New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.195

Fixed point approach to Basset problem

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Received: 31 January 2017, Accepted: 7 February 2017
Published online: 25 August 2017.

Abstract: In the present paper, a sufficient condition for existence and uniqueness of Basset problem is obtained. The theorem on existence and uniqueness is established. This approach permits us to use fixed point iteration method to solve problem for differential equation involving derivatives of nonlinear order.

Keywords: Fixed Point, initial Value Problem, Basset equation, Riemann-Liouville derivative.

1 Introduction

It is well known that differential equations involving derivatives of non-integer order are used in modelling of various physical phenomena in areas like diffusion processes, damping laws, etc. (see [1]-[7]). Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see [1]-[12] and references therein).

Let us give definitions of fractional derivatives and fractional powers of positive operators that will be needed below [11].

Definition 1. *If $x(t) \in C([a,b])$ and $a < x < b$, then*

$$I_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(t)}{(t-s)^{1-\alpha}} dt,$$

where $\alpha \in (-\infty, \infty)$ is called the Riemann-Liouville fractional integral of order α. In the same manner for $\alpha \in (0,1)$

$$D_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{x(s)}{(t-s)^{\alpha}} ds,$$

is called the Riemann-Liouville fractional derivative of order α.

Note that if $x(a) = 0$, then we can write

$$D_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{x'(s)}{(t-s)^{\alpha}} ds.$$

Here,

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} e^{-s} ds \qquad (\alpha > 0).$$

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Consider initial value problem for Basset equation

$$\begin{cases} \frac{dx(t)}{dt} + D_{a+}^{\alpha} x(t) = f(t, x(t)),\ 0 < t < T, \\ 0 < \alpha < 1, \\ x(0) = \varphi. \end{cases} \tag{1}$$

Actually, fractional differential equation corresponds to the Basset problem [3].

We now shortly describe the organization of the paper. In section 2, we give the basic definitions of fixed point, contraction and the basic concepts we need. In section 3, we obtain a sufficient condition for existence and uniqueness of problem and establish the theorem on existence and uniqueness.

2 Fixed point and contraction

Definition 2. *Let $E = (E, d)$ be a metric space. A fixed point of a mapping $A : E \longrightarrow E$ of set E into itself is an element $x \in E$ which is mapped onto itself, that is, $Ax = x$, the image Ax coincides with x.*

Definition 3. *A maping $A : E \longrightarrow E$ is called a contraction on E, if there is a positive real number $\alpha < 1$ such that for all $x, y \in E$*

$$d(Ax, Ay) \leq \alpha d(x, y).$$

Now, we state the existence uniqueness theorem the most important application of the Fixed-Point Theorem to ordinary differential equations. We will consider the initial value problem of the form

$$\frac{dx(t)}{dt} = f(t, x(t))\ ,\ |t - t_0| \leq a,\ x(t_0) = x_0.$$

The problem for ordinary differential equations will be converted to an integral equation, which defines a mapping A, and the conditions of the theorem will imply that A is a contraction such that its fixed point becomes the solution of problem.

Theorem 1. [13] *Assume that f is continuous on the rectangle*

$$D = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}$$

and thus bounded on D, i.e,

$$|f(t, x)| \leq c \text{ for all } (t, x) \in D.$$

Suppose that f satisfies a Lipschitz condition on D with respect to its second argument, that is, there is a constant l such that for $(t, x), (t, y) \in D$

$$|f(t, x) - f(t, y)| \leq l |x - y|.$$

Then, initial value problem has a unique solution x defined on $|t - t_0| \leq \beta$, where

$$\beta < \min\left\{a, \frac{b}{c}, \frac{1}{l}\right\}.$$

This function x is the limit of iterative sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recursive Picard iteration formula

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))\ ds,\ n \in \mathbb{N},$$

where $x_0(t)$ is an arbitrary continuous function. Error bounds are

$$d(x_n,x) \le \frac{\alpha^n}{1-\alpha} e^{2\alpha L} d(x_0,x_1),$$

$$d(x_n,x) \le \frac{\alpha}{1-\alpha} e^{2\alpha L} d(x_{n-1},x_n)\ ,\ n \in \mathbb{N}\ ,$$

where $\alpha = l\beta$.

3 Main results

Firstly, we defined the metric d on $C^1_{[0,T]}$ be the complete space of all continuously differentiable functions defined on the interval $[0,T]$.

Definition 4. *Let $C^1_{[0,T]}$ be the complete space of all continuously differentiable functions defined on the interval $[0,T]$ with the metric d defined by*

$$d(x,y) = \max_{t\in[0,T]} |x(t) - y(t)| + \max_{t\in[0,T]} \left| \frac{dx(t)}{dt} - \frac{dy(t)}{dt} \right| . \tag{2}$$

Now, we consider the following initial value problem for Basset equation

$$\frac{dx(t)}{dt} + D^{\alpha}_{0+} x(t) = f(t,x(t))\ ,\ 0<t<T,\ 0<\alpha<1,\ x(0) = \varphi. \tag{3}$$

In that case below theorem, we give sufficient condition for existence and uniqueness of this problem.

Theorem 2. *Let f be continuous function on the rectangle*

$$B = \{(t,x) :\ t \in [0,T]\} \subseteq \mathbb{R}^2,$$

and $|f(t,x)| \le c$ for all $(t,x) \in B$. Moreover, f satisfies a Lipschitz condition on B with respect to its second argument, i.e, there is a positive constant l such that for arbitrary $(t,x),(t,y) \in B$

$$|f(t,x) - f(t,y)| \le l\,|x-y| \tag{4}$$

is valid. Moreover, let

$$g(\alpha,T,l) = \frac{T^{-\alpha+2}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)} + Tl$$

and suppose that

$$g(\alpha,T,l) < 1. \tag{5}$$

Then, Basset problem (3) has a unique solution $x \in C_{[0,T]}$.

Proof. By integrating both sides of Basset equation (3),

$$\int_0^t \frac{dx(p)}{dt} dp + \int_0^t D^{\alpha}_0 x(p)\, dp = \int_0^t f(p,x(p))\, dp$$

$$x(p)|_0^t + \int_0^t D^{\alpha}_0 x(p)\, dp = \int_0^t f(p,x(p))\, dp$$

$$x(t)-x(0)+\int_0^t D_0^\alpha x(p)\,dp=\int_0^t f(p,x(p))\,dp$$

$$x(t)=-\int_0^t\left(\frac{1}{\Gamma(1-\alpha)}\int_0^p\frac{x'(s)}{(p-s)^\alpha}ds\right)dp+\int_0^t f(p,x(p))\,dp+\varphi.$$

Then we obtain integral equation,

$$x(t)=-\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^p(p-s)^{-\alpha}x'(s)\,ds\,dp+\int_0^t f(p,x(p))\,dp+\varphi. \tag{6}$$

This function x is the limit of the iterative sequence $\{x_n\}_{n=0}^\infty$ defined by the recursive Picard iteration formula

$$x_n(t)=-\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^p(p-s)^{-\alpha}x'_{n-1}(s)\,ds\,dp+\int_0^t f(p,x_{n-1}(p))\,dp+\varphi\text{ , }n\in\mathbb{N} \tag{7}$$

where $x_0(t)$ is an arbitrary continuous function. Error bounds are

$$d(x_n,x)\le\frac{g^n}{1-g}e^{gL}d(x_0,x_1),$$

$$d(x_n,x)\le\frac{g}{1-g}e^{gL}d(x_{n-1},x_n)\text{ , }n\in\mathbb{N}$$

where $\alpha=\frac{l}{L}$. Here, L is a fixed number such that $L>l$. We see that initial value problem (3) can be written in the equivalent integral form (6), which is in the form $x=Ax$, where $A:C^1_{[0,T]}\longrightarrow C^1_{[0,T]}$ is an operator defined by

$$Ax(t)=-\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^p(p-s)^{-\alpha}x'(s)\,ds\,dp+\int_0^t f(p,x(p))\,dp+\varphi, \tag{8}$$

where f is continuous function on the rectangle B. Since,

$$Ax(t)=-\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^p(p-s)^{-\alpha}x'(s)\,ds\,dp+\int_0^t f(p,x(p))\,dp+\varphi$$

under assumptions of theorem, by using (2), we find

$$\begin{aligned}|Ax(t)|&=\left|-\frac{1}{\Gamma(1-\alpha)}\int_0^t\left(\int_0^p(p-s)^{-\alpha}x'(s)\,ds\right)dp+\int_0^t f(p,x(p))\,dp+\varphi\right|\\&\le\frac{1}{|\Gamma(1-\alpha)|}\left|\int_0^t\int_0^p(p-s)^{-\alpha}x'(s)\,ds\,dp\right|+\left|\int_0^t f(p,x(p))\,dp\right|+|\varphi|\\&\le\frac{1}{\Gamma(1-\alpha)}\int_0^t\left(\int_0^p(p-s)^{-\alpha}x'(s)\,ds\right)dp+\int_0^t|f(p,x(p))|\,dp+|\varphi|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(1-\alpha)}\int_0^t\left(\int_0^p (p-s)^{-\alpha}\max_{s\in[0,T]}\left|x'(s)\right|\,ds\right)dp+\int_0^t c\,dp+|\varphi| \\
&= \frac{\max_{s\in[0,T]}\left|x'(s)\right|}{\Gamma(1-\alpha)}\int_0^t\left(\left.\frac{-(p-s)^{-\alpha+1}}{-\alpha+1}\right|_0^p\right)dp\;+\;ct+|\varphi| \\
&= \frac{\max_{s\in[0,T]}\left|x'(s)\right|}{\Gamma(1-\alpha)}\left(\int_0^t\frac{p^{-\alpha+1}}{-\alpha+1}\,dp\right)+ct+|\varphi| \\
&= \frac{\max_{s\in[0,T]}\left|x'(s)\right|}{\Gamma(1-\alpha)}\left(\left.\frac{p^{-\alpha+2}}{(-\alpha+1)(-\alpha+2)}\right|_0^t\right)+ct+|\varphi| \\
&= \frac{\max_{s\in[0,T]}\left|x'(s)\right|}{\Gamma(1-\alpha)}\frac{t^{-\alpha+2}}{(-\alpha+1)(-\alpha+2)}+ct+|\varphi|\,.
\end{aligned}
$$

Thus, we have show that $Ax\in C^1[0,T]$ if $x\in C^1[0,T]$; i.e., A maps the set $C^1[0,T]$ itself. Now, we show that, A is a contraction map on $C^1_{[0,T]}$. Applying the Lipschitz condition (4), we get

$$
\begin{aligned}
|Ax(t)-Ay(t)| = &\left|-\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^p (p-s)^{-\alpha}x'(s)\,ds\,dp+\int_0^t f(p,x(p))\,dp\right. \\
&\left.+\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^p (p-s)^{-\alpha}y'(s)\,ds\,dp-\int_0^t f(p,y(p))\,dp\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)}\left|\int_0^t\int_0^p (p-s)^{-\alpha}\left(x'(s)-y'(s)\right)\,ds\,dp\right|+\int_0^t\left|f(p,x(p))-f(p,y(p))\right|\,dp \\
&\leq \frac{1}{\Gamma(1-\alpha)}\max_{s\in[0,T]}\left|x'(s)-y'(s)\right|\int_0^t\int_0^p (p-s)^{-\alpha}\,ds\,dp+tl\,|x(s)-y(s)| \\
&= \frac{1}{\Gamma(1-\alpha)}\max_{s\in[0,T]}\left|x'(s)-y'(s)\right|\,\frac{t^{-\alpha+2}}{(-\alpha+1)(-\alpha+2)}+tl\,|x(s)-y(s)| \\
&\leq \frac{t^{-\alpha+2}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)}\,d(x,y)+tl\,d(x,y) \\
&= \left(\frac{t^{-\alpha+2}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)}+tl\right)d(x,y)\,.
\end{aligned}
$$

Taking maximum from both sides we have

$$
\begin{aligned}
d(Ax,Ay) &\leq \max_{t\in[0,T]}\left(\frac{t^{-\alpha+2}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)}+tl\right)d(x,y) \\
&= \left(\frac{T^{-\alpha+2}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)}+Tl\right)d(x,y) \\
&\leq g(\alpha,T,l)\,d(x,y)
\end{aligned}
$$

From (4) we see that $g(\alpha,T,l)<1$, so

$$
d(Ax,Ay)\leq d(x,y)\,.
$$

Thus, A is a contraction on $C^1_{[0,T]}$. Therefore, A has a unique fixed point $x \in C^1_{[0,T]}$, that is, a continuous function on $[0,T]$ satisfying $x = Ax$. By (7), we have

$$x(t) = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^p (p-s)^{-\alpha} x'(s)\, ds\, dp + \int_0^t f(p, x(p))\, dp + \varphi. \tag{9}$$

Example 1. Solve initial value problem for Basset equation

$$\frac{dx(t)}{dt} + D^{\alpha}_{0+} x(t) = 1 + \frac{2}{\sqrt{\pi}} \sqrt{x(t)}\,, \quad 0 < t < T,\ x(0) = 0 \tag{10}$$

by the iteration method.

Solution 1. By integrating both sides of Basset equation (10),

$$x(t) = -\frac{1}{\sqrt{\pi}} \int_0^t \int_0^p (p-s)^{-\alpha} x'(s)\, ds\, dp + \int_0^t \left(1 + \frac{2}{\sqrt{\pi}} \sqrt{x(p)}\right) dp$$

Since $f(t, x(t)) = 1 + \frac{2}{\sqrt{\pi}} \sqrt{x(t)}$, $0 < t < T$, $\alpha = \frac{1}{2}$, $\varphi = 0$, we have that $l = \frac{1}{\sqrt{\pi}}$. Let $x_0(t) = 0$. Now, $x_n(t)$ is defined by formula

$$x_n(t) = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^p (p-s)^{-\alpha} x'_{n-1}(s)\, ds\, dp + \int_0^t f(p, x_{n-1}(p))\, dp + \varphi, \qquad n \in \mathbb{N}$$

Then,

$$\begin{aligned} x_1(t) &= -\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \int_0^p (p-s)^{-\frac{1}{2}} x'_0(s)\, ds\, dp + \int_0^t f(p, x_0(p))\, dp + \varphi \\ &= -\frac{1}{\sqrt{\pi}} \int_0^t \int_0^p (p-s)^{-\frac{1}{2}}\, 0\, ds\, dp + \int_0^t \left(1 + \frac{2}{\sqrt{\pi}} \sqrt{0}\right) dp + 0 = t, \end{aligned}$$

$$\begin{aligned} x_2(t) &= -\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \int_0^p (p-s)^{-\frac{1}{2}} x'_1(s)\, ds\, dp + \int_0^t f(p, x_1(p))\, dp + \varphi \\ &= -\frac{1}{\sqrt{\pi}} \int_0^t \int_0^p (p-s)^{-\frac{1}{2}}\, ds\, dp + \int_0^t \left(1 + \frac{2}{\sqrt{\pi}} p^{\frac{1}{2}}\right) dp \\ &= -\frac{1}{\sqrt{\pi}} \int_0^t \left(-2(p-s)^{\frac{1}{2}} \Big|_0^p\right) dp + \left(p + \frac{2}{\sqrt{\pi}} \frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right) \Bigg|_0^t = t. \end{aligned}$$

$$\begin{aligned} x_3(t) &= -\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \int_0^p (p-s)^{-\frac{1}{2}} x'_2(s)\, ds\, dp + \int_0^t f(p, x_2(p))\, dp + \varphi \\ &= \frac{1}{\sqrt{\pi}} \int_0^t \int_0^p (p-s)^{-\frac{1}{2}}\, ds\, dp + \int_0^t \left(1 + \frac{2}{\sqrt{\pi}} p^{\frac{1}{2}}\right) dp = t. \end{aligned}$$

In similar manner, it can be showed

$$x_n(t) = t, \quad n \in \mathbb{N}$$

Hence,

$$x(t) = \lim_{n \longrightarrow \infty} x_n(t) = \lim_{n \longrightarrow \infty} t = t.$$

4 Conclusion

In this work, we consider initial value problem for Basset equation. We obtain a sufficient condition for existence and uniqueness of this problem and establish the theorem on existence and uniqueness. This approach permits us to use fixed point iteration method to solve problem for differential equation involving derivatives of nonlinear order.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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