Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 273 – 281
DOI : 10.15672/hujms.546988

Research Article

Acentralizers of Abelian groups of rank 2

Zahar Mozafar, Bijan Taeri*

Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran

Abstract

Let G be a group. The Acentralizer of an automorphism α of G, is the subgroup of fixed points of α, i.e., $C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$. We show that if G is a finite Abelian p-group of rank 2, where p is an odd prime, then the number of Acentralizers of G is exactly the number of subgroups of G. More precisely, we show that for each subgroup U of G, there exists an automorphism α of G such that $C_G(\alpha) = U$. Also we find the Acentralizers of infinite two-generator Abelian groups.

Mathematics Subject Classification (2010). 20D45, 20D25

Keywords. automorphism, centralizer, acentralizer, finite groups

1. Introduction

Throughout the article, the usual notation will be used, for example $\mathbb{Z}_n$ denotes the cyclic group of integers modulo n, $\mathbb{Z}_n^*$ denotes the group of invertible elements of $\mathbb{Z}_n$. Let G be a group. We denote $\mathrm{cent}(G) = \{C_G(g) \mid g \in G\}$, where $C_G(g)$ is the centralizer of the element g in G. Then for any natural number n, a group is called n-centralizer if $|\mathrm{cent}(G)| = n$. There are some results on finite n-centralizers groups (see [1–7, 10, 13, 15]). The study of n-centralizer infinite groups was initiated in [9]. Let $\mathrm{Aut}(G)$ be the group of automorphisms of G. If $\alpha \in \mathrm{Aut}(G)$, then the Acentralizer of α in G is defined as

$$C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$$

which is a subgroup of G. In particular, if $\alpha = \tau_a$ is an inner automorphisms of G induced by $a \in G$, then $C_G(\tau_a) = C_G(a)$ is the centralizer of a in G. Let $\mathrm{Acent}(G)$ be the set of Acentralizers of G, that is

$$\mathrm{Acent}(G) = \{C_G(\alpha) \mid \alpha \in \mathrm{Aut}(G)\}.$$

The group G is called n-Acentralizer, if $|\mathrm{Acent}(G)| = n$.

It is obvious that G is 1-Acentralizer group if and only if G is a trivial group or $\mathbb{Z}_2$. Nasrabadi and Gholamian [12] proved that G is 2-Acentralizer group if and only if $G \cong \mathbb{Z}_4, \mathbb{Z}_p$ or $\mathbb{Z}_{2p}$ for some odd prime p. Furthermore, they characterized $3, 4, 5$-Acentralizer groups.

*Corresponding Author.
Email addresses: z.mozafar@math.iut.ac.ir (Z. Mozafar), b.taeri@cc.iut.ac.ir (B. Taeri)
Received: 05.06.2018; Accepted: 11.11.2018

Lemma 1.1 ([12]). *Let H and T be finite groups. Then*

$$|\text{Acent}(H)|\,|\text{Acent}(T)| \leq |\text{Acent}(H \times T)|.$$

In addition if $|H|$ and $|T|$ are relatively prime, then

$$|\text{Acent}(H)|\,|\text{Acent}(T)| = |\text{Acent}(H \times T)|.$$

Therefore, if G is a finite nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where p_i, $i = 1, \dots, r$, are distinct primes and $k_i \geq 1$, then

$$|\text{Acent}(G)| = \prod_{i=1}^{r} |\text{Acent}(G_{p_i})|,$$

where G_{p_i}'s are the Sylow p_i-subgroup of G.

Thus in order to find the number of Acentralizers of a finite nilpotent (in particular Abelian) group G, it is enough to find the number of Acentralizers of its Sylow subgroups.

In this paper we compute $|\text{Acent}(G)|$, considering G to be a cyclic group of prime power order and of order $p_1^{k_1} \cdots p_r^{k_r}$, where p_i for $i = 1, \dots, r$ are distinct primes, an elementary Abelian group of prime power order, group of the form $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where m, n are positive integers and p is a prime and finally a free Abelian group of rank 2.

2. Preliminaries

We begin with computing of Acentralizers of finite cyclic groups. We show that if G is a cyclic group of odd order, then $|\text{Acent}(G)|$ is equal to the number of subgroups of G, while if $|G|$ is even, $|\text{Acent}(G)|$ is less than the number of subgroups of G.

Proposition 2.1. Let G be a cyclic group of order $m = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $k_1, \dots, k_r$ are positive integers. Then

$$|\text{Acent}(G)| = \begin{cases} (k_1+1) \cdots (k_r+1) & \text{if } p_1 \neq 2 \\ k_1(k_2+1) \cdots (k_r+1) & \text{if } p_1 = 2 \end{cases}$$

Proof. First let $G = \langle a \rangle$ be a cyclic group of order p^n, where p is an odd prime and n a positive integer. For every $0 \leq k \leq n$, let $G_k = \langle a^{p^{n-k}} \rangle$ be the unique subgroup of G of order p^k. If α is defined as $\alpha(a) = a^{1+p^k}$, then α is an automorphism of G and

$$\alpha(a^{p^{n-k}}) = (a^{p^{n-k}})^{(1+p^k)} = a^{p^{n-k}}$$

and so $C_G(\alpha) = G_k$. Hence every subgroup of G is an Acentralizer of G and $|\text{Acent}(G)| = n + 1$. Similarly we can see that if $p = 2$, then every non-identity subgroup of G is an Acentralizer of G and $|\text{Acent}(G)| = n$.

Now suppose that G is a cyclic group of order $m = p_1^{k_1} \cdots p_r^{k_r}$, where p_i, $i = 1, \dots, r$, are distinct odd primes. Then, by Lemma 1.1, $|\text{Acent}(G)| = (k_1+1) \cdots (k_r+1)$, which is the number of subgroups of G. Also if G is a cyclic group of order $m = 2^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where p_i, $i = 1, \dots, r$, are distinct odd primes, then $|\text{Acent}(G)| = k_1(k_2+1) \cdots (k_r+1)$. Note that in this case the number of subgroups of G is $(k_1+1)(k_2+1) \cdots (k_r+1)$. □

The following question arises naturally.

What is $|\text{Acent}(G)|$, where G is a finite Abelian group?

We show that an elementary Abelian p-group G is a m-Acentralizer group, where m is the number of subgroups of G. The proof of the following result is well-known which is brought for completeness.

Proposition 2.2. Let G be an elementary Abelian group of order p^n. Then $|\mathrm{Acent}(G)|$ is the number of subgroups of G, that is

$$|\mathrm{Acent}(G)| = c_0 + c_1 + \cdots + c_{n-1} + c_n,$$

where $c_0 = 1$ and $c_k = \frac{(p^n-1)(p^n-p)\cdots(p^n-p^{k-1})}{(p^k-1)(p^k-p)\cdots(p^k-p^{k-1})}$ for $k = 1, \ldots, n-1$.

Proof. We note that G is a vector space over $\mathbb{Z}_p$ and for each $1 \le k \le n$, there are c_k subspaces of dimension k in V. To see this, first we count the number of k-element linearly independent subsets in G. Every such set generates a k-dimensional subspace of G. Let $\{v_1, \ldots, v_k\}$ be linearly independent. The vector v_1 (which is a non-zero vector) could be selected in $p^n - 1$ ways, the vector v_2 (which is not a multiple of v_1) in $p^n - p$ ways, …, and v_k (which is not a linear combination of $v_1, v_2, \ldots, v_{k-1}$) in $p^n - p^{k-1}$ ways. So there are $t = (p^n - 1)(p^n - p) \cdots (p^n - p^{k-1})$ lineary independent k-element subsets of G. Every basis of $W := \mathrm{span}\{v_1, \ldots, v_k\}$ generate the same subspace and, as shown above, there are $s = (p^k - 1)(p^k - p) \cdots (p^k - p^{k-1})$ basis of W. Therefore there are t/s distinct k-dimensional subspaces of G.

We show that for every subspace W of V, there exists $\alpha \in \mathrm{Aut}(V)$ such that α induces identity just on W, that is $C_V(\alpha) = W$. Let W be a k-dimensional subspace of V. Then there exists a subspace U of V such that $V = W \oplus U$. Let $\mathscr{A} = \{w_1, \ldots, w_k, u_1, \ldots, u_t\}$ be a basis of V, where $\{w_1, \ldots, w_k\}$ is a basis of W and $\{u_1, \ldots, u_t\}$ be a basis of U. If $t \ge 2$, then we can define α on V as follows: $\alpha(w_i) = w_i$, $i = 1, \ldots, k$, $\alpha(u_1) = u_1 + u_2$, $\alpha(u_i) = u_{i+1}$, $i = 2, \ldots, t-1$, $\alpha(u_t) = u_1$. Thus α is an automorphism of V inducing identity just on W. If $t = 1$ then α can defined as $\alpha(w_i) = w_i$, $i = 1, \ldots, k$, $\alpha(u_1) = u_1 + w_1$. If $t = 0$, that is $W = V$, then α is the identity automorphism. □

We can generalize the above result. In fact we can show that if V is a vector space over any field, then every subspace of V is a centralizer of an automorphism of V.

We need to know the structure of subgroups of direct products. We briefly recall the discussions on pages 34-36 of [14] about subgroups of direct products. A subgroup D of $G = H \times K$ such that $DH = G = DK$ and $D \cap H = \{1\} = D \cap K$ is called a diagonal in G (with respect to H and K). If $H \cong K$ and $\delta : H \longrightarrow K$ is an isomorphism, then

$$D(\delta) = D(H, \delta) = \{x\delta(x) \mid x \in H\}$$

is a diagonal in G (with respect to H and K). Conversely, if D is a diagonal in G (with respect to H and K), then there exists a unique isomorphism $\delta : H \longrightarrow K$ such that $D = D(\delta)$. Thus there is a bijection between diagonals (with respect to H and K) and isomorphisms of H to K.

Every subgroup U of a direct product $G = H \times K$ is a diagonal in a certain section of G. More precisely, there is natural isomorphism

$$\frac{UK \cap H}{U \cap H} \cong \frac{UH \cap K}{U \cap K}.$$

Conversely, let $W_1 \trianglelefteq U_H \le H$ and $W_2 \trianglelefteq U_K \le K$ be subgroups of direct factors. For every isomorphism $\delta : \frac{U_H}{W_1} \longrightarrow \frac{U_K}{W_2}$ there exists a subgroup $U \le G$ such that $U_H = UK \cap H, U_K = UH \cap K, W_1 = U \cap H$ and $W_2 = U \cap K$, namely

$$U = D(U_H, \delta) = \{xy \mid x \in U_H, y \in \delta(xW_1)\}.$$

Thus in order to recover the subgroups of $G = H \times K$ we need the isomorphisms between the sections (i.e., intervals in the subgroup lattice) in $[\{1\}, H]$ respectively $[\{1\}, K]$. Also every subgroup of a direct product $G = H \times K$ is a direct of the form $H_1 \times K_1$, where $H_1 \le H$ and $K_1 \le K$ or is a diagonal.

Set $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $1 \le m \le n$. First of all, we have the direct product of chains of length m respectively n, that is, $(m+1)(n+1)$ subgroups. Second, we have m

sections of order p from the first direct factor and n sections of order p from the second direct factor. Thus for each pair of 1-segments correspond to the isomorphisms $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ of these sections we have $p-1$ diagonals i.e., $mn(p-1)$.

Third, we have $m-1$ sections of order p^2 from the first direct factor and $n-1$ sections of order p^2 from the second direct factor. Thus for each pair of 2-segments correspond to the isomorphisms $\mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_{p^2}$ of these sections we have p^2-p diagonals i.e., $(m-1)(n-1)(p^2-p)$.

In general, for every $k=0,1,\ldots,n-m$ we have $m-(k-1)$ sections of order p^k from the first direct factor and $n-(k-1)$ sections of order p^k from the second direct factor. Thus for each pair of k-segments correspond to the isomorphisms $\mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^k}$ of these sections we have p^k-p^{k-1} diagonals i.e., $(m-k+1)(n-k+1)(p^k-p^{k-1})$. Thus we have the following result.

Theorem 2.3 ([8])**.** *Let $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m \le n$. Then, the number of subgroups of G is*

$$(m+1)(n+1)+\sum_{k=0}^{m-1}(m-k)(n-k)(p^{k+1}-p^k).$$

Fix an isomorphism

$$G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$$

with $1 \le m \le n$ and let $\mathbb{Z}_{p^m} \cong \langle a\rangle$, $\mathbb{Z}_{p^n} \cong \langle b\rangle$. Given an endomorphism $\alpha : G \longrightarrow G$ we get $\alpha(a)=a^ib^j$ and $\alpha(b)=a^rb^s$, for some integers $0 \le i,r < p^m$ and $0 \le j,s < p^n$. We indicate this situation by a matrix $\begin{bmatrix} i & r \\ j & s \end{bmatrix}$. Observe that the relations $a^{p^m}=1$ and $b^{p^n}=1$ yield $j \equiv 0 \pmod{p^{n-m}}$. Note that if $n=m$, then certainly $\mathrm{Aut}(G)=\mathrm{GL}_2(p^m)$, the group of invertible 2 by 2 matrices over the ring $\mathbb{Z}_{p^m}$ of integers $\pmod{p^m}$.

Theorem 2.4 ([11, Corollary 3])**.** *Let $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m < n$. Then, the matrix $\begin{bmatrix} i & r \\ j & s \end{bmatrix}$ represents*

(1) *an endomorphism of G if and only if $i \in \mathbb{Z}_{p^m}$, $j \equiv 0 \pmod{p^{n-m}}$, $r \in \mathbb{Z}_{p^n}$ and $s \in \mathbb{Z}_{p^n}$;*

(2) *an automorphism of G if and only if $i \in \mathbb{Z}^*_{p^m}$, $j \equiv 0 \pmod{p^{n-m}}$, $r \in \mathbb{Z}_{p^n}$ and $s \in \mathbb{Z}^*_{p^n}$.*

3. Main results

In this section we compute $|\mathrm{Acent}(G)|$, where $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. First we show that if p is odd, then $|\mathrm{Acent}(G)|$ is equal to total number of subgroups of G.

Theorem 3.1. Let $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $m \le n$ and p is odd prime, then $|\mathrm{Acent}(G)|$ is equal to the number of subgroups of G, that is

$$|\mathrm{Acent}(G)|=(m+1)(n+1)+\sum_{k=0}^{m-1}(m-k)(n-k)(p^{k+1}-p^k).$$

Proof. Let $G=A\times B$, $A=\langle a\rangle \cong \mathbb{Z}_{p^m}$, $B=\langle b\rangle \cong \mathbb{Z}_{p^n}$, $m \le n$. Let α be an automorphism of G such that $\alpha(a)=a^ib^j$ and $\alpha(b)=a^rb^s$, where $0 \le i,r < p^m$ and $0 \le j,s < p^n$. By Theorem 2.4, $\gcd(i,p^m)=1$, $\gcd(s,p^n)=1$, and $j \equiv 0 \pmod{p^{n-m}}$. Since

$$\begin{aligned}\alpha(a^xb^y)&=(\alpha(a))^x(\alpha(b))^y\\&=(a^ib^j)^x(a^rb^s)^y\\&=a^{ix+ry}b^{jx+sy}\end{aligned}$$

we have

$$\begin{aligned} C_G(\alpha) &= \{a^x b^y \mid \alpha(a^x b^y) = a^x b^y\} \\ &= \{a^x b^y \mid a^{ix+ry} b^{jx+sy} = a^x b^y\}. \end{aligned}$$

Hence the elements of $C_G(\alpha)$ is of the form $a^x b^y$, where (x, y) is a solution of the following equation

$$\begin{cases} ix + ry = x & \pmod{p^m}, \\ jx + sy = y & \pmod{p^n} \end{cases}$$

that is

$$\begin{cases} (i-1)x + ry = 0 & \pmod{p^m}, \\ jx + (s-1)y = 0 & \pmod{p^n}. \end{cases} \tag{1}$$

Let $A_u = \langle a^{p^{m-u}} \rangle$ be the unique subgroup of A of order p^u, $u = 0, 1, \ldots, m$; and let $B_v = \langle b^{p^{n-v}} \rangle$ be the unique subgroup of B of order p^v, $v = 0, 1, \ldots, n$. Then G has $(m+1)(n+1)$ subgroups of the form $A_u \times B_v$, $u = 0, \ldots, m$, $v = 0, \ldots, n$. For every $u = 0, \ldots, m$ and $v = 0, \ldots, n$ we find an automorphism α of G inducing identity just on $A_u \times B_v$. If we choose $i = 1 + p^u$, $j = 0$, $r = 0$, and $s = 1 + p^v$ then α defined by $\alpha(a) = a^{1+p^u}$ and $\alpha(b) = b^{1+p^v}$ is an automorphism of G, such that $C_G(\alpha) = A_u \times B_v$.

For every $k = 1, \ldots, m$ we have the diagonals corresponding to the automorphisms $\mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^k}$, which give $p^k - p^{k-1}$ diagonals for each pair of k-segments. So there are $(m-k+1)(n-k+1)(p^k - p^{k-1})$ diagonal subgroups corresponding to the automorphisms between sections of order p^k. We find these subgroups explicitly and automorphisms of G inducing identity just on these subgroups.

For every $u = k, \ldots, m$, $v = k, \ldots, n$ and for every t with $\gcd(p^k, t) = 1$, the isomorphism

$$\begin{aligned} \delta_k : A_u/A_{u-k} &\longrightarrow B_v/B_{v-k} \\ a^{p^{m-u}} A_{u-k} &\mapsto b^{p^{n-v}t} B_{v-k} \end{aligned}$$

gives a diagonal subgroup

$$\begin{aligned} D_{u,v,t} &= \{xy \mid x \in A_u,\ y \in \delta_k(xA_{u-k})\} \\ &= \{a^{p^{m-u}\ell_1} y \mid 1 \le \ell_1 \le p^u,\ y \in b^{p^{n-v}\ell_1 t} B_{v-k}\} \\ &= \{a^{p^{m-u}\ell_1} b^{p^{n-v}\ell_1 t} b^{p^{n-(v-k)}\ell_2} \mid 1 \le \ell_1 \le p^u,\ 1 \le \ell_2 \le p^{v-k}\} \\ &= \{a^{p^{m-u}\ell_1} b^{p^{n-v}(\ell_1 \cdot t + \ell_2 \cdot p^k)} \mid 1 \le \ell_1 \le p^u,\ 1 \le \ell_2 \le p^{v-k}\}. \end{aligned}$$

We find an automorphism of G inducing identity just on $D_{u,v,t}$. We must choose i, j, r, s such that $(p^{m-u}\ell_1, p^{n-v}(\ell_1 t + \ell_2 p^k))$ is a solution of (1) that is

$$\begin{cases} p^{m-u}(i-1)\ell_1 + p^{n-v} r(\ell_1 t + \ell_2 p^k) = 0 & \pmod{p^m}, \\ p^{m-u} j\ell_1 + p^{n-v}(s-1)(\ell_1 t + \ell_2 p^k) = 0 & \pmod{p^n}. \end{cases}$$

If we choose $i = 1 + p^u$, $s = 1 + p^v$, $j = p^{n-m+u}$, and $r = p^{m-n+v}$, then α, defined by $\alpha(a) = a^{1+p^u} b^{p^{n-m+u}}$ and $\alpha(b) = a^{p^{m-n+v}} b^{1+p^v}$, is an automorphism of G such that $C_G(\alpha) = D_{u,v,t}$.

Thus we have shown for every subgroup M of G there exists $\alpha \in \mathrm{Aut}(G)$ such that $C_G(\alpha) = M$. Hence $|\mathrm{Acent}(G)|$ is equal to total number of subgroups of G and the proof is completed. □

To compute $|\mathrm{Acent}(\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n})|$, we need to find the subgroups of $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, which are not Acentralizers.

Lemma 3.2. Let $G = A \times B$, $A = \langle a \rangle \cong \mathbb{Z}_{2^m}$, $B = \langle b \rangle \cong \mathbb{Z}_{2^n}$, $m \leq n$. The following subgroups are not Acentralizers of G.

(1) $A_u = \langle a^{2^{m-u}} \rangle$, where $u = 0, 1, \ldots, m$,
(2) $B_v = \langle b^{2^{n-v}} \rangle$, where $v = 1, 2, \ldots, n-m-1$, and
(3) $D_{u,v,t}$, where $k = v \leq u$, $u = k, \ldots, m$, $v = k, \ldots, n$ for every t with $\gcd(2^k, t) = 1$ and $k = 1, \ldots, m$.

Proof. First we show that the element $b^{2^{n-1}}$ is a unique element of order 2 in G, which is fixed by every automorphism of G. Let α be an automorphism of G. We know that $\alpha(a) = a^i b^j$ and $\alpha(b) = a^r b^s$, where $0 \leq i, r < 2^m$ and $0 \leq j, s < 2^n$. By Theorem 2.4, $\gcd(i, 2^m) = 1$, $\gcd(s, 2^n) = 1$, and $j \equiv 0 \pmod{2^{n-m}}$; so $i-1$ and $s-1$ are even. Therefore,

$$\begin{aligned}
\alpha(b^{2^{n-1}}) &= (a^r b^s)^{2^{n-1}} \\
&= a^{2^{n-1}r} b^{2^{n-1}s} \\
&= (a^{2^m})^{2^{n-m-1}r} b^{2^{n-1}(s-1)} b^{2^{n-1}} \\
&= b^{2^{n-1}}.
\end{aligned}$$

Since $b^{2^{n-1}} \notin A_u$, $u = 0, \ldots, m$, and $b^{2^{n-1}} \notin D_{u,v,t}$, $v \leq u$, it follows that A_u and $D_{u,v,t}$ are not Acentralizers.

We show the centralizer of α is not equal to B_v, $v = 1, 2, \ldots, n-m-1$. Suppose that $C_G(\alpha) = B_v$. Then $b^{2^{n-v}} = \alpha(b^{2^{n-v}}) = a^{2^{n-v}r} b^{2^{n-v}s}$. Since $m+1 \leq n-v \leq n-1$, $a^{2^{n-v}r} = 1$. Therefore $b^{2^{n-v}} = b^{2^{n-v}s}$ so $b^{2^{n-v}(s-1)} = 1$. Hence $2^{n-v}(s-1) \equiv 0 \pmod{2^n}$. If $s = 1$, then $\alpha(b) = a^r b$ and so $\alpha(b^{2^m}) = a^{2^m r} b^{2^m} = b^{2^m}$. But $b^{2^m} \notin B_v$. Thus $s \neq 1$. If $j = 0$, then $\alpha(a) = a^i$ and so $\alpha(a^{2^{m-1}}) = a^{2^{m-1}i} = a^{2^{m-1}(i-1)} a^{2^{m-1}} = a^{2^{m-1}}$. But $a^{2^{m-1}} \notin B_v$. Hence $j \neq 0$. Since $\gcd(2^n, s) = 1$, there exsits t with $t = 1, \ldots, n-1$, such that $\gcd(2^n, s-1) = 2^t$. If $n-m \leq t \leq n-1$, then $\alpha(b^{2^m}) = a^{2^m r} b^{2^m s} = b^{2^m(s-1)} b^{2^m} = b^{2^m}$. But $b^{2^m} \notin B_v$. Thus $1 \leq t \leq n-m-1$. Hence $s - 1 = 2^t k'$ where k' is odd.

If $j = 2^{n-m}h$, where h is odd, then

$$\begin{aligned}
\alpha(a^{2^{m-1}} b^{2^{n-t-1}}) &= \alpha(a)^{2^{m-1}} \alpha(b)^{2^{n-t-1}} \\
&= a^{2^{m-1}i} b^{2^{m-1}j} a^{2^{n-t-1}r} b^{2^{n-t-1}s} \\
&= a^{2^{m-1}} a^{2^{m-1}(i-1)} b^{2^{m-1}j} b^{2^{n-t-1}s} \\
&= a^{2^{m-1}} b^{2^{m-1}j} b^{2^{n-t-1}(s-1)} b^{2^{n-t-1}} \\
&= a^{2^{m-1}} b^{2^{m-1}j + 2^{n-t-1}(s-1)} b^{2^{n-t-1}}.
\end{aligned}$$

Since $2^{m-1}j + 2^{n-t-1}(s-1) = 2^{n-1}h + 2^{n-1}k' = 2^{n-1}(h+k')$ and $h+k'$ is even, $2^{m-1}j + 2^{n-t-1}(s-1) \equiv 0 \pmod{2^n}$, we have $\alpha(a^{2^{m-1}} b^{2^{n-t-1}}) = a^{2^{m-1}} b^{2^{n-t-1}}$. But $a^{2^{m-1}} b^{2^{n-t-1}} \notin B_v$.

If $j = 2^{n-m}h$, where h is even, then

$$\begin{aligned}
\alpha(a^{2^{m-1}} b^{2^{n-t}}) &= \alpha(a)^{2^{m-1}} \alpha(b)^{2^{n-t}} \\
&= a^{2^{m-1}i} b^{2^{m-1}j} a^{2^{n-t}r} b^{2^{n-t}s} \\
&= a^{2^{m-1}} a^{2^{m-1}(i-1)} b^{2^{m-1}j}\ b^{2^{n-t}s} \\
&= a^{2^{m-1}} b^{2^{m-1}j} b^{2^{n-t}(s-1)} b^{2^{n-t}} \\
&= a^{2^{m-1}} b^{2^{m-1}j + 2^{n-t}(s-1)} b^{2^{n-t}}.
\end{aligned}$$

Since $2^{m-1}j + 2^{n-t}(s-1) = 2^{n-1}h + 2^n k' = 2^{n-1}(h + 2k')$ and $h + 2k'$ is even, $2^{m-1}j + 2^{n-t}(s-1) \equiv 0 \pmod{2^n}$. Hence $\alpha(a^{2^{m-1}} b^{2^{n-t}}) = a^{2^{m-1}} b^{2^{n-t}}$. But $a^{2^{m-1}} b^{2^{n-t}} \notin B_v$. Thus B_v is not an Acentralizer. □

In the following theorem we show that $|\mathrm{Acent}(G)|$, where $G \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$ is less than the number of subgroups of G.

Theorem 3.3. Let $G \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, where $m \leq n$, then

$$|\mathrm{Acent}(G)| = (m+1)(n+1) + \sum_{k=0}^{m-1}(m-k)(n-k)(2^{k+1} - 2^k) - (n - m - 2 + 2^{m+1}).$$

Proof. Using the notation of the proof of Theorem 3.1, we have,

$$\begin{cases} (i-1)x + ry = 0 & \pmod{2^m}, \\ jx + (s-1)y = 0 & \pmod{2^n}. \end{cases} \tag{2}$$

Let $A_u = \langle a^{2^{m-u}} \rangle$ be the unique subgroup of A of order 2^u, $u = 0, 1, \ldots, m$; and let $B_v = \langle b^{2^{n-v}} \rangle$ be the unique subgroup of B of order 2^v, $v = 0, 1, \ldots, n$. Then G has $(m+1)(n+1)$ subgroups of the form $A_u \times B_v$, $u = 0, \ldots, m$, $v = 0, \ldots, n$. By Lemma 3.2 , $A_u = \langle a^{2^{m-u}} \rangle$ for $u = 0, 1, \ldots, m$ and $B_v = \langle b^{2^{n-v}} \rangle$ for $v = 1, \ldots, n-m-1$ are not Acentralizers. For every $u = 1, \ldots, m$ and $v = 1, \ldots, n$ we find an automorphism α of G inducing identity just on $A_u \times B_v$. If we choose $i = 1 + 2^u$, $j = 0$, $r = 0$, and $s = 1 + 2^v$ then α defined by $\alpha(a) = a^{1+2^u}$ and $\alpha(b) = b^{1+2^v}$ is an automorphism of G, such that $C_G(\alpha) = A_u \times B_v$. For $u = 0$ and $v = n-m, \ldots, n$ we find an automorphism α of G inducing identity just on $A_u \times B_v$. If we choose $i = 1$, $j = 2^{n-m}$, $r = 2^{m-n+v}$, and $s = 1$ then α defined by $\alpha(a) = ab^{2^{n-m}}$ and $\alpha(b) = a^{2^{m-n+v}}b$ is an automorphism of G, such that $C_G(\alpha) = A_u \times B_v$.
Also by Lemma 3.2, $D_{u,v,t}$, $v \leq u$ are not Acentralizers. For other $D_{u,v,t}$ the proof is similar to Theorem 3.1. Hence

$$\begin{aligned} |\mathrm{Acent}(G)| &= (m+1)(n+1) + \sum_{k=0}^{m-1}(m-k)(n-k)(p^{k+1} - p^k) \\ &\quad -[(m+1) + (n-m-1) + [(2-1) + \ldots + (2^m - 1)]] \\ &= (m+1)(n+1) + \sum_{k=0}^{m-1}(m-k)(n-k)(p^{k+1} - p^k) \\ &\quad -[n - m - 2 + 2^{m+1}] \end{aligned}$$

and the resut follows. □

In the rest of the paper we find the Acentralizers of infinite two generator Abelian groups. We start with free Abelian groups. Let G be a free Abelian group of rank 2. Note that $\mathrm{Aut}(G) = \mathrm{GL}_2(\mathbb{Z})$, the group of invertible 2 by 2 matrices over $\mathbb{Z}$. If $\{a, b\}$ is a basis of G and α is an automorphism of G, then $\alpha(a) = a^i b^j$ and $\alpha(b) = a^r b^s$, where $i, j, r, s \in \mathbb{Z}$, and $is - jr \neq 0$. Since

$$C_G(\alpha) = \{a^x b^y \mid a^{ix+ry} b^{jx+sy} = a^x b^y\},$$

the elements of $C_G(\alpha)$ is of the form $a^x b^y$, where (x, y) is a solution of the following equation

$$\begin{cases} (i-1)x + ry = 0 \\ jx + (s-1)y = 0. \end{cases}$$

Let H be a non-trivial subgroup of G. First suppose that $\mathrm{rank}(H) = 1$. Then there exists a basis $\{a, b\}$ of G such that $\{a^u\}$, where u is a positive integer, is a basis of H. If $u = 1$, then $H = \langle a \rangle$ and so $H = C_G(\alpha)$, where α is an automorphism of G defined by $\alpha(a) = a$ and $\alpha(b) = ab$. We claim that if $u > 1$, then there is no automorphism α with $C_G(\alpha) = H$. Suppose that $C_G(\alpha) = H$, for some $\alpha \in \mathrm{Aut}(G)$. Since $a^u = \alpha(a^u) = a^{iu} b^{ju}$, $j = 0$, and $i = 1$ and so $\alpha(a) = a$. Thus $\langle a \rangle \leq C_G(\alpha) = H$, and so $u = 1$, which is contradiction.

Suppose that $\mathrm{rank}(H) = 2$. Then there exists a basis $\{a, b\}$ of G such that $\{a^u, b^v\}$, where u and v are positive integers with $u \mid v$, is a basis of H. We find an automorphism α such that $H = C_G(\alpha)$. If $u+v+1 \neq 0$, then we define $\alpha(a) = a^{1+v}b^{-v}$ and $\alpha(b) = a^{-u}b^{1+u}$ (that is $i-1 = v$, $j = -v$, $r = -u$, and $s-1 = u$). If $u+v+1 = 0$, then we define $\alpha(a) = a^{u^2+u+1}b^{u^2+u-2}$ and $\alpha(b) = a^{-u^2}b^{1-u^2}$ (that is $i-1 = u^2+u$, $j = u^2+u-2$, $r = -u^2$, and $s-1 = 1-u^2$). In any case it is easy to see that $H = C_G(\alpha)$.

Let $G = A \times B$, where $A = \langle a \rangle \cong \mathbb{Z}$ and $B = \langle b \rangle \cong \mathbb{Z}_n$. If α is an automorphism of G, then $\alpha(a) = a^i b^j$ and since $\alpha(b)$ is of finite order, $\alpha(b) = b^s$, where $\gcd(n, s) = 1$. Since B is a characteristic subgroup of G, it follows that a subgroup of A is not an Acentralizer of G. Suppose that C is a subgroup of G and α is an automorphism of G such that $C = C_G(\alpha)$. Since $a^x b^y \in C_G(\alpha)$ if and only if $a^x b^y = a^{ix} b^{jx} b^{sy}$, it follows that the elements of $C_G(\alpha)$ are in the form $a^x b^y$, where

$$\begin{cases} (i-1)x = 0 \\ jx + (s-1)y = 0 \pmod{n}. \end{cases} \tag{3}$$

Case I: If $i \neq 1$, then $x = 0$. So $C = C_G(\alpha)$ is a subgroup of B. For any divisor d of n, let $B_d = \langle b^{n/d} \rangle$ be the unique subgroup of order d. It is easy to see that such automorphism exists. In fact if we define $\alpha(a) = a^2 b$ and $\alpha(b) = b^{1+d}$, then α is an automorphism of G and $C_G(\alpha) = B_d$.
Case II: If $x \neq 0$, then $i = 1$. Let $t = \gcd(j, n)$. Then $\alpha(a^{n/t}) = a^{n/t} b^{nj/t} = a^{n/t} (b^{j/t})^n$ and so $a^{n/t} \in C_G(\alpha)$. If $a^\ell = \alpha(a^\ell)$, then $a^\ell = a^\ell b^{j\ell}$ and $n \mid j\ell$. Therefore $\frac{n}{t} \mid \frac{j}{t}\ell$ and so $\frac{n}{t} \mid \ell$. Hence x is a multiple of n/t. It follows that $b^y \in C_G(\alpha)$ and $n \mid (s-1)y$. Let $v = \gcd(s-1, n)$. Then $\frac{n}{v} \mid \frac{s-1}{v} y$ and so $\frac{n}{v} \mid y$. Hence $b^{n/v} \in C_G(\alpha)$. It follows that $C_G(\alpha) = \langle a^{n/t} \rangle \times \langle b^{n/v} \rangle$.

It is easy to see that such automorphism exists. In fact, if t and v are two arbitrary divisors of n then $\alpha(a) = ab^t$ and $\alpha(b) = b^{1+v}$ defines an automorphism of G and $C_G(\alpha) = \langle a^{n/t} \rangle \times \langle b^{n/v} \rangle$.

Acknowledgment. The authors gratefully thank the anonymous referee for the constructive comments and recommendations which definitely helped to improve the readability and quality of the paper.

References

[1] A. Abdollahi, S.M.J. Amiri, and A.M. Hassanabadi, *Groups with specific number of centralizers*, Houston J. Math. **33** (1), 43–57, 2007.

[2] S.M.J. Amiri and H. Rostami, *Centralizers and the maximum size of the pairwise noncommuting elements in finite groups*, Hacet. J. Math Stat. **46** (2), 193–198, 2017.

[3] A.R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloq. **7** (2), 139–146, 2000.

[4] A.R. Ashrafi, *Counting the centralizers of some finite groups*, J. Korean Comput. Appl. Math. **7** (1), 115–124, 2000.

[5] A.R. Ashrafi and B. Taeri, *On finite groups with a certain number of centralizers*, J. Appl. Math. Comput. **17** (12), 217–227, 2005.

[6] A.R. Ashrafi and B. Taeri, *On finite groups with exactly seven element centralizers*, J. Appl. Math. Comput. **22** (1-2), 403–410, 2006.

[7] S.M. Belcastro and G.J. Sherman, *Counting centralizers in finite groups*, Math. Mag. **67** (5), 366–374, 1994.

[8] Gr.G. Călugăreanu, *The total number of subgroups of a finite Abelian group*, Sci. Math. Jpn. **60** (1), 157–167, 2004.

[9] J. Dutta, *A characterization of 4-centralizer groups*, Chin. J. Math. (N.Y.) Article ID:871072, 2 pages, 2013.

[10] J. Dutta, *On a problem posed by Belcastro and Sherman*, Kyungpook Math. J. **56** (1), 121–123, 2016.

[11] M. Golasiński and D.L. Goncalves, *On automorphisms of finite Abelian p-groups*, Math. Slovaca **58** (4), 405–412, 2008.

[12] M.M. Nasrabadi and A. Gholamian, *On finite n-Acentralizers groups*, Comm. Algebra **43** (2), 378–383, 2015.

[13] R.K. Nath, *Commutativity degree of a class of finite groups and consequences*, Bull. Aust. Math. Soc. **88** (3), 448–452, 2013.

[14] R. Schmidt, *Subgroup lattices of groups*, De Gruyter Exp. Math., 14. Walter de Gruyter, 1994.

[15] M. Zarrin, *On element-centralizers in finite groups*, Arch. Math. (Basel) **93**, 497–503, 2009.