

On the numerical solution of Hammerstein integral equations using shifted Chebyshev polynomials of the third kind method

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Abstract: In this paper, shifted Chebyshev polynomials of the third kind method is presented to solve numerically the Fredholm, Volterra-Hammerstein integral equations. The proposed method converts the equation system of linear or non-linear algebraic equations, which can be solved. Some numerical examples are included to demonstrate the validity and applicability of the proposed technique. All computations are done using Mathematica 7.

Keywords: Fredholm-Hammerstein integral equations, Volterra integral equation, shifted Chebyshev polynomials of the third kind method.

1 Introduction

Integral equations have proved itself as one of the most important branches of mathematics. Integral equations are one of the most useful mathematical tools in both pure and applied mathematics. The theory of integral equations has close contacts with many different areas of mathematics. Foremost among these are differential equations and operator theory. Many problems in the fields of ordinary and partial differential equations can be recast as integral equations ([1], [3], [4], [18]). Integral equations arise naturally in physics, chemistry, biology, engineering and many physical phenomena ([1], [5]). The principle investigators of the theory of integral equations are Vito Volterra (1860-1940) and Ivar Fredholm (1866-1927), together with David Hilbert (1862-1943) and Erhard Schmidt (1876-1959). There are several methods for approximating the solution of linear and non-linear integral equations ([10]-[14]).

In [6] the author using Legendre pseudo-spectral method to approximate and exact solutions of the fractional-order delay differential equations. In [7] the author using Legendre pseudo-spectral method for solving the fractional diffusion equation. In [9] the author using implementation of the operational matrix of fractional derivative for solving nonlinear multi-order fractional differential equations. In [16] the author using homotopy analysis method for solving the bi-harmonic equation.

The Chebyshev polynomials are mostly used to solve problems of differential equations or integral equations. Chebyshev polynomials are used to introduce an efficient medication of homotopy perturbation method [8]. Also, the polynomial approximation is used to solve high-order linear Fredholm integro-differential equations with constant coefficient [11] and others ([15], [17]).

We consider the Hammerstein integral equations in the forms [17]

$$x(t) = f(t) + \lambda_1 \int_0^1 k_1(t, s)F(x(s)) ds + \lambda_2 \int_0^t k_2(t, s)G(x(s)) ds \quad (1)$$

where $f(t)$, $k_1(t, s)$ and $k_2(t, s)$ are given functions, $t \leq 0, s \leq 1$ and λ_1, λ_2 are arbitrary constant.

2 Some Properties of Chebyshev Polynomials of the Third Kind

2.1 Chebyshev Polynomials of the Third Kind

The Chebyshev polynomials $V_n(x)$ of the third kind are orthogonal polynomials of degree n in x defined on $[-1, 1]$ [2, 20].

$$V_n = \frac{\cos(n+1/2)\Theta}{\cos\frac{\Theta}{2}},$$

where $x = \cos \Theta$ and $\Theta \in [0, \pi]$. The polynomials $V_n(x)$ are orthogonal on $[-1, 1]$ with respect to the inner product:

$$\langle V_n(x), V_m(x) \rangle = \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} V_n(x) V_m(x) dx = \begin{cases} \pi, & \text{for } n = m \\ 0, & \text{for } n \neq m \end{cases} \quad (2)$$

where $\sqrt{\frac{1+x}{1-x}}$ is weight function corresponding to $V_n(x)$. The polynomials $V_n(x)$ may be generated by using the recurrence relations

$$\begin{aligned} V_{n+1}(x) &= 2xV_n(x) - V_{n-1}(x), \\ V_0(x) &= 1, V_1(x) = 2x - 1, n = 1, 2, \dots \end{aligned}$$

The analytical form of the Chebyshev polynomials of the third kind $V_n(x)$ of degree n are given by

$$V_n(x) = \sum_{k=0}^{\lfloor \frac{2n+1}{2} \rfloor} (-1)^k (2)^{n-k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} (x+1)^{n-k}, n \in \mathbb{Z}^+, \quad (3)$$

where $\lfloor \frac{2n+1}{2} \rfloor$ denotes the integer part of $\frac{(2n+1)}{2}$.

2.2 The shifted Chebyshev polynomials of the third kind

In order to use these polynomials on the interval $[0, 1]$, we define the so called shifted Chebyshev polynomials of the third kind by the introducing the change of variable $V(x) = 2x - 1$ [20]. The shifted Chebyshev polynomials of the third kind are defined as $V_n^*(x) = V_n(2x - 1)$.

These polynomials are orthogonal on the support interval $[0, 1]$ as the following inner product

$$\langle V_n^*(x), V_m^*(x) \rangle = \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} V_n^*(x) V_m^*(x) dx = \begin{cases} \frac{\pi}{2}, & \text{for } n = m \\ 0, & \text{for } n \neq m \end{cases} \quad (4)$$

where $\sqrt{\frac{x}{1-x}}$ is the weight function corresponding to $V_n^*(x)$ and normalized by the requirement that $V_n^*(1) = 1$.

The polynomials $V_n^*(x)$ may be generated by using the recurrence relations

$$\begin{aligned} V_{n-1}^*(x) &= 2(2x-1)V_n^*(x) - V_{n-1}^*(x), \\ V_0^*(x) &= 1, V_1^*(x) = 4x-3, n = 1, 2, \dots \end{aligned}$$

The analytical form of the shifted Chebyshev polynomials of the third kind $V_n^*(x)$ of degree n in x given by:

$$V_n^*(x) = \sum_{k=0}^n (-1)^k 2^{2n-2k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} (x)^{n-k}, n \in \mathbb{Z}^+, \tag{5}$$

In the special method, the square integrable function $g(x)$ in $[0, 1]$, is represented by an infinite expansion of the shifted Chebyshev polynomials of the third kind as following:

$$g(x) = \sum_{i=0}^{\infty} b_i V_i^*(x), \tag{6}$$

where b_i is a chosen sequence of prescribed basis function. One then proceeds some how to estimate as many as possible of the coefficients b_i , thus approximating $g(x)$ by a finite sum of $(m+1)$ -terms such as:

$$g_m(x) = \sum_{i=0}^m b_i V_i^*(x), \tag{7}$$

where the coefficients $b_i, i = 0, 1, \dots$ are given by

$$b_i = \frac{1}{\pi} \int_{-1}^1 g\left(\frac{x+1}{2}\right) V_i(x) \sqrt{\frac{1+x}{1-x}} dx, \tag{8}$$

where the coefficients $b_i, i = 0, 1, \dots$ are given by

$$b_i = \frac{2}{\pi} \int_0^1 g(x) V_i^*(x) \sqrt{\frac{x}{1-x}} dx, \tag{9}$$

3 Procedure Solution Using the Proposed Numerical Method

We consider the Fredholm-Volterra integral equation (1). The function $x(t)$ may be expanded by infinite series of the shifted Chebyshev polynomials of the third kind as follows [17]:

$$x(t) = \sum_{n=0}^{\infty} c_n V_n^*(t), \tag{10}$$

where $c_n = (x(t), V_n^*(t))$. If we consider truncation series in Eq.(10), we obtain

$$\begin{aligned} x(t) &\simeq \sum_{n=0}^N c_n V_n^*(t) \\ &= c^T V^*(t), \end{aligned} \tag{11}$$

such that c and $V^*(t)$ are matrices given by

$$c = [c_0, c_1, \dots, c_N], \quad V^*(t) = [V_0^*(t), V_1^*(t), \dots, V_N^*(t)]^T. \quad (12)$$

Then, we substitute the approximation Eq.(11) into Eq.(1), we get

$$\sum_{n=0}^N c_n V_n^*(t) = f(t) + \lambda_1 \int_0^1 k_1(t, s) F(c^T V^*(s)) ds + \lambda_2 \int_0^t k_2(t, s) G(c^T V^*(s)) ds. \quad (13)$$

Now, to use the shifted Chebyshev polynomials of the third kind which is a matrix method based on the shifted Chebyshev polynomials points depended by

$$t_i = -1 + \frac{2}{N}i, \quad i = 0, 1, 2, \dots, N. \quad (14)$$

We collocate Eq.(13) with the points Eq.(14) to obtain

$$\sum_{n=0}^N c_n V_n^*(t_j) = f(t_j) + \lambda_1 \int_0^1 k_1(t_j, s) F(c^T V^*(s)) ds + \lambda_2 \int_0^{t_j} k_2(t_j, s) G(c^T V^*(s)) ds. \quad (15)$$

The integral term in Eq.(15) can be found using composite trapezoidal integration technique as.

$$\int_0^1 k_1(t_j, s) F(c^T V^*(s)) ds \cong \frac{h}{2} \left[\Omega_1(S_0) + \Omega_1(S_m) + 2 \sum_{k=1}^{m-1} \Omega_1(S_k) \right], \quad (16)$$

where $\Omega_1(S) = k_1(t_j, s) F(c^T V^*(s))$, $h = \frac{1}{m}$, for an arbitrary integer m , $S_i = ih$, $i = 0, 1, \dots, m$ and

$$\int_0^{t_j} k_2(t_j, s) G(c^T V^*(s)) ds \cong \frac{h_j}{2} \left[\Omega_2(\bar{S}_0) + \Omega_2(\bar{S}_m) + 2 \sum_{k=1}^{m-1} \Omega_2(\bar{S}_k) \right], \quad (17)$$

where $\Omega_2(S) = k_2(t_j, s) G(c^T V^*(s))$, $h_j = \frac{t_j}{m}$, for an arbitrary integer m , $\bar{S}_i = ih$, $i = 0, 1, \dots, m$. Eq.(15) gives $(N + 1)$ system of linear or non-linear algebraic equations, which can be solved for c_k , $k = 0, 1, \dots, N$. So the unknown function $x(t)$ can be found.

4 Numerical Implementation

In this section to a chive the validity, the accuracy and our theoretical discussion of the proposed method we give some computational results of numerical examples.

Example 1. Consider Eq.(1) with the following functions and coefficients [17]

$$f(t) = t^3 - (6 - 2e)e^t, \quad \lambda_1 = 1, \quad \lambda_2 = 0$$

$$k_1(t, s) = e^{t+s}, \quad k_2(t, s) = 0, \quad F(x(s)) = x(s), \quad G(x(s)) = 0$$

Eq.(1) takes the form

$$x(t) = t^3 - (6 - 2e)e^t + \int_0^1 e^{(s+t)} x(s) ds \quad (18)$$

We apply the suggested method with $N = 4$ and approximate the solution $x(t)$ as follows

$$x_N(t) = \sum_{i=0}^4 c_i V_i^*(t). \tag{19}$$

By the same procedure in the previous section and using Eq.(15) we have

$$\sum_{i=0}^4 c_i V_i^*(t_j) - \frac{h}{2} \left[\Omega(s_0) + \Omega(s_m) + 2 \sum_{k=1}^{m-1} \Omega(s_k) \right] = f(t_j), \quad j = 0, 1, 2, 3, 4 \tag{20}$$

where

$$\Omega(s) = e^{(s+t_j)} \sum_{i=0}^4 c_i V_i^*(s)$$

and the nodes $s_{i+1} = s_i + h, i = 0, 1, \dots, m, s_0 = 0$ and $h = \frac{1}{m}$. Eq.(20) represents linear system of $N + 1$ algebraic equations in the coefficients c_i .

By solving it using the conjugate gradient method, we obtain

$$c_0 = 0.70894, c_1 = 0.251995, c_2 = 0.03710, c_3 = 0.00195, c_4 = -4.0319 \times 10^{-12}.$$

Therefore, the approximate solution is given by

$$x_4(t) = \sum_{n=0}^4 c_n V_n^*(t) = 0.70894V_0^*(t) + 0.251995V_1^*(t) + 0.03710V_2^*(t) + 0.00195V_3^*(t) - 4.0319 \times 10^{-12}V_4^*(t)$$

The exact solution of this example is $x(t) = t^3$. The behavior of the approximate solution using the proposed method

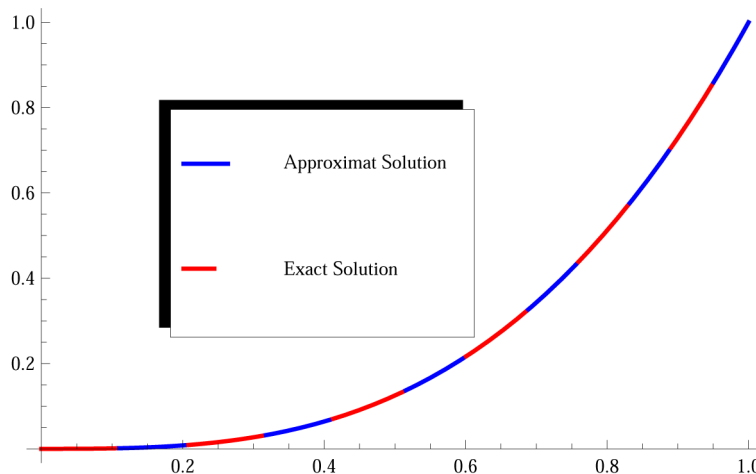


Fig. 1: The behavior of the exact solution and the approximate solution at $N = 4$.

with $N = 4$ and the exact solution in figure 1.

From this figure 1, it is clear that the proposed method can be considered as an efficient method to solve the linear integral equation.

Example 2. Consider the following integral equation [17]

$$x(t) = 2te^t - e^t + 1 - \int_0^t (s+t)e^{x(s)} ds, \quad (21)$$

where $f(t) = 2te^t - e^t + 1$, $k_1(t, s) = 0$, $k_2(t, s) = s + t$, $\lambda_1 = 0$, $\lambda_2 = -1$, $F(x(s)) = 0$ and $G(x(s)) = e^{x(s)}$. We apply the suggested method with $N = 4$ and approximate the solution $x(t)$ as follows

$$x_N(t) = \sum_{i=0}^4 c_i V_i^*(t) \quad (22)$$

By the same procedure in the previous section and using Eq.(15), we have

$$\sum_{i=0}^4 c_i V_i^*(t_j) + \frac{h_j}{2} \left[\Omega(s_0) + \Omega(s_m) + 2 \sum_{k=1}^{m-1} \Omega(s_k) \right] = f(t_j), \quad j = 0, 1, 2, 3, 4 \quad (23)$$

where the nodes $s_{i+1} = s_i + h$, $i = 0, 1, \dots, m$, $s_0 = 0$ and $h_j = \frac{t_j}{m}$.

$$\Omega(s) = (s + t_j) e^{\sum_{i=0}^4 c_i V_i^*(s)}$$

Eq.(23) presents non-linear system of $N + 1$ algebraic equation in the coefficients c_i . By solving it by using the Newton iteration method with suitable initial solution. We obtain

$$c_0 = 0.87499, c_1 = 0.125, c_2 = -2.349 \times 10^{-8}, c_3 = 2.736 \times 10^{-9}, c_4 = 6.610 \times 10^{-10}.$$

Therefore, the approximate solution as follows

$$x_4(t) = \sum_{i=0}^4 c_i V_i^*(t) = 0.87499V_0^*(t) + 0.125V_1^*(t) - 2.349 \times 10^{-8}V_2^*(t) + 2.736 \times 10^{-9}V_3^*(t) + 6.610 \times 10^{-10}V_4^*(t)$$

The exact solution of this example is $x(t) = t$. The behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution are present in in figure 2. From this figure 2, it is clear that the proposed method can be considered as an efficient method to solve the non-linear integral equation.

Example 3. Consider the following integral equation [17]

$$x(t) = te + 1 - \int_0^1 (s+t)e^{x(s)} ds,$$

The exact solution of this problem is $x(t) = t$.

We apply the suggested method with $N = 4$ and approximate the solution $x(t)$ as in Eq.(24) and the same procedure in the previous section and using Eq.(15), we have

$$\sum_{i=0}^3 c_i V_i^*(t_j) + \frac{h}{2} \left[\Omega(s_0) + \Omega(s_m) + 2 \sum_{k=1}^{m-1} \Omega(s_k) \right] = f(t_j), \quad j = 0, 1, 2, 3 \quad (24)$$

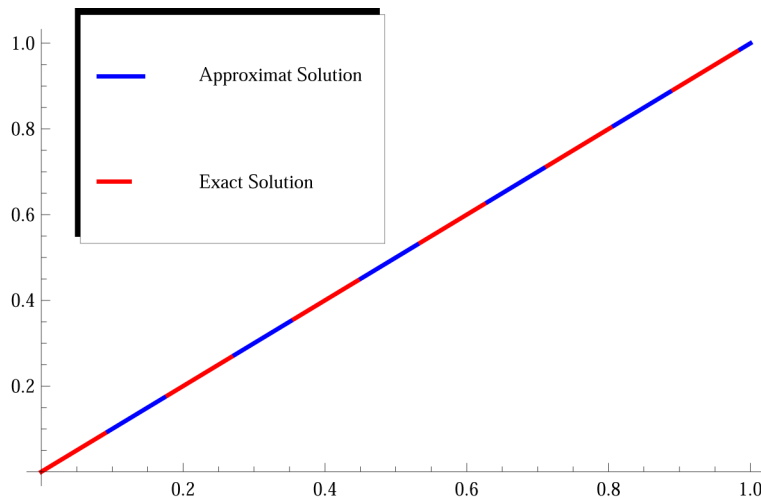


Fig. 2: The behavior of the exact solution and the approximate solution at $N = 4$.

where the nodes $s_{i+1} = s_i + h, i = 0, 1, \dots, m, s_0 = 0$ and $h = \frac{1}{m}$ and

$$\Omega(s) = (s + t_j)e^{\sum_{i=0}^3 c_j V_i^*(s)}$$

Eq.(25) presents non-linear system of algebraic. By solving it by using the well known Newton iteration method with suitable initial solution. we obtain

$$c_0 = 0.875, c_1 = 0.125, c_2 = -2.4120 \times 10^{-16}, c_3 = 7.167 \times 10^{-17}, c_4 = 3.527 \times 10^{-17}.$$

Therefore, the approximate solution of this example can be found using Eq.(24) as follows

$$x_4(t) = \sum_{n=0}^4 c_n V_n^*(t) = 0.875V_0^*(t) + 0.125V_1^*(t) - 2.4120 \times 10^{-16}V_2^*(t) + 7.167 \times 10^{-17}V_3^*(t) + 3.527 \times 10^{-17}V_4^*(t) \tag{25}$$

The exact solution of this example is $x(t) = t$. The behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution are present in in figure 3. From this figure 3, its is clear that the proposed method can be considered as an efficient methods to solve the non-linear integral equation.

Example 4. Consider the following integral equation [17]

$$x(t) = \frac{t}{2} - \frac{t^3}{6} - \frac{1}{3} + \int_0^1 (s+t)x(s) ds + \int_0^t (s-t)x(s) ds \tag{26}$$

The exact solution of this problem is $x(t) = t$. We apply the suggested method with $N = 4$ and approximate the solution $x(t)$ as follows

$$x_N(t) = \sum_{i=0}^4 c_i v_i^*(t) \tag{27}$$

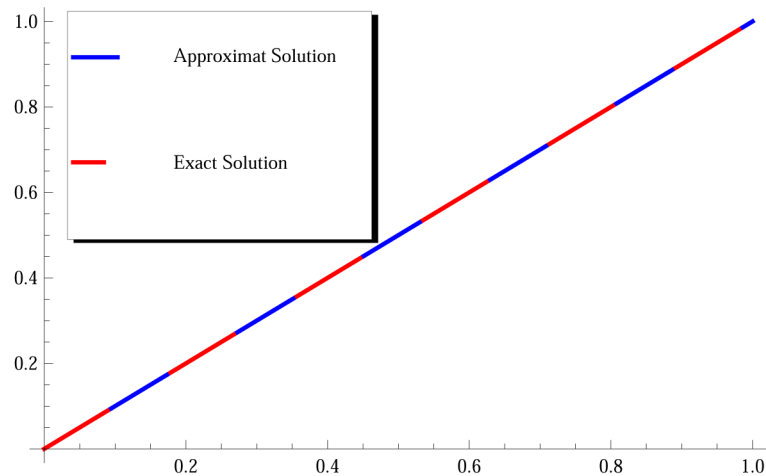


Fig. 3: The behavior of the exact solution and the approximate solution at $N = 4$.

By the same procedure in the previous section and using Eq.(15), we have

$$\sum_{i=0}^4 c_i V_i^*(t_j) - \frac{h}{2} \left[\Omega_1(\bar{s}_0) + \Omega_1(\bar{s}_m) + 2 \sum_{k=1}^{m-1} \Omega_1(\bar{s}_k) \right] - \frac{h_j}{2} \left[\Omega_2(\bar{s}_0) + \Omega_2(\bar{s}_m) + 2 \sum_{k=1}^{m-1} \Omega_2(\bar{s}_k) \right] = f(t_j), \quad j = 0, 1, 2, 3, 4 \quad (28)$$

where $\bar{s}_{i+1} = \bar{s}_i + h, s_{i+1} = s_i + h_j, i = 0, 1, 2, \dots, m, s_0 = \bar{s}_0 = 0, h = \frac{1}{m}, h_j = \frac{t_j}{m}$ and $\Omega_1(s) = (s+t_j) \sum_{i=0}^4 c_i V_i^*(s), \Omega_2(s) = (t_j - s) \sum_{i=0}^4 c_i V_i^*(s)$. Eq.(29) presents linear system $N + 1$ algebraic equations. By solving it using Newton iteration method, we obtain

$$c_0 = 0.875011, c_1 = 0.125, c_2 = 6.30883 \times 10^{-8}, c_3 = 8.3096 \times 10^{-11}, c_4 = 8.4650 \times 10^{-11}.$$

Therefore, the approximate solution as follows

$$x_4(t) = \sum_{n=0}^4 c_n V_n^*(t) = 0.875011 V_0^*(t) + 0.125 V_1^*(t) + 6.30883 \times 10^{-8} V_2^*(t) + 8.3096 \times 10^{-11} V_3^*(t) + 8.4650 \times 10^{-11} V_4^*(t)$$

The behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution are present in figure 4. From this figure 4, its is clear that the proposed method can be considered as an efficient methods to solve the linear Hammerstein integral equations.

Example 5. Consider the following integral equation [19]

$$x(t) = e^{t+1} - \int_0^1 (x^3(s)) e^{(t-2s)} ds, \quad (29)$$

where $f(t) = e^{t+1}$ and $k_2(t, s) = 0, k_1(t, s) = e^{(t-2s)}, \lambda_1 = -1, \lambda_2 = 0, F(x(s)) = x^3(s), G(x(s)) = 0$. It is and easy to verify that the exact solution of this problem is $x(t) = e^t$. We apply the suggested method with $N = 4$ and approximate the solution $x(t)$ as follows

$$x_4(t) = \sum_{i=0}^4 c_i V_i^*(t) \quad (30)$$

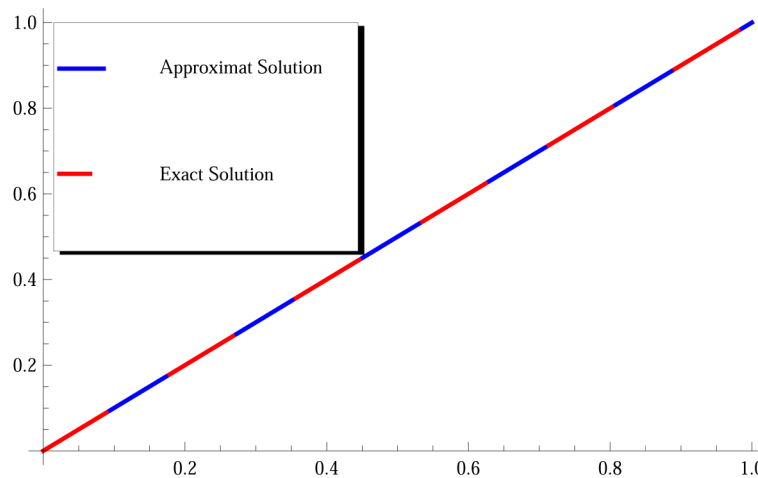


Fig. 4: The behavior of the exact solution and the approximate solution at $N = 4$.

By the same procedure in the previous section and using Eq.(15), we have

$$\sum_{i=0}^4 c_i V_i^*(t_j) + \frac{h}{2} \left[\Omega(s_0) + \Omega(s_m) + 2 \sum_{k=1}^{m-1} \Omega(s_k) \right] = f(t_j), \quad j = 0, 1, 2, 3, 4 \tag{31}$$

where the nodes $s_{i+1} = s_i + h, i = 0, 1, \dots, m, s_0 = 0$ and $h = \frac{1}{m}$.

$$\Omega(s) = e^{(t_j - 2s)} \left(\sum_{i=0}^4 c_i V_i^*(s) \right)^3$$

Eq.(32) presents non-linear system of $N + 1$ algebraic equation in the coefficients c_i . By solving it by using the Newton iteration method with suitable initial solution, we obtain

$$c_0 = 2.41690, c_1 = 0.283323, c_2 = 0.0173170, c_3 = 0.0007112644, c_4 = 0.0000216190.$$

Therefore, the approximate solution as follows

$$x_4(t) = \sum_{n=0}^4 c_n V_n^*(t) = 2.41690V_0^*(t) + 0.283323V_1^*(t) + 0.0173170V_2^*(t) + 0.0007112644V_3^*(t) + 0.0000216190V_4^*(t)$$

The behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution are present in in figure 5. From this figure 5, its is clear that the proposed method can be considered as an efficient method.

5 Conclusion

In this paper, we approximate method for the solution of linear and non-linear Fredholm and Volterra integral equations in the most general form has been proposed and investigated. The presented method which is based on the shifted Chebyshev polynomials of the third kind is proposed. A comparison of the exact solution reveals that the presented method is very effective and convenient. The numerical results show that the accuracy improves with increasing N , hence for better results, using number N is recommended. Also, from the obtained approximate solution, we can conclude that

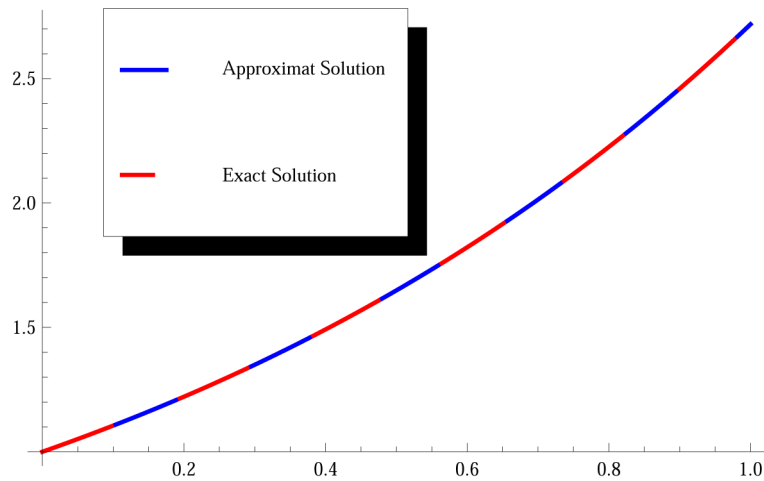


Fig. 5: The behavior of the exact solution and the approximate solution at $N = 4$.

Table 1: Comparison between Exact solution and approximate solution and error for Example 5

x	Exact solution	Approximate solution	Error
0	1	1	0
0.1	1.10517	1.10517	1.75803×10^{-6}
0.2	1.2142	1.2142	3.32845×10^{-7}
0.3	1.34986	1.34986	1.07348×10^{-6}
0.4	1.49182	1.49182	4.06538×10^{-7}
0.5	1.69872	1.69872	6.91612×10^{-8}
0.6	1.82212	1.82212	4.96547×10^{-7}
0.7	2.01375	2.01375	1.29584×10^{-6}
0.8	2.22554	2.22554	6.06484×10^{-7}
0.9	2.45966	2.45966	1.63799×10^{-6}
1	2.71828	2.71828	7.40761×10^{-7}

the proposed method gives the solution in an excellent agreement with the exact solution. All computations are done using Mathematica 7 programs.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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