8\left\{1+(\lambda-\lambda_1)C_x^2+\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}$ and
$u_{2(opt)}^{*} = \frac{\bar{Y}\left[8\rho C_y+C_x\left\{-4+(\lambda-\lambda_1)C_x^2+3(\lambda-\lambda_1)\rho C_x C_y+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}\right]}{8\bar{X}C_x\left\{1+(\lambda-\lambda_1)C_x^2+\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}$.
The minimum *MSE* of $\hat{\hat{Y}}_{P1}^{*}$, at optimum values of u_1^{*} and u_2^{*}, is given by

(6.4)

$$MSE_{\min}\left(\hat{\hat{Y}}_{P1}^{*}\right) \cong \frac{\bar{Y}^2\left\{-25\left(\lambda - \lambda_1\right)^2 C_x^4 + 16\left\{-\lambda + \left(\lambda - \lambda_1\right)\rho^2\right\}\left\{-4 + \left(\lambda - \lambda_1\right) C_x^2\right\} C_y^2\right\}}{64\left\{1 + \left(\lambda - \lambda_1\right) C_x^2 + \left(\lambda - \lambda\rho^2 + \lambda_1\rho^2\right) C_y^2\right\}}.$$

6.2. Second proposed estimator. On the line of (3.7), the second proposed estimator under double sampling is given by

$$\hat{\hat{Y}}_{P2}^{*} = \left[v_1^{*}\frac{1}{2}\bar{y}\left\{\exp\left(\frac{\bar{x}_1 - \bar{x}}{\bar{x}_1 + \bar{x}}\right) + \exp\left(\frac{\bar{x} - \bar{x}_1}{\bar{x}_1 + \bar{x}}\right)\right\} + v_2^{*}\left(\bar{x}_1 - \bar{x}\right)\right]\exp\left(\frac{\bar{x}_1 - \bar{x}}{\bar{x}_1 + \bar{x}}\right), \quad (6.5)$$

where v_1^{*} and v_2^{*} are two unknown constants.
The expressions for *Bias* and *MSE* of $\hat{\hat{Y}}_{P2}^{*}$, to first order of approximation, are given by

$$Bias\left(\hat{\hat{Y}}_{P2}^{*}\right) \cong \frac{1}{2}\left[-2\bar{Y} + \bar{Y}\left\{2 + \left(\lambda - \lambda_1\right) C_x\left(C_x - \rho C_y\right)\right\} v_1^{*} + \bar{X}\left(\lambda - \lambda_1\right) C_x^2 v_2^{*}\right] \quad (6.6)$$

and

$$MSE\left(\hat{\hat{Y}}_{P2}^{*}\right) \cong \bar{Y}^2 + \frac{1}{4}\bar{Y}^2\left\{4 + 5\left(\lambda - \lambda_1\right) C_x^2 + 8\left(-\lambda + \lambda_1\right)\rho C_x C_y + 4\lambda C_y^2\right\} v_1^{*2}$$
$$+\bar{X}\left(\lambda - \lambda_1\right) C_x^2 v_2^{*}\left(-\bar{Y} + \bar{X} v_2^{*}\right) + \bar{Y} v_1^{*}\left\{-2\bar{Y} - \left(\lambda - \lambda_1\right) C_x\left(C_x - \rho C_y\right)\left(\bar{Y} - 2\bar{X} v_2^{*}\right)\right\}. \quad (6.7)$$

The optimum values of v_1^{*} and v_2^{*}, obtained by minimizing the *MSE* of $\hat{\hat{Y}}_{P2}^{*}$, are given by
$v_{1(opt)}^{*} = \frac{4}{4+(\lambda-\lambda_1)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2}$ and $v_{2(opt)}^{*} = \frac{\bar{Y}}{2\bar{X}}\left(1 + \frac{-8C_x+8\rho C_y}{C_x\left\{4+(\lambda-\lambda_1)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}\right)$.
The minimum *MSE* of $\hat{\hat{Y}}_{P2}^{*}$, at optimum values of v_1^{*} and v_2^{*}, is given by

(6.8)

$$MSE_{\min}\left(\hat{\bar{Y}}_{P2}^{*}\right) \cong \frac{1}{4}\bar{Y}^2\left\{4+\left(-\lambda+\lambda_1\right)C_x^2-\frac{16}{4+\left(\lambda-\lambda_1\right)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2}\right\}.$$

6.3. Third proposed estimator. On the line of (3.12), the third proposed estimator of the population mean under double sampling is given by

(6.9)

$$\hat{\bar{Y}}_{P3}^{*}=\left[w_1^{*}\frac{1}{4}\bar{y}\left(\frac{\bar{x}_1}{\bar{x}}+\frac{\bar{x}}{\bar{x}_1}\right)\left\{\exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right)+\exp\left(\frac{\bar{x}-\bar{x}_1}{\bar{x}_1+\bar{x}}\right)\right\}+w_2^{*}\left(\bar{x}_1-\bar{x}\right)\right]\exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right),$$

where w_1^{*} and w_2^{*} are two unknown constants.
The expressions for *Bias* and *MSE* of $\hat{\bar{Y}}_{P3}^{*}$, to first order of approximation, are given by

(6.10)

$$Bias\left(\hat{\bar{Y}}_{P3}^{*}\right) \cong \frac{1}{2}\left[-2\bar{Y}+\bar{Y}\left\{2+\left(\lambda-\lambda_1\right)C_x\left(2C_x-\rho C_y\right)\right\}w_1^{*}+\bar{X}\left(\lambda-\lambda_1\right)C_x^2w_2^{*}\right]$$

and

$$MSE\left(\hat{\bar{Y}}_{P3}^{*}\right) \cong \bar{Y}^2+\frac{1}{4}\bar{Y}^2\left(4+9\left(\lambda-\lambda_1\right)C_x^2+8\left(-\lambda+\lambda_1\right)\rho C_xC_y+4\lambda C_y^2\right)w_1^{*2}$$

(6.11) $$+\bar{X}\left(\lambda-\lambda_1\right)C_x^2w_2^{*}\left(-\bar{Y}+\bar{X}w_2^{*}\right)+\bar{Y}w_1^{*}\left[-2\bar{Y}+\left(\lambda-\lambda_1\right)C_x\left\{\rho C_y\left(\bar{Y}-2\bar{X}w_2^{*}\right)-2C_x\left(\bar{Y}-\bar{X}w_2^{*}\right)\right\}\right].$$

The optimum values of w_1^{*} and w_2^{*}, obtained by minimizing the *MSE* of $\hat{\bar{Y}}_{P3}^{*}$, are given by
$w_{1(opt)}^{*}=\frac{4+2(\lambda-\lambda_1)C_x^2}{4+5(\lambda-\lambda_1)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2}$ and $w_{2(opt)}^{*}=\frac{\bar{Y}\left[8\rho C_y+C_x\left\{-4+(\lambda-\lambda_1)C_x^2+4(\lambda-\lambda_1)\rho C_xC_y+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}\right]}{2\bar{X}C_x\left\{4+5(\lambda-\lambda_1)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}$.
The minimum *MSE* of $\hat{\bar{Y}}_{P3}^{*}$, at optimum values of w_1^{*} and w_2^{*}, is given by

(6.12)

$$MSE_{\min}\left(\hat{\bar{Y}}_{P3}^{*}\right) \cong \frac{\bar{Y}^2\left\{-9\left(\lambda-\lambda_1\right)^2C_x^4+4\left\{-\lambda+\left(\lambda-\lambda_1\right)\rho^2\right\}\left(-4+\left(\lambda-\lambda_1\right)C_x^2\right)C_y^2\right\}}{4\left\{4+5\left(\lambda-\lambda_1\right)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}.$$

Remarks: Expressions given in (6.4), (6.8) and (6.12) contain the unknown population parameters, which can be estimated either from the sample values or through repeated survey or by experience gathered in due course of time.

7. Efficiency comparisons under two-phase sampling

In this section, we compare the proposed estimators with the existing estimators of population mean based on double sampling scheme.

(a) **Comparison with difference type estimator**

(i) From (5.2) and (6.4), $MSE_{\min}\left(\hat{\bar{Y}}_{P1}^{*}\right) < Var_{\min}\left(\hat{\bar{Y}}_{D}^{*}\right)$, if

$$\frac{\left\{5\bar{Y}\left(\lambda-\lambda_1\right)C_x^2+8\bar{Y}\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}^2}{64\left\{1+\left(\lambda-\lambda_1\right)C_x^2+\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}>0.$$

(ii) From (5.2) and (6.8), $MSE_{\min}\left(\hat{\bar{Y}}_{P2}^{*}\right) < Var_{\min}\left(\hat{\bar{Y}}_{D}^{*}\right)$, if

$$\frac{\bar{Y}^2}{4}\left[-4+\left(\lambda-\lambda_1\right)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2+\frac{16}{4+\left(\lambda-\lambda_1\right)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2}\right]>0.$$

(iii) From (5.2) and (6.12), $MSE_{\min}\left(\hat{\bar{Y}}_{P3}^{*}\right) < Var_{\min}\left(\hat{\bar{Y}}_{D}^{*}\right)$, if

$$\frac{\left\{3\bar{Y}\left(\lambda-\lambda_1\right)C_x^2+4\bar{Y}\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}^2}{4\left\{4+5\left(\lambda-\lambda_1\right)C_x^2+4\left(\lambda-\lambda\rho^2+\lambda_1\rho^2\right)C_y^2\right\}}>0.$$

Note: When conditions (i)-(iii) are satisfied, the proposed estimators $\hat{\hat{Y}}^{*}_{Pi}(i=1,2,3)$ perform better than difference type estimator $\hat{\hat{Y}}^{*}_{D}$.

(b) Comparison with Gupta and Shabbir [3] estimator

(iv) From (5.6) and (6.4), $MSE_{\min}\left(\hat{\hat{Y}}^{*}_{P1}\right) < MSE_{\min}\left(\hat{\hat{Y}}^{*}_{GS}\right)$, if

$$\frac{\bar{Y}^2\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{-1+(\lambda-\lambda_1)\tau^2C_x^2\right\}C_y^2}{1-(\lambda-\lambda_1)\tau^2C_x^2+\left\{\lambda-(\lambda-\lambda_1)\rho^2\right\}C_y^2}$$
$$-\frac{\bar{Y}^2\left[-25(\lambda-\lambda_1)^2C_x^4+16\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{-4+(\lambda-\lambda_1)C_x^2\right\}C_y^2\right]}{64\left\{1+(\lambda-\lambda_1)C_x^2+(\lambda-\lambda\rho^2+\lambda_1\rho^2)C_y^2\right\}}>0.$$

(v) From (5.6) and (6.8), $MSE_{\min}\left(\hat{\hat{Y}}^{*}_{P2}\right) < MSE_{\min}\left(\hat{\hat{Y}}^{*}_{GS}\right)$, if

$$\frac{1}{4}\bar{Y}^2\left(-4+(\lambda-\lambda_1)C_x^2-\frac{4\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{-1+(\lambda-\lambda_1)\tau^2C_x^2\right\}C_y^2}{-1+(\lambda-\lambda)\tau^2C_x^2+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}\right.$$
$$\left.+\frac{16}{4+(\lambda-\lambda_1)C_x^2+4(\lambda-\lambda\rho^2+\lambda_1\rho^2)C_y^2}\right)>0.$$

(vi) From (5.6) and (6.12), $MSE_{\min}\left(\hat{\hat{Y}}^{*}_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}^{*}_{GS}\right)$, if

$$\frac{\bar{Y}^2}{4}\left(\frac{4\left\{\lambda-(\lambda-\lambda_1)\rho^2\right\}\left\{-1+(\lambda-\lambda_1)\tau^2C_x^2\right\}C_y^2}{-1+(\lambda-\lambda_1)\tau^2C_x^2+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}\right.$$
$$\left.-\frac{-9(\lambda-\lambda_1)^2C_x^4+4\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{-4+(\lambda-\lambda_1)C_x^2\right\}C_y^2}{4+5(\lambda-\lambda_1)C_x^2+4(\lambda-\lambda\rho^2+\lambda_1\rho^2)C_y^2}\right)>0,$$

Note: When conditions (iv)-(vi) are satisfied, the proposed estimators $\hat{\hat{Y}}^{*}_{Pi}(i=1,2,3)$ perform better than the Gupta and Shabbir [3] estimator $\hat{\hat{Y}}^{*}_{GS}$.

(c) Comparison with Grover and Kaur [2] estimator

(vii) From (5.10) and (6.4), $MSE_{\min}\left(\hat{\hat{Y}}^{*}_{P1}\right) < MSE_{\min}\left(\hat{\hat{Y}}^{*}_{GK}\right)$, if

$$\frac{\bar{Y}^2(\lambda-\lambda_1)^2C_x^2\left[(\lambda-\lambda_1)C_x^2\left\{24+(-\lambda+\lambda_1)C_x^2\right\}-8\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{8+(\lambda-\lambda_1)C_x^2\right\}C_y^2\right]}{64\left[-1+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2\right]\left[-1+(-\lambda+\lambda_1)C_x^2+\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2\right]}>0,$$

when above condition is satisfied, the estimator $\hat{\hat{Y}}^{*}_{P1}$ is more efficient than $\hat{\hat{Y}}^{*}_{GK}$.

(viii) From (5.10) and (6.8), $MSE_{\min}\left(\hat{\hat{Y}}^{*}_{P2}\right) < MSE_{\min}\left(\hat{\hat{Y}}^{*}_{GK}\right)$, if

$$\frac{\bar{Y}^2}{4}\left(-4+\frac{16(-\lambda+\lambda_1)C_x^2+(\lambda-\lambda_1)^2C_x^4+64\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}{-16+16\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2}\right.$$
$$\left.+\frac{16}{4+(\lambda-\lambda_1)C_x^2+4\left\{\lambda-\lambda\rho^2+\lambda_1\rho^2\right\}C_y^2}\right)>0,$$

when above condition is satisfied, the estimator $\hat{\hat{Y}}^{*}_{P2}$ is more efficient than $\hat{\hat{Y}}^{*}_{GK}$.

(ix) From (5.10) and (6.12), $MSE_{\min}\left(\hat{\hat{Y}}^{*}_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}^{*}_{GK}\right)$, if

$$\frac{5\bar{Y}^2(\lambda-\lambda_1)C_x^2\left[(\lambda-\lambda_1)C_x^2\left\{28+(-\lambda+\lambda_1)C_x^2\right\}-4\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}\left\{16+3(\lambda-\lambda_1)C_x^2\right\}C_y^2\right]}{64\left[1-\left\{-\lambda+(\lambda-\lambda_1)\rho^2\right\}C_y^2\right]\left[4+5(\lambda-\lambda_1)C_x^2+4(\lambda-\lambda\rho^2+\lambda_1\rho^2)C_y^2\right]}>0,$$

Note: When conditions (vii)-(ix) are satisfied, the proposed estimators $\hat{\hat{Y}}^*_{Pi} (i = 1, 2, 3)$ perform better than the Grover and Kaur [2] estimator $\hat{\hat{Y}}^*_{GK}$.

(d) **Comparisons among proposed estimators**

(x) From (6.4) and (6.8), $MSE_{\min}\left(\hat{\hat{Y}}^*_{P2}\right) < MSE_{\min}\left(\hat{\hat{Y}}^*_{p1}\right)$, if

$$\frac{\bar{Y}^2}{64}\left(-64 + \frac{256}{4 + (\lambda - \lambda_1) C_x^2 + 4(\lambda - \lambda\rho^2 + \lambda_1\rho^2) C_y^2}\right.$$
$$\left. + \frac{(\lambda - \lambda_1) C_x^2 \left\{16 + 9(-\lambda + \lambda_1) C_x^2\right\} + 64\left(\lambda - \lambda\rho^2 + \lambda_1\rho^2\right) C_y^2}{1 + (\lambda - \lambda_1) C_x^2 + (\lambda - \lambda\rho^2 + \lambda_1\rho^2) C_y^2}\right) > 0,$$

when above condition is satisfied, the estimator $\hat{\hat{Y}}^*_{P2}$ is more efficient than $\hat{\hat{Y}}^*_{P1}$.

(xi) From (6.4) and (6.12), $MSE_{\min}\left(\hat{\hat{Y}}^*_{P3}\right) < MSE_{\min}\left(\hat{\hat{Y}}^*_{P1}\right)$, if

$$\frac{\bar{Y}^2 (\lambda - \lambda_1) C_x^2 \left[(\lambda - \lambda_1) C_x^2 \left\{44 + 19(\lambda - \lambda_1) C_x^2\right\} - 4\left\{-\lambda + (\lambda - \lambda_1)\rho^2\right\}\left\{16 + 7(\lambda - \lambda_1) C_x^2\right\} C_y^2\right]}{64\left[1 + (\lambda - \lambda_1) C_x^2 + \left\{\lambda - \lambda\rho^2 + \lambda_1\rho^2\right\} C_y^2\right]\left[4 + 5(\lambda - \lambda_1) C_x^2 + 4\left\{\lambda - \lambda\rho^2 + \lambda_1\rho^2\right\} C_y^2\right]} > 0,$$

when above condition is satisfied, the estimator $\hat{\hat{Y}}^*_{P3}$ is more efficient than $\hat{\hat{Y}}^*_{P1}$.

(xii) From (6.8) and (6.12), $MSE_{\min}\left(\hat{\hat{Y}}^*_{p3}\right) < MSE_{\min}\left(\hat{\hat{Y}}^*_{p2}\right)$, if

$$\frac{\bar{Y}^2}{4}\left(4 - \frac{16}{4 + (\lambda - \lambda_1) C_x^2 + 4(\lambda - \lambda\rho^2 + \lambda_1\rho^2) C_y^2}\right.$$
$$\left. + \frac{4\left[(-\lambda + \lambda_1) C_x^2 + (\lambda - \lambda_1)^2 C_x^4 + 4\left\{-\lambda + (\lambda - \lambda_1)\rho^2\right\} C_y^2\right]}{4 + 5(\lambda - \lambda_1) C_x^2 + 4(\lambda - \lambda\rho^2 + \lambda_1\rho^2) C_y^2}\right) > 0,$$

when above condition is satisfied, the estimator $\hat{\hat{Y}}^*_{P3}$ is more efficient than $\hat{\hat{Y}}^*_{P2}$.

8. Empirical Study

The empirical study is based on three populations under: (i) complete information case and (ii) incomplete information case.

8.1. Complete auxiliary information. In this section, we compare the estimators numerically by using different real life data sets. The values of minimum MSEs of the estimators are given in Tables 1-3 based on the Populations I-III, respectively.

Population 1: [source: Kadilar and Cingi [5]].
The summary statistics are: $N = 200$, $n = 50$, $\bar{Y} = 500$, $\bar{X} = 25$, $C_y = 15$, $C_x = 2$, $\rho = 0.90$, $\beta_{2x} = 50$, $\lambda = 0.015$.

Population 2: [source: Kadilar and Cingi [6]].
Let y =level of apple production (1 unit = 100 tones) and x =number of trees (1 unit = 100 trees). The data statistics are: $N = 106$, $n = 20$, $\bar{Y} = 2212.59$, $\bar{X} = 27421.70$, $C_y = 5.22$, $C_x = 2.10$, $\rho = 0.86$, $\beta_{2x} = 34.57$, $\lambda = 0.040566$.

Population 3: [source: Kadilar and Cingi [7]].
Let y =level of apple production (1 unit = 100 tones) and x = number of trees. The data statistics are: $N = 104$, $n = 20$, $\bar{Y} = 6.254$, $\bar{X} = 13931.683$, $C_y = 1.866$, $C_x = 1.653$,

$\rho = 0.865$, $\beta_{2x} = 17.516$, $\lambda = 0.040385$.

Under complete information case, the minimum MSE values of the proposed and existing estimators are given in Table 1.
For $\hat{\bar{Y}}_{GS(1)}$ with $(a = 1, b = \rho)$, $\hat{\bar{Y}}_{GS(2)}$ with $(a = 1, b = C_x)$, $\hat{\bar{Y}}_{GS(3)}$ with $(a = 1, \beta_{2x})$,

Table 1. Minimum *MSE* values of different estimators (complete information).

Estimator	Population-I	Population-II	Population-III
$\hat{\bar{Y}}_D$	160313.00	1409112.00	1.38
$\hat{\bar{Y}}_{GS(1)}$	95468.40	1043370.00	1.33
$\hat{\bar{Y}}_{GS(2)}$	95650.43	1043380.00	1.33
$\hat{\bar{Y}}_{GS(3)}$	97421.62	1043510.00	1.33
$\hat{\bar{Y}}_{GS(4)}$	95308.27	1043370.00	1.33
$\hat{\bar{Y}}_{GS(5)}$	97099.35	1043440.00	1.33
$\hat{\bar{Y}}_{GK}$	96203.40	1043340.00	1.29
$\hat{\bar{Y}}_{P1}$	92612.00	876024.00	1.01
$\hat{\bar{Y}}_{P2}$	95306.60	1002810.00	1.24
$\hat{\bar{Y}}_{P3}$	91712.50	832286.00	0.92

$\hat{\bar{Y}}_{GS(4)}$ with $(a = \beta_{2x}, b = C_x)$, and $\hat{\bar{Y}}_{GS(5)}$ with $(a = C_x, b = \beta_{2x})$.

8.2. Summary statistics under two-phase sampling (partial information). Population 1: [source: Kadilar and Cingi [5]].
The summary statistics are: $N = 200$, $n_1 = 90$, $n = 50$, $\bar{Y} = 500$, $\bar{X} = 25$, $C_y = 15$, $C_x = 2$, $\rho = 0.90$, $\beta_{2x} = 50$, $\lambda = 0.015$.

Population 2: [source: Kadilar and Cingi [6]].
Let y =level of apple production (1 unit = 100 tones) and x =number of trees (1 unit = 100 trees). The summary statistics are: $N = 106$, $n_1 = 40$, $n = 20$, $\bar{Y} = 2212.59$, $\bar{X} = 27421.70$, $C_y = 5.22$, $C_x = 2.10$, $\rho = 0.86$, $\beta_{2x} = 34.57$, $\lambda = 0.040566$.

Population 3: [source: Kadilar and Cingi [7]].
Let y =level of apple production (1 unit = 100 tones) and x = number of trees. The summary statistics are: $N = 104$, $n_1 = 40$, $n = 20$, $\bar{Y} = 6.254$, $\bar{X} = 13931.683$, $C_y = 1.866$, $C_x = 1.653$, $\rho = 0.865$, $\beta_{2x} = 17.516$, $\lambda = 0.040385$.

The values of minimum MSEs of the proposed and existing estimators constructed under two-phase sampling for all populations are given in Table 2. For $\hat{\bar{Y}}^*_{GS(1)}$ with $(a = 1, b = \rho)$, $\hat{\bar{Y}}^*_{GS(2)}$ with $(a = 1, b = C_x)$, $\hat{\bar{Y}}^*_{GS(3)}$ with $(a = 1, \beta_{2x})$, $\hat{\bar{Y}}^*_{GS(4)}$ with $(a = \beta_{2x}, b = C_x)$, and $\hat{\bar{Y}}^*_{GS(5)}$ with $(a = C_x, b = \beta_{2x})$.
It is worth mentioning here that for each of the three populations, the proposed estimators $\hat{\bar{Y}}_{Pi}$ and $\hat{\bar{Y}}^*_{Pi}$ $(i = 1, 2, 3)$ perform better than the existing estimators. It is observed that the proposed estimator $\hat{\bar{Y}}_{P3}$ and $\hat{\bar{Y}}^*_{P3}$ are more efficient than their counterparts considered here.

Table 2. Minimum *MSE* values of different estimators in double sampling (partial information)

Estimator	Population-I	Population-II	Population-III
$\hat{\bar{Y}}^*_D$	438750.00	2944860.00	2.95
$\hat{\bar{Y}}^*_{GS(1)}$	155854.00	1757000.00	2.73
$\hat{\bar{Y}}^*_{GS(2)}$	156129.00	1757010.00	2.73
$\hat{\bar{Y}}^*_{GS(3)}$	158855.00	1757220.00	2.73
$\hat{\bar{Y}}^*_{GS(4)}$	155613.00	1757000.00	2.73
$\hat{\bar{Y}}^*_{GS(5)}$	158351.00	1757100.00	2.73
$\hat{\bar{Y}}^*_{GK}$	157838.00	1787510.00	2.67
$\hat{\bar{Y}}^*_{P1}$	155785.00	1659340.00	2.47
$\hat{\bar{Y}}^*_{P2}$	157326.00	1755550.00	2.65
$\hat{\bar{Y}}^*_{P3}$	155271.00	1627170.00	2.41

9. Conclusion

In this paper, we proposed some improved exponential type estimators of finite population mean when complete and partial auxiliary information is available. The proposed estimators perform better than all other competitor estimators considered here. It is to be noted the suggested estimators although biased but are always better than the unbiased difference type estimator of the finite population mean. Based on both theoretical and numerical comparisons, the proposed estimators are more precise than their counterparts. The work can easily be extended to improve the estimation of finite population mean using information on auxiliary attributes, stratified random sampling and other sampling designs. Finally, we recommend the use of $\hat{\bar{Y}}_{P3}$ and $\hat{\bar{Y}}^*_{P3}$ for efficient estimation of the population mean under simple and two-phase sampling schemes, respectively.

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