

**THE APPROXIMATE SOLUTION OF HIGH-ORDER LINEAR  
DELAY EQUATIONS WITH VARIABLE COEFFICIENTS IN  
TERMS OF SHIFTED CHEBYSHEV POLYNOMIALS**

**YÜKSEK MERTEBEDEN DEĞİŞKEN KATSAYILI DELAY  
DENKLEMLERİNİN CHEBYSHEV POLİNOMLARI İLE  
YAKLAŞIK ÇÖZÜMLERİ**

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**ABSTRACT**

This paper presents a numerical method for the approximate solution of  $m$ .th-order linear delay equations with variable coefficients under the mixed conditions in terms of shifted Chebyshev polynomials. The technique we have used is an improved Chebyshev collocation method. In addition, examples that illustrate the pertinent features of the method are presented and the results of study are discussed.

**Keywords:** Shifted Chebyshev polynomials and series, delay equations, Chebyshev collocation method

**ÖZET**

Bu çalışmada  $m$ .mertebeden değişken katsayılı lineer delay denklemlerinin karışık koşullar altında Chebyshev polinomları ile numerik çözümleri verilmiştir. Burada önerilen yöntem Chebyshev sıralama yönteminin geliştirilmiş halidir. Yöntemin hassasiyetini belirtmek için örnekler verilmiş ve bulunan sonuçlar tartışılmıştır.

**Anahtar Kelimeler:** Ötelenmiş Chebyshev polinomları ve serileri, delay denklemleri, Chebshev sıralama yöntemi

**1. INTRODUCTION**

It is well known that linear delay equations have been considered by many authors (El-Safty and Abo Hassa, 1990; Arıkoğlu and Özkol, 2006; Derfel, 1980; Gulsu and Sezer, 2005; Xiong and Liang, 2007). The past couple decades have seen a dramatic increase

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in the application of delay models to problems in biology, physics and engineering (Zhou et. al., 2006; Zhang et. al., 2006, Duman et. al., 2009). In the field of delay equation the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has been growing more and more. Based on the obtained method, we shall give sufficient approximate solution of the linear delay difference Eq.(1). The results can extend and improve the recent works. An example is given to demonstrate the effectiveness of the results.

In recent years, Chebyshev matrix and Chebyshev collocation methods have been given to find polynomial solutions of differential, integral and integro-differential equations by many authors (Sezer, 1996; Gulsu and Sezer, 2005).

Our purpose in this study is to develop and to apply the Chebyshev collocation methods to the high-order linear delay equation with variable coefficients

$$\sum_{k=0}^m P_k(t) y(t+k) = f(t), \quad k \geq 0, k \in N \quad (1)$$

under the initial conditions

$$\sum_{k=0}^{m-1} [a_{ik} y(a+k) + b_{ik} y(b+k) + c_{ik} y(c+k)] = \lambda_i, \quad i = 0, 1, \dots, m-1 \quad (2)$$

where  $P_k(t)$  and  $f(t)$  are analytical functions,  $a_{ik}, b_{ik}, c_{ik}$  and  $\lambda_i$  are real or complex constants. The aim of this study is to get solution as truncated Chebyshev series defined by

$$y(t) = \sum_{n=0}^N a_n T_n^*(t), \quad T_n^*(t) = \cos(n\theta), \quad 2t-1 = \cos \theta, \quad 0 \leq t \leq 1 \quad (3)$$

where  $T_n^*(t)$  denotes the shifted Chebyshev polynomials of the first kind,  $\sum'$  denotes a sum whose first term is halved,  $a_n$  ( $0 \leq n \leq N$ )

are unknown Chebyshev coefficients and  $N$  is chosen any positive integer such that  $N \geq m$ .

The rest of this paper is organized as follows. Higher-order linear delay equation with variable coefficients and fundamental relations are presented in Section 2. The new scheme is based on Chebyshev collocation method. The method of finding approximate solution is described in Section 3. To support our findings, we present result of numerical experiments in Section 4. Section 5 concludes this article with a brief summary. Finally some references are introduced at the end.

## 2. FUNDAMENTAL RELATIONS

Let us consider the  $m$ th-order linear delay difference equation with variable coefficients (1) and find the matrix forms of each term in the equation. First we can convert the solution  $y(t)$  defined by a truncated Chebyshev series (3) to matrix forms

$$y(t) = \mathbf{T}^*(t)\mathbf{A}, \quad y(t+k) = \mathbf{T}^*(t+k)\mathbf{A} \quad (4)$$

where

$$\mathbf{T}^*(t) = [T_0^*(t) \ T_1^*(t) \ \dots \ T_N^*(t)]$$

$$\mathbf{A} = \left[ \frac{1}{2} a_0 \ a_1 \ \dots \ a_N \right]^T$$

On the other hand, it is well known (Synder, 1966), that the relation between the powers  $t^n$  and the shifted Chebyshev polynomials  $T_n^*(t)$  is

$$t^n = 2^{-2n+1} \sum_{k=0}^n \binom{2n}{k} T_{n-k}^*(t), \quad 0 \leq t \leq 1 \quad (5)$$

By using the expression (5) and taking  $n=0,1,\dots,N$  we find the corresponding matrix relation as follows

$$(\mathbf{X}(t))^T = \mathbf{D}(\mathbf{T}^*(t))^T \quad \text{and} \quad \mathbf{X}(t) = \mathbf{T}^*(t)\mathbf{D}^T \quad (6)$$

where

$$\mathbf{X}(t) = [1 \ t \ \dots \ t^N]$$

$$\mathbf{D} = \begin{bmatrix} 2^0 \binom{0}{0} & 0 & 0 & 0 & \dots & 0 \\ 2^{-2} \binom{2}{1} & 2^{-1} \binom{2}{2} & 0 & 0 & \dots & 0 \\ 2^{-4} \binom{4}{2} & 2^{-3} \binom{4}{3} & 2^{-3} \binom{4}{4} & 0 & \dots & 0 \\ 2^{-6} \binom{6}{3} & 2^{-5} \binom{6}{4} & 2^{-5} \binom{6}{5} & 2^{-5} \binom{6}{6} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{-2N} \binom{2N}{N} & 2^{-2N+1} \binom{2N}{N+1} & 2^{-2N+1} \binom{2N}{N+2} & 2^{-2N+1} \binom{2N}{N+3} & \dots & 2^{-2N+1} \binom{2N}{2N} \end{bmatrix} \quad (7)$$

Then, by taking into account (6) we obtain

$$\mathbf{T}^*(t) = \mathbf{X}(t)(\mathbf{D}^{-1})^T \quad (8)$$

To obtain the matrix  $\mathbf{X}(t+k)$  in terms of the matrix  $\mathbf{X}(t)$ , we can use the following relation:

$$\begin{aligned} \mathbf{X}(t) &= [1 \ t \ t^2 \ \dots \ t^N], \quad \mathbf{X}(t+k) = [1 \ t+k \ (t+k)^2 \ \dots \ (t+k)^N] \\ \mathbf{X}(t+k) &= \mathbf{X}(t)\mathbf{B}_k \end{aligned} \quad (9)$$

where

$$\mathbf{B}_k = \begin{bmatrix} \binom{0}{0} k^0 & \binom{1}{0} k^1 & \binom{2}{0} k^2 & \dots & \binom{N}{0} k^N \\ 0 & \binom{1}{1} k^0 & \binom{2}{1} k^1 & \dots & \binom{N}{1} k^{N-1} \\ 0 & 0 & \binom{2}{2} k^0 & \dots & \binom{N}{2} k^{N-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \binom{N}{N} k^0 \end{bmatrix} \quad (10)$$

Consequently, by substituting the matrix forms (7) and (8) into (4), we have the matrix relation of solution

$$y(t+k) = \mathbf{T}^*(t+k)\mathbf{A} = \mathbf{X}(t+k)(\mathbf{D}^T)^{-1}\mathbf{A} \quad (11)$$

and by means of (4), (7) and (11), the matrix relation is

$$y(t+k) = \mathbf{X}(t)\mathbf{B}_k(\mathbf{D}^T)^{-1}\mathbf{A} \quad (12)$$

### 3. METHOD OF SOLUTION

In this section, we consider high order linear delay equation in (1) and approximate to solution by means of finite Chebyshev series defined in (3). The aim is to find Chebyshev coefficients, which are the matrix  $\mathbf{A}$ . For this purpose, substituting the matrix relations (12) into Eq. (1) and then simplifying, we obtain the fundamental matrix equation

$$\sum_{k=0}^m P_k(t)\mathbf{X}(t)\mathbf{B}_k(\mathbf{D}^T)^{-1}\mathbf{A} = f(t) \quad (13)$$

By using in Eq. (13) collocation points  $t_i$  defined by

$$t_i = \frac{i}{N}, i = 0, 1, \dots, N \quad (14)$$

we get the system of matrix equations

$$\sum_0^m P_k(t_i)\mathbf{X}(t_i)\mathbf{B}_k(\mathbf{D}^T)^{-1}\mathbf{A} = f(t_i), i = 0, 1, \dots, N \quad (15)$$

or briefly the fundamental matrix equation

$$\sum_{k=0}^m \mathbf{P}_k \mathbf{X} \mathbf{B}_k (\mathbf{D}^T)^{-1} \mathbf{A} = \mathbf{F} \quad (16)$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(t_0) & 0 & \dots & \dots & 0 \\ 0 & P_k(t_1) & \dots & \dots & 0 \\ \dots & \dots & & & \dots \\ \dots & \dots & & & \dots \\ 0 & 0 & \dots & \dots & P_k(t_N) \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \cdot \\ \cdot \\ f(t_N) \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} X(t_0) \\ X(t_1) \\ \cdot \\ \cdot \\ X(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdot & \cdot & \cdot & t_0^N \\ 1 & t_1 & t_1^2 & \cdot & \cdot & \cdot & t_1^N \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 1 & t_N & t_N^2 & \cdot & \cdot & \cdot & t_N^N \end{bmatrix}$$

Hence, the fundamental matrix equation (15) corresponding to Eq. (1) can be written in the form

$$\mathbf{W}\mathbf{A} = \mathbf{F} \text{ or } [\mathbf{W}; \mathbf{F}], \quad \mathbf{W} = [w_{i,j}], \quad i, j = 0, 1, \dots, N \quad (17)$$

where

$$\mathbf{W} = \sum_{k=0}^m \mathbf{P}_k \mathbf{X} \mathbf{B}_k (\mathbf{D}^T)^{-1} \quad (18)$$

Here, Eq.(16) corresponds to a system of  $(N+1)$  linear algebraic equations with unknown Chebyshev coefficients  $a_0, a_1, \dots, a_N$ . We can obtain the corresponding matrix forms for the conditions (2), by means of the relation (12),

$$\sum_{k=0}^{m-1} [a_{ik} \mathbf{X}(a) + b_{ik} \mathbf{X}(b) + c_{ik} \mathbf{X}(c)] \mathbf{B}_k (\mathbf{D}^T)^{-1} \mathbf{A} = [\lambda_i], \quad i = 0, 1, 2, \dots, m-1 \quad (19)$$

On the other hand, the matrix form for conditions can be written as

$$\mathbf{U}_i \mathbf{A} = [\lambda_i] \text{ or } [\mathbf{U}_i; \lambda_i], \quad i = 0, 1, 2, \dots, m-1 \quad (20)$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m-1} [a_{ik} \mathbf{X}(a) + b_{ik} \mathbf{X}(b) + c_{ik} \mathbf{X}(c)] \mathbf{B}_k (\mathbf{D}^T)^{-1} \quad (21)$$

and

$$\mathbf{U}_i = [u_{i0} \ u_{i1} \ u_{i2} \ \dots \ u_{iN}], \quad i = 0, 1, 2, \dots, m-1 \quad (22)$$

To obtain the solution of Eq. (1) under conditions (2), by replacing the row matrices (17) by the last  $m$  rows of the matrix (22), we have the new augmented matrix,

$$\tilde{\mathbf{W}}; \tilde{\mathbf{F}} = \begin{bmatrix} w_{00} & w_{01} & \cdot & \cdot & \cdot & w_{0N} & ; & f(t_0) \\ w_{10} & w_{11} & \cdot & \cdot & \cdot & w_{1N} & ; & f(t_1) \\ \dots & \dots & & & & \dots & ; & \dots \\ w_{N-m,0} & w_{N-m,1} & \cdot & \cdot & \cdot & w_{N-m,N} & ; & f(t_{N-m}) \\ u_{00} & u_{01} & \cdot & \cdot & \cdot & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdot & \cdot & \cdot & u_{1N} & ; & \lambda_1 \\ \dots & \dots & & & & \dots & ; & \dots \\ u_{m-1,0} & u_{m-1,1} & \cdot & \cdot & \cdot & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix} \quad (23)$$

If  $\text{rank} \tilde{\mathbf{W}} = \text{rank}[\tilde{\mathbf{W}}; \tilde{\mathbf{F}}] = N + 1$ , then we can write

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-1} \tilde{\mathbf{F}} \quad (24)$$

Thus the matrix  $\mathbf{A}$  (thereby the coefficients  $a_0, a_1, \dots, a_N$ ) is uniquely determined. Also the Eq. (1) with conditions (2) has a unique solution. This solution is given by truncated Chebyshev series (3). We use the relative error to measure the difference between the numerical and analytic solutions. The result with  $N=4(1)6$  using the Chebyshev collocation method discussed in Section 2 are shown in Table1.

We can easily check the accuracy of the method. Since the truncated Chebyshev series (3) is an approximate solution of Eq.(1), when the solution  $y_N(t)$  is substituted in Eq.(1), the resulting equation must be satisfied approximately, that is, for  $t = t_q \in [0,1]$ ,  $q = 0,1,2,\dots$

$$E(t_q) = \left| \sum_{k=0}^m P_k(t) y(t+k) - f(t) \right| \cong 0 \quad (25)$$

and  $E(t_q) \leq 10^{-k_q}$  ( $k_q$  positive integer).

If  $\max 10^{-k_q} = 10^{-k}$  ( $k$  positive integer) is prescribed, then the truncation limit  $N$  is increased until the difference  $E(t_q)$  at each of

the points becomes smaller than the prescribed  $10^{-k}$ . On the other hand, the error can be estimated by the function

$$E_N(t) = \sum_{k=0}^m P_k(t)y(t+k) - f(t) \quad (26)$$

If  $E_N(t) \rightarrow 0$ , when  $N$  is sufficiently large enough, then the error decreases.

#### 4. ILLUSTRATIVE EXAMPLE

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in Maple9. The absolute errors in Tables are the values of  $|y(x) - y_N(x)|$  at selected points.

**Example 1.** Let us first consider the second order linear delay difference equation with variable coefficients

$$(t-1)y(t+2) + (2-3t)y(t+1) + 2ty(t) = 1$$

with

$$y(0) = 2, y(1) = 2$$

and seek the solution  $y(t)$  as a truncated Chebyshev series

$$y(t) = \sum_{n=0}^N a_n T_n^*(t)$$

So that  $P_0(t)=2t$ ,  $P_1(t)=2-3t$ ,  $P_2(t)=t-1$ ,  $g(t)=1$ . Then, for  $N=4$ , the collocation points are

$$t_0=0, t_1=1/4, t_2=1/2, t_3=3/4, t_4=1$$

and the fundamental matrix equation of the problem is defined by

$$\{\mathbf{P}_0 \mathbf{X} \mathbf{B}_0 + \mathbf{P}_1 \mathbf{X} \mathbf{B}_1 + \mathbf{P}_2 \mathbf{X} \mathbf{B}_2\} (\mathbf{D}^T)^{-1} \mathbf{A} = \mathbf{F}$$

where  $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{X}$  are matrices of order  $(5 \times 5)$  defined by



$$\mathbf{P}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \mathbf{P}_1 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 5/4 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{P}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3/4 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 \\ 1 & 2/4 & 4/16 & 8/64 & 16/256 \\ 1 & 3/4 & 9/16 & 27/64 & 81/256 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

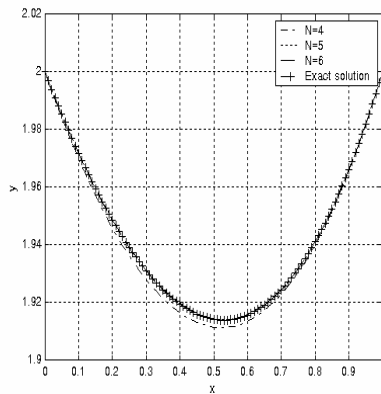
$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 3/8 & 1/2 & 1/8 & 0 & 0 \\ 5/16 & 15/32 & 3/16 & 1/32 & 0 \\ 35/128 & 7/16 & 7/32 & 1/16 & 1/128 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & 1 & 4 & 12 & 32 \\ 0 & 0 & 1 & 6 & 24 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

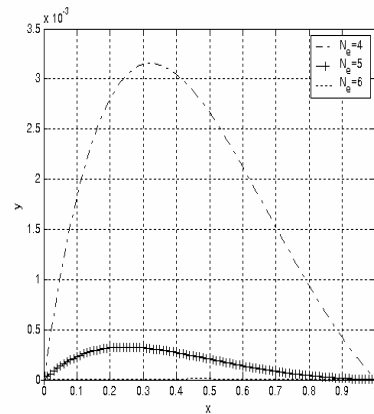
If these matrices are substituted in (16), it is obtained linear algebraic system. This system yields the approximate solution of the problem. In Table1, the resulting values using the present method together with various N and also the exact values of  $y = 2^t - t + 1$ .

**Table 1.** Error analysis of Example 1 for the t value

t	Exact Solution	Present Method					
		N=4	N <sub>e</sub> =4	N=5	N <sub>e</sub> =5	N=6	N <sub>e</sub> =6
0.0	2.00000	2.00000	0.00000	2.00000	0.00000	2.00000	0.00000
0.1	1.97177	1.96995	0.0018	1.97201	0.00034	1.97177	0.00000
0.2	1.94870	1.94591	0.0028	1.94901	0.00031	1.94870	0.00000
0.3	1.93114	1.92799	0.0031	1.93146	0.00032	1.93115	0.00001
0.4	1.91951	1.91645	0.0031	1.91978	0.00027	1.91952	0.00001
0.5	1.91421	1.91155	0.0027	1.91441	0.00020	1.91423	0.00002
0.6	1.91572	1.91359	0.0021	1.91586	0.00014	1.91572	0.00000
0.7	1.92450	1.92299	0.0015	1.92459	0.00009	1.92452	0.00002
0.8	1.94110	1.94017	0.0093	1.94114	0.00004	1.94111	0.00001
0.9	1.96607	1.96565	0.0042	1.96609	0.00002	1.96606	0.00001
1.0	2.00000	1.99999	0.00001	1.99999	0.00001	2.00000	0.00000



**Fig.1.** Numerical and exact solution of the Example1 for N=4,5,6



**Fig.2.** Error function of Example1 for various N.

Fig.1 shows the resulting graph of solution of Example1 for  $N = 4,5,6$  and it is compared with exact solution. In Fig.2 we plot error function for Example1.

**Example 2.** Let us find the Chebyshev series solution of the following first order linear delay equation

$$y(t+1) - y(t) = \sin(t+1) - \sin(t)$$

with  $y(0) = 1, y'(0) = 1$ . The exact solution of this problem is  $y(t) = \sin t$ . Using the procedure in Section 3 for the interval  $t \in [0,1]$  and taking  $N=8,9$  and 10 the matrices in Eq.(18) are computed. Hence linear algebraic system is gained. This system is approximately solved using the Maple9.

We display a plot of absolute difference exact and approximate solutions in Fig.3. and error functions for various  $N$  is shown in Fig.4. The solution of the linear delay equation is obtained for  $N=8, 9, 10$ . The difference between the respective solutions is of the order of  $10^{-6}$  and the accuracy increases as the  $N$  is increased.

**Table 2.** Error analysis of Example 2 for the  $t$  value

t	Exact Solution	Present Method					
		N=8	N <sub>e</sub> =8	N=9	N <sub>e</sub> =9	N=10	N <sub>e</sub> =10
0.0	0.000000	0.370E-6	0.37E-6	0.109E-6	0.10E-6	0.117E-6	0.11E-6
0.1	0.099833	0.099801	3.20E-5	0.099821	1.20E-5	0.099791	4.30E-5
0.2	0.198669	0.198594	7.50E-5	0.198560	1.09E-4	0.198607	6.20E-5
0.3	0.295520	0.295402	1.18E-4	0.295256	2.64E-4	0.295467	5.31E-5
0.4	0.389418	0.389273	1.45E-4	0.389003	4.15E-4	0.389403	1.50E-5
0.5	0.479426	0.479284	1.42E-4	0.478931	4.95E-4	0.479458	3.21E-5
0.6	0.564642	0.564532	1.10E-4	0.564161	4.81E-4	0.564717	7.50E-5
0.7	0.644218	0.644152	6.60E-5	0.643842	3.76E-4	0.644311	9.32E-5
0.8	0.717356	0.717334	2.20E-5	0.717130	2.26E-4	0.717438	8.22E-5
0.9	0.783327	0.783330	9.30E-5	0.783243	8.40E-5	0.783372	4.51E-5
1.0	0.841470	0.841470	0.00000	0.841467	3.00E-6	0.841468	0.20E-5

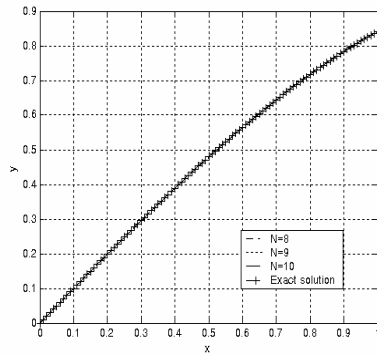


Fig.3. Numerical and exact solution of the Example2 for N=8,9,10

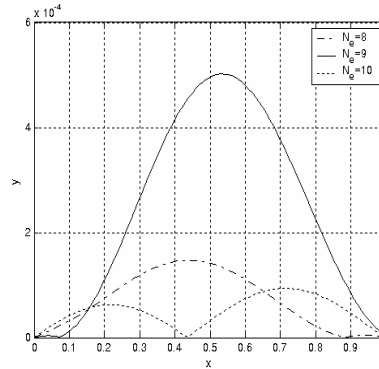


Fig.4. Error function of Example2 for various N.

**Example 3.** Consider another linear delay equation

$$y(t + 2) + y(t) = e^{(t+2)} + e^t$$

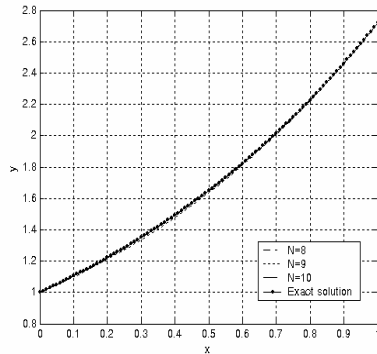
We follow the same procedure as in Example1 to find the solution of delay equation with the conditions

$$y(0) = 1, y(1) = e$$

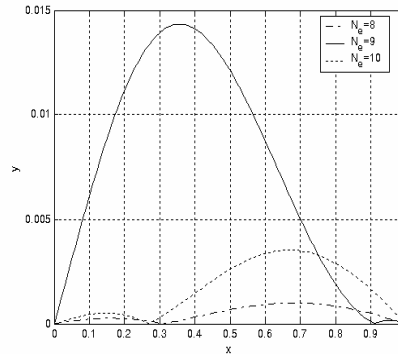
The exact solution of the problem is given by  $y(t) = \exp(t)$ . For numerical results, see Table 3. We display a plot of absolute difference exact and approximate solutions in Fig.5. and error functions for various N is shown in Fig.6. This plot clearly indicates that when we increase the truncation limit N, we have less error.

**Table 3.** Error analysis of Example 3 for the t value

t	Exact Solution	Present Method					
		N=8	N <sub>e</sub> =8	N=9	N <sub>e</sub> =9	N=10	N <sub>e</sub> =10
0.0	1.00000	0.99999	0.00001	0.99999	0.00001	1.000000	0.00000
0.1	1.10517	1.10495	0.00022	1.09904	0.00613	1.10473	0.00044
0.2	1.22140	1.22115	0.00025	1.21024	0.01116	1.22099	0.00041
0.3	1.34985	1.34982	0.00003	1.33593	0.01392	1.35016	0.00031
0.4	1.49182	1.49214	0.00032	1.47771	0.01411	1.49328	0.00146
0.5	1.64872	1.64939	0.00067	1.63660	0.01212	1.65133	0.00261
0.6	1.82211	1.82305	0.00094	1.81339	0.00872	1.82549	0.00338
0.7	2.01375	2.01475	0.00100	2.00872	0.00503	2.01724	0.00349
0.8	2.22554	2.22638	0.00084	2.22365	0.00189	2.22843	0.00289
0.9	2.45960	2.46009	0.00049	2.45948	0.00012	2.46126	0.00166
1.0	2.71828	2.71827	0.00001	2.71828	0.00000	2.71828	0.00000



**Fig.5.** Numerical and exact solution  
Example 3 for  $N=8, 9, 10$



**Fig.6.** Error function of of the  
Example 3 for various  $N$ .

**Example 4.** We consider another linear delay equation to demonstrate that the Chebyshev polynomials are powerful to approximate the solution to desired accuracy. The equation we consider is

$$(t + 3)y(t + 2) - 2(t + 2)y(t + 1) - (t + 1)y(t) = 0$$

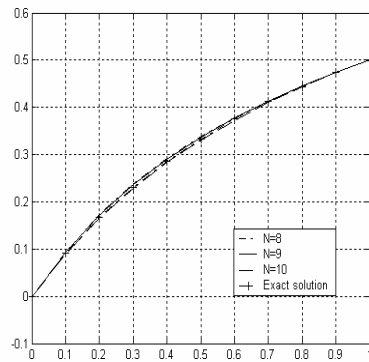
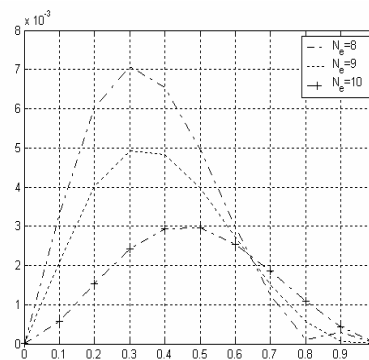
with the conditions

$$y(0) = 0, \quad y(1) = 1/2$$

We again use Chebyshev polynomials to approximate the solution of problem and compare it with the exact solution given by  $y = \frac{t}{t+1}$  following the procedure given in Section 3. The comparison of the solutions given above with the exact solution of the problem is given in Table 4. We plot the approximate solutions by this method and the exact solution in Fig.7 and the error functions in Fig.8.

**Table 4.** Error analysis of Example 5 for the t value

t	Exact Solution	N=8	N <sub>e</sub> =8	Present Method		N=10	N <sub>e</sub> =10
				N=9	N <sub>e</sub> =9		
0	0.000000	0.349E-6	0.349E-6	0.330E-7	0.331E-7	0.736E-6	0.736E-6
0.1	0.090909	0.094226	0.003317	0.093012	0.002103	0.090332	0.000577
0.2	0.16667	0.172692	0.006022	0.170669	0.003999	0.165128	0.001542
0.3	0.23077	0.237849	0.007079	0.235699	0.004929	0.228341	0.002429
0.4	0.28571	0.292247	0.006537	0.290537	0.004827	0.282772	0.002938
0.5	0.33333	0.338277	0.004947	0.337292	0.003962	0.330370	0.002960
0.6	0.37500	0.377987	0.002987	0.377728	0.002728	0.372454	0.002546
0.7	0.41176	0.413010	0.001250	0.413269	0.001509	0.409911	0.001849
0.8	0.44444	0.444540	0.000100	0.445006	0.000566	0.443360	0.001080
0.9	0.47368	0.473384	0.000296	0.473737	0.000057	0.473260	0.000420
1	0.50000	0.500003	0.000003	0.500003	0.000003	0.500005	0.000005

**Fig.7.** Numerical and exact solution of the Example 4 for N=8,9,10**Fig.8.** Error function of Example 4 for various N.**Example 5** (Sezer, 2005, Example 2)

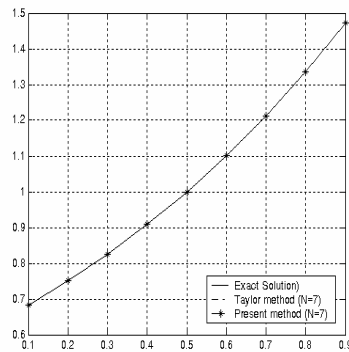
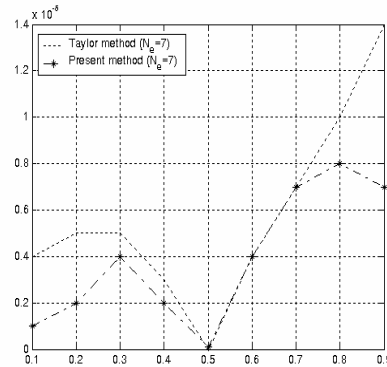
Let us find the Chebyshev series solution of the following first order linear delay equation

$$y(t+1) - y(t) = e^t$$

with conditions  $y(1/2) = 1$

**Table 5.** Absolute errors obtained for Example5

t	Exact Solution	Taylor method		Present method	
		N=7	N <sub>e</sub> =7	N=7	N <sub>e</sub> =7
0.1	0.683664	0.683668	0.400E-5	0.683663	0.100E-5
0.2	0.751306	0.751311	0.500E-5	0.751308	0.200E-5
0.3	0.826064	0.826069	0.500E-5	0.826068	0.400E-5
0.4	0.908687	0.908690	0.300E-5	0.908689	0.200E-5
0.5	1.000000	1.000000	0.000E-0	0.999999	0.100E-6
0.6	1.100916	1.100912	0.400E-5	1.100912	0.400E-5
0.7	1.212445	1.212438	0.700E-5	1.212438	0.700E-5
0.8	1.335701	1.335691	0.100E-4	1.335693	0.800E-5
0.9	1.471918	1.471904	0.140E-4	1.471911	0.700E-5

**Fig.9.** Numerical solution for different method**Fig.10.** Comparison of error function for different method

The solution of linear delay equation is obtained for  $N=7$ . For numerical results, see Table 5. We display a plot of Taylor matrix method and Exact solution for  $N=7$  in Fig.9 and we compare errors Taylor matrix method and Present method for  $N=7$  in Fig.10. It seems that the solutions almost identical. One can obtain a better approximation to the numerical solutions by adding new terms to the series in Eq.(3).

**Example 6.** Let us find the Chebyshev series solution of third order linear delay equation

$$t^2 y(t) - y(t+1) - 2y(t+2) + 2ty(t+3) = t^4 - 3t^2 + 2t + 19$$

with conditions

$$y(0) = -2, y(1/2) = -7/4, y(1) = -1$$

and the exact solution  $y = t^2 - 2$ . Using the procedure in Section 3, we find the approximate solution of this equation for  $N=4$  which is the same with the exact solution.

## 5. CONCLUSION

In recent years, the studies of high order linear delay difference equation have attracted the attention of many mathematicians and physicists. The Chebyshev collocation methods are used to solve the high order linear delay equation numerically. A considerable advantage of the method is that the Chebyshev polynomial coefficients of the solution are found very easily by using computer programs. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. For this reason, this process is much faster than the other methods. Illustrative examples are included to demonstrate the validity and applicability of the technique, and performed on the computer using a program written in Maple9. To get the best approximating solution of the equation, we take more forms from the Chebyshev expansion of functions, that is, the truncation limit  $N$  must be chosen large enough. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial functions. Illustrative examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contributed to the good agreement between approximate and exact values in the numerical example.

As a result, the power of the employed method is confirmed. We assured the correctness of the obtained solutions by putting them



back into the original equation with the aid of Maple, it provides an extra measure of confidence in the results. We predict that the Chebyshev expansion method will be a promising method for investigating exact analytic solutions to linear delay equations. The method can also be extended to the system of linear delay equations with variable coefficients, but some modifications are required.

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