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A NEW RESULT FOR WEIGHTED ARITHMETIC MEAN SUMMABILITY FACTORS OF INFINITE SERIES INVOLVING ALMOST INCREASING SEQUENCES

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Abstract. In this paper, a known theorem dealing with weighted mean summability methods of non-decreasing sequences has been generalized for $|A, p_n; \delta|_k$ summability factors of almost increasing sequences. Also, some new results have been obtained concerning $\left|\bar{N}, p_n\right|_k$, $\left|\bar{N}, p_n; \delta\right|_k$ and $|C, 1; \delta|_k$ summability factors.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). We denote u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [9]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

A series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [10]),

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{3}$$

If we take $\delta = 0$, then we have $|C, \alpha|_k$ summability (see [12]).
Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

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The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (w_n) of the weighted arithmetic mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]). The $(\bar{N}, p_n)$ mean of (s_n) reduces to the Cesàro mean $(C, 1)$ when $(p_n) = 1$; to the logarithmic mean $(\ell, 1)$ when $(p_n) = \frac{1}{n+1}$ [17]. $(\bar{N}, p_n)$ means were used in many applications of summability theory such as Tauberian and Korovkin type- theorems (see e.g. [18], [19] and [2]).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5]),

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid \Delta w_{n-1} \mid^k < \infty. \tag{6}$$

where

$$\Delta w_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{7}$$

In the special case if we take $\delta = 0$, we have $|\bar{N}, p_n|_k$ summability (see [3]). When $p_n = 1$ for all values of n, $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ summability. Also if we take $\delta = 0$ and $k = 1$, then we have $|\bar{N}, p_n|$ summability.

Let $A = (a_{nv})$ be a normal matrix. i.e., a lower triangular matrix of nonzero diagonal entries. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{8}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{9}$$

Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{10}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv}\sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv}$$

$$= \sum_{i=0}^{n} a_i \bar{a}_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} a_v. \tag{11}$$

Since $\bar{a}_{n-1,n} = \sum_{i=n}^{n-1} a_{n-1,i} = 0$,

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \bar{a}_{nv} a_v - \sum_{v=0}^{n-1} \bar{a}_{n-1,v} a_v$$

$$= \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) a_v + \bar{a}_{n-1,n} a_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{12}$$

The series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [16])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \bar{\Delta} A_n(s) \right|^k < \infty \tag{13}$$

where

$$\Delta A_n(s) = A_n(s) - A_{n+1}(s), \quad \textit{and} \quad \bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

By a weighted mean matrix we state

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq \text{n} \\ 0 & v > n, \end{cases}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$.

If we take $\delta = 0$, then $|A, p_n; \delta|_k$ summability is the same as $|A, p_n|_k$ summability (see [20]) and if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ summability. Also, if we take $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then $|A, p_n; \delta|_k$ summability is the same as $|C, 1|_k$ summability.

2. The Known Results

Quite recently, Bor has proved the following theorems concerning on weighted arithmetic mean summability factors of infinite series.

Theorem 1. [4] *Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that*

$$|\Delta \lambda_n| \leq \beta_n, \tag{14}$$

$$\beta_n \to 0 \quad \textit{as} \quad n \to \infty \tag{15}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \tag{16}$$

$$|\lambda_n| X_n = O(1). \tag{17}$$

If

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{18}$$

and (p_n) *is a sequence that*

$$P_n = O(np_n), \tag{19}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{20}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem 2. [6] *Let* (X_n) *be a positive non-decreasing sequence. If the sequences* (X_n), (β_n), (λ_n), (p_n) *satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, and*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{21}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left((\frac{P_v}{p_v})^{\delta k} \frac{1}{P_v}\right) \quad as \quad m \to \infty, \tag{22}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Theorem 3. [7] *Let* (X_n) *be a positive non-decreasing sequence. If the sequences* (X_n), (β_n), (λ_n), *and* (p_n) *satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, and*

$$\sum_{n=1}^{m} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{23}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Theorem 4. [7] *Let* (X_n) *be a positive non-decreasing sequence. If the sequences* (X_n), (β_n), (λ_n), *and* (p_n) *satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, and*

$$\sum_{n=1}^{m} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{24}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem 5. [8] *Let* (X_n) *be a positive non-decreasing sequence. If the sequences* (X_n) , (β_n), (λ_n), *and* (p_n) *satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, condition (22) of Theorem 2, and*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{25}$$

then the series $\sum a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $|\bar{N}, p_n; \delta|_k$, $k \geq 1$, $0 \leq \delta < 1/k$.

We need the following lemmas.

Lemma 6. [13] *Under the conditions on* (X_n), (β_n), *and* (λ_n) *as expressed in the statement of Theorem 1, we have the following:*

$$nX_n\beta_n = O(1), \tag{26}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{27}$$

Lemma 7. [15] *If the conditions (19) and (20) of Theorem 1 are satisfied, then* $\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right)$.

Remark 8. *Under the conditions on the sequence* (λ_n) *of Theorem 1, we have that* (λ_n) *is bounded and* $\Delta\lambda_n = O(1/n)$ *(see* [4]*).*

3. The Main Results

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (z_n) and two positive constants C and B such that $Cz_n \leq b_n \leq Bz_n$ (see [1]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. In this paper we generalize Theorem 5 to $|A, p_n; \delta|_k$ summability method using almost increasing sequences and normal matrix instead of non-decreasing sequences and weighted mean matrix, respectively. The following our main theorem is generalized the above results concerning $|\bar{N}, p_n|_k$ and $|\bar{N}, p_n; \delta|_k$ summability methods.

Theorem 9. [22] *Let* $k \geq 1$ *and* $0 \leq \delta < 1/k$. *Let* $A = (a_{nv})$ *be a positive normal matrix such that*

$$\overline{a}_{n0} = 1,\ n = 0, 1, ..., \tag{28}$$

$$a_{n-1,v} \geq a_{nv},\ \ for\ n \geq v+1, \tag{29}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{30}$$

$$\sum_{v=1}^{n-1} a_{vv}\hat{a}_{n,v+1} = O(a_{nn}), \tag{31}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\} \quad as \quad m \to \infty, \tag{32}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad as \quad m \to \infty. \tag{33}$$

Let (X_n) be an almost increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy all the conditions of Theorem 5, then the series $\sum a_n \frac{P_n\lambda_n}{np_n}$ is summable $|A, p_n; \delta|_k$, $k \geq 1$, $0 \leq \delta < 1/k$.

4. Proof of Theorem 9

Proof. Let (V_n) denotes the A-transform of the series $\sum a_n \frac{P_n\lambda_n}{np_n}$. Then, by the definition, we have that

$$\bar{\Delta}V_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \frac{P_v\lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}P_v\lambda_v}{vp_v}\right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn}P_n\lambda_n}{np_n} \sum_{r=1}^{n} a_r$$

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}P_v\lambda_v}{vp_v}\right) s_v + \frac{\hat{a}_{nn}P_n\lambda_n}{np_n} s_n,$$

$$\bar{\Delta}V_n = \frac{a_{nn}P_n\lambda_n}{np_n} s_n + \sum_{v=1}^{n-1} \frac{P_v\lambda_v}{vp_v} \Delta_v(\hat{a}_{nv}) s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_v \Delta\left(\frac{P_v}{vp_v}\right) s_v$$
$$+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_{v+1}}{(v+1)p_{v+1}} \Delta\lambda_v s_v$$
$$\bar{\Delta}V_n = V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.$$

To complete the proof of Theorem 9, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |V_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\, V_{n,1} \,|^k \leq \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^k \left(\frac{P_n}{p_n}\right)^k |\lambda_n|^k \frac{|s_n|^k}{n^k}$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\lambda_n|^k \frac{|s_n|^k}{n^k} = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{n^{k-1}}{n^k} |\lambda_n|^{k-1} |\lambda_n||s_n|^k$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{1}{n} \frac{1}{X_n^{k-1}} |\lambda_n||s_n|^k$$
$$= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|s_v|^k}{vX_v^{k-1}} + O(1)|\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{nX_n^{k-1}}$$

$$= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m = O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m = O(1)$$

as $m \to \infty$.By applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$ and as in $V_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid V_{n,2} \mid^k = \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|\sum_{v=1}^{n-1} \frac{P_v \lambda_v}{v p_v} \Delta_v(\hat{a}_{nv}) s_v\right|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |s_v|^k \frac{1}{v^k} \left(\frac{P_v}{p_v}\right)^k\right\} \times \left\{\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |s_v|^k \frac{1}{v^k} \left(\frac{P_v}{p_v}\right)^k$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} |\lambda_v|^{k-1} |\lambda_v||s_v|^k \frac{1}{v^k} \left(\frac{P_v}{p_v}\right)^k \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})|$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} a_{vv} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v^k} \left(\frac{P_v}{p_v}\right)^k$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v^k} \left(\frac{P_v}{p_v}\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v^k} v^{k-1}$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v} = O(1) \quad as \quad m \to \infty.$$

Also, by using conditions of Theorem 9, we obtain that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid V_{n,3} \mid^k = \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|\sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta\left(\frac{P_v}{v p_v}\right) \lambda_v s_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{\sum_{v=1}^{n-1} a_{vv}^{1-k} \hat{a}_{n,v+1} |\lambda_v|^k |s_v|^k \frac{1}{v^k}\right\} \times \left\{\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1}\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^{k-1} \hat{a}_{n,v+1} |\lambda_v|^k |s_v|^k \frac{1}{v^k}$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} |\lambda_v|^{k-1} |\lambda_v||s_v|^k \frac{1}{v} \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v} = O(1) \quad as \quad m\to\infty.$$

Finally, by virtue of the hypotheses of Theorem 9, by Lemma 6, we have $v\beta_v = O(\frac{1}{X_v})$, then

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid V_{n,4}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\frac{P_{v+1}}{(v+1)p_{v+1}}\Delta\lambda_v s_v\right|^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}a_{vv}^{1-k}\hat{a}_{n,v+1}|\Delta\lambda_v|^k s_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}\hat{a}_{n,v+1}\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}a_{vv}^{1-k}\hat{a}_{n,v+1}|\Delta\lambda_v|^k|s_v|^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^{k-1}|s_v|^k|\Delta\lambda_v|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|s_v|^k(v\beta_v)^{k-1}\beta_v = O(1)\sum_{v=1}^{m}v\beta_v|s_v|^k\frac{1}{vX_v^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|s_r|^k}{rX_r^{k-1}} + O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{vX_v^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m = O(1)\sum_{v=1}^{m-1}|(v+1)\Delta\beta_v - \beta_v|X_v$$
$$+ O(1)m\beta_m X_m$$
$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}X_v\beta_v + O(1)m\beta_m X_m = O(1) \quad as \quad m\to\infty.$$

□

This completes the proof of Theorem 9.

Conclusion 10. *If we take $\delta = 0$ in Theorem 9, then Theorem 9 reduces to $|A, p_n|_k$ summability theorem (see* [21]*).*
Let (X_n) be a positive non-decreasing sequence. The following results have been obtained.
1. If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 9, then Theorem 9 reduces to Theorem 5.
2. If we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 9, then we obtain Theorem 4 and

if we put $\delta = 0$ and $k = 1$ in Theorem 5, we have a known result of Mishra and Srivastava dealing with $\left|\bar{N}, p_n\right|$ summability factors of infinite series (see [15]*).*
3. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 9, then we obtain a known result of Mishra and Srivastava concerning the $|C, 1; \delta|_k$ summability factors of infinite series.
4. If we take $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 9, then we obtain a known result of Mishra and Srivastava concerning the $|C, 1|_k$ summability factors of infinite series (see [14]*).*

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