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Research Article

Weighted boundedness for Toeplitz type operator associated to singular integral operator with variable Calderón-Zygmund kernel

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Abstract

In this paper, we establish the weighted sharp maximal function inequalities for the Toeplitz type operator associated to the singular integral operator with variable Calderón-Zygmund kernel. As an application, we obtain the boundedness of the operator on weighted Lebesgue and Morrey spaces.

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1. Introduction and preliminaries

As the development of singular integral operators(see [9, 30]), their commutators have been well studied. In [5, 28, 29], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [3]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [12, 25], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1, 11], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained (also see [10]). In [2], Calderón and Zygmund introduce some singular integral operators with variable kernel and discuss their boundedness. In [17–19, 31], the authors obtain the boundedness for the commutators generated by the singular integral operators with variable kernel and BMO functions. In [21], the authors prove the boundedness for the multilinear oscillatory singular integral operators generated by the operators and BMO functions. In [14, 15, 20], some Toeplitz type operators associated to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained.

On the other hand, the classical Morrey space was introduced by Morrey in [23] to investigate the local behavior of solutions to second order elliptic partial differential equations (also see [24]). As the Morrey space may be considered as an extension of the Lebesgue

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space, it is natural and important to study the boundedness of operator on the Morrey spaces. The boundedness of the maximal operator, the singular integral operator, the fractional integral operator and their commutators on Morrey spaces have been studied by many authors (see [6,7,13,16,22]). In [16], Komori and Shirai studied the boundedness of these operators on weighted Morrey spaces.

Motivated by these, in this paper, we will study the Toeplitz type operator generated by the singular integral operator with variable Calderón-Zygmund kernel and the weighted Lipschitz and BMO functions.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [9,30])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta^{\#}(f)(x) = M^{\#}(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$ and $1 \le p < \infty$, set

$$M_{\eta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\eta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

For $0 < \eta < n$, $1 \le p < \infty$ and the non-negative weight function w, set

$$M_{\eta,p,w}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y) dy \right)^{1/p}.$$

We write $M_{\eta,p,w}(f) = M_{p,w}(f)$ if $\eta = 0$.

The A_p weight is defined by (see [9])

$$A_p = \left\{ 0 < w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{0 < w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e.\}.$$

Given a non-negative weight function w. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions b such that

$$||b||_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} dy \right)^{1/p} < \infty,$$

and the weighted BMO space $BMO(w)$ is the space of functions b such that

$$||b||_{BMO(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} dy \right)^{1/p} < \infty.$$

We write $BMO(w) = BMO(R^n)$ if $w = 0$.

Remark.(1) It has been known that (see [8]), for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck||b||_{Lip_\beta(w)} w(x) w(2^kQ)^{\beta/n}.$$

(2) It has been known that(see [9]), for $b \in BMO(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck||b||_{BMO(w)} w(x).$$

(3) Let $b \in Lip_\beta(w)$ or $b \in BMO(w)$ and $w \in A_1$. By [8,9], we know that spaces $Lip_\beta(w)$ or $BMO(w)$ coincide and the norms $||b||_{Lip_\beta(w)}$ or $||b||_{BMO(w)}$ are equivalent with respect to different values $1 \le p < \infty$.

Definition 1.1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for} \quad t \ge 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1 \le p < \infty$,

$$||f||_{L^{p,\eta,\varphi}(w)} = \sup_{x \in R^n,\ d>0} \left(\frac{1}{\varphi(d)^{1-p\eta/n}} \int_{Q(x,d)} |f(y)|^p w(y) dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\eta,\varphi}(R^n, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\eta,\varphi}(w)} < \infty\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey spaces (see [26,27]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue spaces (see [9]).

In this paper, we will study some singular integral operators as following (see [2]).

Definition 1.2. Let $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel if
(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;
(b) Ω is homogeneous of degree zero;
(c) $\int_\Sigma \Omega(x) x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition 1.3. Let $K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if
(d) $K(x, \cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;
(e) $\max_{|\gamma| \le 2n} \left|\left| \frac{\partial^{|\gamma|}}{\partial^\gamma y} \Omega(x, y) \right|\right|_{L^\infty(R^n \times \Sigma)} = L < \infty$.

Moreover, let b be a locally integrable function on R^n and T be the singular integral operator with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where $K(x, x-y) = \frac{\Omega(x,x-y)}{|x-y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernel. The Toeplitz type operator associated to T is defined by

$$T_b = \sum_{k=1}^m T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are the singular integral operator with variable Calderón-Zygmund kernel T or $\pm I$(the identity operator), $T^{k,2}$ are the linear operators, $k = 1, ..., m$, $M_b(f) = bf$.

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operator T_b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [28, 29]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator T_b. As the application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the Toeplitz type operator T_b.

2. Theorems and lemmas

We shall prove the following theorems.

Theorem 2.1. *Let T be the singular integral operator as in* **Definition 1.3**, $w \in A_1$, $0 < \eta < 1$, $1 < s < \infty$, $0 < \beta < 1$ *and* $b \in Lip_\beta(w)$. *If* $T_1(g) = 0$ *for any* $g \in L^r(R^n)(1 < r < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_\eta^\#(T_b(f))(\tilde{x}) \le C||b||_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).$$

Theorem 2.2. *Let T be the singular integral operator as in* **Definition 1.3**, $w \in A_1$, $0 < \eta < 1$, $1 < s < \infty$ *and* $b \in BMO(w)$. *If* $T_1(g) = 0$ *for any* $g \in L^r(R^n)(1 < r < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_\eta^\#(T_b(f))(\tilde{x}) \le C||b||_{BMO(w)} w(\tilde{x}) \sum_{k=1}^m M_{s,w}(T^{k,2}(f))(\tilde{x}).$$

Theorem 2.3. *Let T be the singular integral operator as in* **Definition 1.3**, $w \in A_1$, $0 < \beta < 1$, $b \in Lip_\beta(w)$, $1 < p < n/\beta$ *and* $1/q = 1/p - \beta/n$. *If* $T_1(g) = 0$ *for any* $g \in L^r(R^n)(1 < r < \infty)$ *and* $T^{k,2}$ *are the bounded linear operators on* $L^p(R^n, w)$ *for* $1 < p < \infty$ *and* $w \in A_1 (1 \le k \le m)$, *then* T_b *is bounded from* $L^p(R^n, w)$ *to* $L^q(R^n, w^{1-q})$.

Theorem 2.4. *Let T be the singular integral operator as in* **Definition 1.3**, $w \in A_1$, $0 < \eta < 1$, $0 < \beta < 1$ *and* $b \in Lip_\beta(w)$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $0 < D < 2^n$. *If* $T_1(g) = 0$ *for any* $g \in L^r(R^n)(1 < r < \infty)$ *and* $T^{k,2}$ *are the bounded linear operators on* $L^{p,\varphi}(R^n, w)$ *for* $1 < p < \infty$ *and* $w \in A_1 (1 \le k \le m)$, *then* T_b *is bounded from* $L^{p,\beta,\varphi}(R^n, w)$ *to* $L^{q,\varphi}(R^n, w^{1-q})$.

Theorem 2.5. *Let T be the singular integral operator as in* **Definition 1.3**, $1 < p < \infty$ *and* $b \in BMO(w)$. *If* $T_1(g) = 0$ *for any* $g \in L^r(R^n)(1 < r < \infty)$ *and* $T^{k,2}$ *are the bounded operators on* $L^p(R^n, w)$ *for* $1 < p < \infty$ *and* $w \in A_1 (1 \le k \le m)$, *then* T_b *is bounded from* $L^p(R^n, w)$ *to* $L^p(R^n, w^{1-p})$.

Theorem 2.6. *Let T be the singular integral operator as in* **Definition 1.3**, $0 < D < 2^n$, $1 < p < \infty$ *and* $b \in BMO(w)$. *If* $T_1(g) = 0$ *for any* $g \in L^r(R^n)(1 < r < \infty)$ *and* $T^{k,2}$ *are the bounded operators on* $L^{p,\varphi}(R^n, w)$ *for* $1 < p < \infty$ *and* $w \in A_1 (1 \le k \le m)$, *then* T_b *is bounded from* $L^{p,\varphi}(R^n, w)$ *to* $L^{p,\varphi}(R^n, w^{1-p})$.

To prove the theorems, we need the following lemmas.

Lemma 2.7. *(see [9, p.485]) Let* $0 < p < q < \infty$ *and for any function* $f \ge 0$. *We define that, for* $1/r = 1/p - 1/q$,

$$||f||_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \qquad N_{p,q}(f) = \sup_Q ||f\chi_Q||_{L^p}/||\chi_Q||_{L^r},$$

where the sup is taken for all measurable sets Q *with* $0 < |Q| < \infty$. *Then*

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 2.8. *(see [2]) Let T be the singular integral operator as* **Definition 1.3**. *Then T is bounded on $L^p(R^n, w)$ for $w \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.*

Lemma 2.9. *(see [8, 9]) Let $0 \le \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then*

$$||M_{\eta,s,w}(f)||_{L^q(w)} \le C||f||_{L^p(w)}.$$

Lemma 2.10. *(see [9]) Let $0 < p, \eta < \infty$ and $w \in \cup_{1\le r<\infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p w(x)dx \le C \int_{R^n} M_\eta^\#(f)(x)^p w(x)dx.$$

Lemma 2.11. *Let $0 < p < \infty$, $0 < \eta < \infty$, $0 < D < 2^n$ and $w \in A_1$. Then, for any smooth function f for which the left-hand side is finite,*

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \le C||M_\eta^\#(f)||_{L^{p,\varphi}(w)}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ by [4]. By Lemma 2.10, we have, for $f \in L^{p,\varphi}(R^n, w)$,

$$\begin{aligned}
&\int_Q |M_\eta(f)(y)|^p w(y)dy \\
= &\int_{R^n} |M_\eta(f)(y)|^p w(y)\chi_Q(y)dy \\
\le &\int_{R^n} |M_\eta(f)(y)|^p M(w\chi_Q)(y)dy \\
\le &\ C\int_{R^n} |M_\eta^\#(f)(y)|^p M(w\chi_Q)(y)dy \\
= &\ C\left(\int_Q |M_\eta^\#(f)(y)|^p M(w\chi_Q)(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M_\eta^\#(f)(y)|^p M(w\chi_Q)(y)dy\right) \\
\le &\ C\left(\int_Q |M_\eta^\#(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M_\eta^\#(f)(y)|^p \frac{w(Q)}{|2^{k+1}Q|}dy\right) \\
\le &\ C\left(\int_Q |M_\eta^\#(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M_\eta^\#(f)(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right) \\
\le &\ C\left(\int_Q |M_\eta^\#(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M_\eta^\#(f)(y)|^p \frac{w(y)}{2^{nk}}dy\right) \\
\le &\ C||M_\eta^\#(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi(2^{k+1}d) \\
\le &\ C||M_\eta^\#(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty (2^{-n}D)^k\varphi(d) \\
\le &\ C||M_\eta^\#(f)||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M_\eta(f)(x)^p w(x)dx\right)^{1/p} \le C\left(\frac{1}{\varphi(d)}\int_Q M_\eta^\#(f)(x)^p w(x)dx\right)^{1/p}$$

and

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \le C||M_\eta^\#(f)||_{L^{p,\varphi}(w)}.$$

This finishes the proof. □

Lemma 2.12. *Let $0 \le \eta < n$, $0 < D < 2^n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then*

$$||M_{\eta,s,w}(f)||_{L^{q,\varphi}(w)} \le C||f||_{L^{p,\eta,\varphi}(w)}.$$

The proof of the Lemma is similar to that of Lemma 2.11 by Lemma 2.9, we omit the details.

3. Proofs of theorems

Proof of Theorem 2.1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|^\eta \, dx\right)^{1/\eta} \leq C||b||_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T_1(g) = 0$,

$$T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|^\eta \, dx\right)^{1/\eta} \\
\leq\ & C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^\eta dx\right)^{1/\eta} = I_1 + I_2.
\end{aligned}$$

For I_1, by the weak (L^1, L^1) boundedness of T (see Lemma 2.8) and Kolmogoro's inequality (see Lemma 2.7), we obtain

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta} \\
\leq\ & \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{||T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q||_{L^\eta}}{||\chi_Q||_{L^{\eta/(1-\eta)}}} \\
\leq\ & \frac{C}{|Q|}||T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)||_{WL^1} \\
\leq\ & \frac{C}{|Q|}\int_{R^n} |M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|dx \\
\leq\ & \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}|w(x)^{-1/s}|T^{k,2}(f)(x)|w(x)^{1/s}dx \\
\leq\ & \frac{C}{|Q|}\left(\int_{2Q} |b(x) - b_{2Q}|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\int_{2Q} |T^{k,2}(f)(x)|^s w(x)dx\right)^{1/s} \\
\leq\ & \frac{C}{|Q|}||b||_{Lip_\beta(w)} w(2Q)^{1/s'+\beta/n} w(2Q)^{1/s-\beta/n} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}) \\
\leq\ & C||b||_{Lip_\beta(w)} \frac{w(2Q)}{|2Q|} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}) \\
\leq\ & C||b||_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 \ &\leq\ C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta} \\
&\leq\ C||b||_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For I_2, by [2], we know that

$$T(f)(x)=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}a_{uv}(x)\int_{R^n}\frac{Y_{uv}(x-y)}{|x-y|^n}f(y)dy,$$

where $g_u\le Cu^{n-2}$, $|a_{uv}(x_0)|\le Cu^{-2n}$, $|a_{uv}(x)-a_{uv}(x_0)|\le Cu^{-2n+1}|x-x_0|/|x_0-y|$, $|Y_{uv}(x-y)|\le Cu^{n/2-1}$ for $x\in Q$ and

$$\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\le Cu^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

for $|x-y|>2|x_0-x|>0$. Thus, notice $w\in A_1\subset A_s$, we get, for $x\in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|\\
\le\ &\int_{(2Q)^c}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dy\\
=\ &\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\left|\frac{a_{uv}(x)Y_{uv}(x-y)}{|x-y|^n}-\frac{a_{uv}(x_0)Y_{uv}(x_0-y)}{|x_0-y|^n}\right||T^{k,2}(f)(y)|dy\\
\le\ &C\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}|a_{uv}(x)-a_{uv}(x_0)|\frac{|Y_{uv}(x-y)|}{|x-y|^n}|T^{k,2}(f)(y)|dy\\
+\ &C\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}|a_{uv}(x_0)|\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right||T^{k,2}(f)(y)|dy\\
\le\ &C\sum_{u=1}^{\infty}u^{-2n+1}u^{n/2-1}u^{n-2}\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|T^{k,2}(f)(y)|dy\\
+\ &C\sum_{u=1}^{\infty}u^{-2n}u^{n/2}u^{n-2}\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|T^{k,2}(f)(y)|dy\\
\le\ &C\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}+b_{2^{j+1}Q}-b_{2Q}|w(y)^{-1/s}|T^{k,2}(f)(y)|w(y)^{1/s}dy\\
\le\ &C\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}\left(\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}|^{s'}w(y)^{1-s'}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^sw(y)dy\right)^{1/s}\\
&+\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}|b_{2^{j+1}Q}-b_{2Q}|\left(\int_{2^{j+1}Q}w(y)^{-1/(s-1)}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^sw(y)dy\right)^{1/s}\\
\le\ &C\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}||b||_{Lip_\beta(w)}w(2^{j+1}Q)^{1/s'+\beta/n}w(2^{j+1}Q)^{1/s-\beta/n}M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})\\
&+C\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}||b||_{Lip_\beta(w)}w(\tilde{x})jw(2^{j+1}Q)^{\beta/n}w(2^{j+1}Q)^{1/s-\beta/n}M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})\\
&\times\frac{|2^{j+1}Q|}{w(2^{j+1}Q)^{1/s}}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}w(y)dy\right)^{1/s}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}w(y)^{-1/(s-1)}dy\right)^{(s-1)/s}\\
\le\ &C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j}\\
\le\ &C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned} I_2 &\leq \frac{1}{|Q|}\int_Q \sum_{k=1}^m |T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|dx \\ &\leq C||b||_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}). \end{aligned}$$

These complete the proof of Theorem 2.1 □

Proof of Theorem 2.2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|^\eta\, dx\right)^{1/\eta} \leq C||b||_{BMO(w)}w(\tilde{x})\sum_{k=1}^m M_{s,w}(T^{k,2}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 2.1, we have

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|^\eta\, dx\right)^{1/\eta} \\ \leq\ & C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^\eta dx\right)^{1/\eta} = I_3 + I_4. \end{aligned}$$

By using the same argument as in the proof of Theorem 2.1, we get

$$\begin{aligned} I_3 &\leq C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta} \\ &\leq C\sum_{k=1}^m \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\frac{||T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q||_{L^\eta}}{||\chi_Q||_{L^{\eta/(1-\eta)}}} \\ &\leq C\sum_{k=1}^m \frac{C}{|Q|}||T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)||_{WL^1} \\ &\leq C\sum_{k=1}^m \frac{C}{|Q|}\int_{R^n} |M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|dx \\ &\leq C\sum_{k=1}^m \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}|w(x)^{-1/s}|T^{k,2}(f)(x)|w(x)^{1/s}dx \\ &\leq C\sum_{k=1}^m \frac{C}{|Q|}\left(\int_{2Q} |b(x) - b_{2Q}|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\int_{2Q} |T^{k,2}(f)(x)|^s w(x)dx\right)^{1/s} \\ &\leq C\sum_{k=1}^m \frac{w(2Q)}{|2Q|}\left(\frac{1}{w(2Q)}\int_{2Q} |b(x) - b_{2Q}|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\frac{1}{w(2Q)}\int_{2Q} |T^{k,2}(f)(x)|^s w(x)dx\right)^{1/s} \\ &\leq C||b||_{BMO(w)}w(\tilde{x})\sum_{k=1}^m M_{s,w}(T^{k,2}(f))(\tilde{x}), \end{aligned}$$

$$\begin{aligned}
I_4 &\le \sum_{k=1}^m \frac{C}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dydx \\
&\le \sum_{k=1}^m \frac{C}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y)-b_{2Q}|\sum_{u=1}^\infty\sum_{v=1}^{g_u}|a_{uv}(x)-a_{uv}(x_0)|\frac{|Y_{uv}(x-y)|}{|x-y|^n}|T^{k,2}(f)(y)|dydx \\
&+ \sum_{k=1}^m \frac{C}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y)-b_{2Q}|\sum_{u=1}^\infty\sum_{v=1}^{g_u}|a_{uv}(x_0)| \\
&\quad\times\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right||T^{k,2}(f)(y)|dydx \\
&\le \sum_{k=1}^m \frac{C}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|T^{k,2}(f)(y)|dydx \\
&\le C\sum_{k=1}^m\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}}\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}+b_{2^{j+1}Q}-b_{2Q}|w(y)^{-1/s}|T^{k,2}(f)(y)|w(y)^{1/s}dy \\
&\le C\sum_{k=1}^m\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}}\left(\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}|^{s'}w(y)^{1-s'}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s w(y)dy\right)^{1/s} \\
&+\sum_{k=1}^m\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}}|b_{2^{j+1}Q}-b_{2Q}|\left(\int_{2^{j+1}Q}w(y)^{-1/(s-1)}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s w(y)dy\right)^{1/s} \\
&\le C\sum_{k=1}^m\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}}||b||_{BMO(w)}w(2^{j+1}Q)M_{s,w}(T^{k,2}(f))(\tilde{x}) \\
&+C\sum_{k=1}^m\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}}||b||_{BMO(w)}w(\tilde{x})jw(2^{j+1}Q)^{1/s}M_{s,w}(T^{k,2}(f))(\tilde{x}) \\
&\times\frac{|2^{j+1}Q|}{w(2^{j+1}Q)^{1/s}}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}w(y)dy\right)^{1/s}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}w(y)^{-1/(s-1)}dy\right)^{(s-1)/s} \\
&\le C||b||_{BMO(w)}w(\tilde{x})\sum_{k=1}^m M_{s,w}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^\infty j2^{-j} \\
&\le C||b||_{BMO(w)}w(\tilde{x})\sum_{k=1}^m M_{s,w}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2.2 □

Proof of Theorem 2.3. Choose $1 < s < p$ in Theorem 2.1 and notice $w^{1-q} \in A_1$, then we have, by Lemmas 2.9 and 2.10,

$$\begin{aligned}
&||T_b(f)||_{L^q(w^{1-q})} \le \|M_\eta(T_b(f))\|_{L^q(w^{1-q})} \le C\|M_\eta^\#(T_b(f))\|_{L^q(w^{1-q})} \\
\le&\ C||b||_{Lip_\beta(w)}\sum_{k=1}^m \|wM_{\beta,s,w}(T^{k,2}(f))\|_{L^q(w^{1-q})} \\
=&\ C||b||_{Lip_\beta(w)}\sum_{k=1}^m \|M_{\beta,s,w}(T^{k,2}(f))\|_{L^q(w)} \\
\le&\ C||b||_{Lip_\beta(w)}\sum_{k=1}^m \|T^{k,2}(f)\|_{L^p(w)} \\
\le&\ C||b||_{Lip_\beta(w)}\|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 2.3 □

Proof of Theorem 2.4. Choose $1 < s < p$ in Theorem 2.1 and notice $w^{1-q} \in A_1$, then we have, by Lemmas 2.11 and 2.12,

$$\begin{aligned} & ||T_b(f)||_{L^{q,\varphi}(w^{1-q})} \leq \|M_\eta(T_b(f))\|_{L^{q,\varphi}(w^{1-q})} \leq C\|M_\eta^{\#}(T_b(f))\|_{L^{q,\varphi}(w^{1-q})} \\ \leq\ & C||b||_{Lip_\beta(w)} \sum_{k=1}^{m} \|wM_{\beta,s,w}(T^{k,2}(f))\|_{L^{q,\varphi}(w^{1-q})} \\ =\ & C||b||_{Lip_\beta(w)} \sum_{k=1}^{m} \|M_{\beta,s,w}(T^{k,2}(f))\|_{L^{q,\varphi}(w)} \\ \leq\ & C||b||_{Lip_\beta(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^{p,\beta,\varphi}(w)} \\ \leq\ & C||b||_{Lip_\beta(w)}\|f\|_{L^{p,\beta,\varphi}(w)}. \end{aligned}$$

This completes the proof of Theorem 2.4 □

Proof of Theorem 2.5. Choose $1 < s < p$ in Theorem 2.2 and notice $w^{1-p} \in A_1$, then we have, by Lemmas 2.9 and 2.10,

$$\begin{aligned} & ||T_b(f)||_{L^p(w^{1-p})} \leq \|M_\eta(T_b(f))\|_{L^p(w^{1-p})} \leq C\|M_\eta^{\#}(T_b(f))\|_{L^p(w^{1-p})} \\ \leq\ & C||b||_{BMO(w)} \sum_{k=1}^{m} \|wM_{s,w}(T^{k,2}(f))\|_{L^p(w^{1-p})} \\ =\ & C||b||_{BMO(w)} \sum_{k=1}^{m} \|M_{s,w}(T^{k,2}(f))\|_{L^p(w)} \\ \leq\ & C||b||_{BMO(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p(w)} \\ \leq\ & C||b||_{BMO(w)}\|f\|_{L^p(w)}. \end{aligned}$$

This completes the proof of Theorem 2.5 □

Proof of Theorem 2.6. Choose $1 < s < p$ in Theorem 2.2 and notice $w^{1-p} \in A_1$, then we have, by Lemmas 2.11 and 2.12,

$$\begin{aligned} & ||T_b(f)||_{L^{p,\varphi}(w^{1-p})} \leq \|M_\eta(T_b(f))\|_{L^{p,\varphi}(w^{1-p})} \leq C\|M_\eta^{\#}(T_b(f))\|_{L^{p,\varphi}(w^{1-p})} \\ \leq\ & C||b||_{BMO(w)} \sum_{k=1}^{m} \|wM_{s,w}(T^{k,2}(f))\|_{L^{p,\varphi}(w^{1-p})} \\ =\ & C||b||_{BMO(w)} \sum_{k=1}^{m} \|M_{s,w}(T^{k,2}(f))\|_{L^{p,\varphi}(w)} \\ \leq\ & C||b||_{BMO(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^{p,\varphi}(w)} \\ \leq\ & C||b||_{BMO(w)}\|f\|_{L^{p,\varphi}(w)}. \end{aligned}$$

This completes the proof of Theorem 2.6. □

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