EÜFBED - Fen Bilimleri Enstitüsü Dergisi Cilt-Sayı: 3-2 Yıl: 2010 263-271

SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES ABSOLUTE VALUES ARE QUASI-CONVEX

TÜREVİNİN MUTLAK DEĞERİ QUASI-KONVEKS OLAN FONKSİYONLAR İÇİN BAZI HERMITE-HADAMARD TİPİ EŞİTSİZLİKLER

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Geliş Tarihi: 22 Haziran 2010 **Kabul Tarihi:** 2 Aralık 2010

ABSTRACT

In this paper we establish some estimates of the right hand side of Hermite-Hadamard type inequality for functions whose derivatives absolute values are quasi-convex.

Key words: Quasi-convex functions, hölder inequality, power mean inequality.

ÖZET

Bu makalede, türevlerinin mutlak değeri quasi-konveks olan fonksiyonlar için Hermite-Hadamard tipi eşitsizliklerin sağ taraflarıyla ilgili bazı hesaplamalar oluşturduk.

Anahtar kelimeler: Quasi-konveks fonksiyonlar, hölder eşitsizliği, power mean eşitsizliği.

1. INTRODUCTION

Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and a,b $\in I$, with $a \le b$. The following inequality, known as the *Hermite-Hadamard* inequality for convex functions, holds:

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$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

In recent years many authors have established several inequalities connected to *Hermite Hadamard's* inequality. For recent results, refinements, counterparts, generalizations and new *Hermite-Hadamard* type inequalities [see Dragomir, 1992; Kırmacı, 2004; Tseng *et al.*, 2010 and Yang *et al.*, 2004).

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f:[a,b] \to \mathbb{R}$ is said to be quasi-convex on [a,b] if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}.$$

for any x,y $\in$ [a,b] and $\lambda \in$ [0,1]. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see Ion, 2007).

Recently, Ion (2007) established two inequalities for functions whose first derivatives in absolute value are quasi-convex. Namely, he obtained the following results:

Theorem 1: Assume $a, b \in \mathbb{R}$ with $a < b$ and $f:[a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $|f'|$ is quasi-convex on [a,b] then the following inequality holds true

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{4}\max\{|f'(a)|,|f'(b)|\}.$$

Theorem 2: Assume $a, b \in \mathbb{R}$ with $a < b$ and $f:[a,b] \to \mathbb{R}$ is a differentiable function on (a,b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/(p-1)}$ is quasi-convex on [a,b] then the following inequality holds true

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{p/(p-1)},|f'(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}$$

Alomari *et al*. obtained the following results (Alomari *et al*. (2010)).

Theorem 3: Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where a,b $\in I$ with $a < b$. If $|f'|$ is quasi-convex on [a,b], then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{8}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(a)|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(b)|\right\}\right]. \quad (1.1)$$

Theorem 4: Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where a,b $\in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on [a,b], for $p > 1$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},|f'(a)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \quad (1.2)$$

$$+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].$$

Theorem 5: Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, a,b $\in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex on [a,b], $q \ge 1$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\le \frac{b-a}{8}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(a)|^q\right\}\right)^{\frac{1}{q}}+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right\}\right)\right]. \quad (1.3)$$

For recent results related to quasi-convex functions, we refer interest of readers to (see Alomari *et al.*, 2009a; Alomari *et al.* 2009b; Alomari and Darus, 2010a; Alomari and Darus, 2010b; Dragomir and Pearce 1998; Set *et al.*, 2010; Sarıkaya *et al.*, 2010; Tseng *et al.*, 2003).

The main purpose of this study is to generalize the Theorem 3, Theorem 4 and Theorem 5 for quasi-convex functions using the Lemma 1.

2. Hermite-Hadamard Type Inequalities For Quasi-Convex Functions

In order to prove our main theorems, we need the following lemma which was proved by Kavurmacı *et al* (2010).

Lemma 1: Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°(the interior of I) where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du$$

$$=\frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt.$$

Theorem 6: : Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°(the interior of I) such that $f' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is quasi-convex on [a,b], then the following inequality holds:

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{(x-a)^2}{2(b-a)}\max\{|f'(x)|,|f'(a)|\}+\frac{(b-x)^2}{2(b-a)}\max\{|f'(x)|,|f'(b)|\}.$$

Proof: From Lemma 1, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|dt$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\max\{\left|f'(x)\right|,\left|f'(a)\right|\}dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\max\{\left|f'(x)\right|,\left|f'(b)\right|\}dt$$

$$\leq \frac{(x-a)^2}{2(b-a)}\max\{\left|f'(x)\right|,\left|f'(a)\right|\}+\frac{(b-x)^2}{2(b-a)}\max\{\left|f'(x)\right|,\left|f'(b)\right|\}.$$

which completes the prof.

Remark 1: *In Theorem 6, if choose* $x=\frac{a+b}{2}$*, we obtain (1.1) inequality.*

Theorem 7: Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°(the interior of I) such that $f'\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $\left|f'\right|^{\frac{p}{p-1}}$ is quasi-convex on [a,b], $p>1$, then the following inequality holds:

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^{\frac{p}{p-1}},\left|f'(a)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}.$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof: From Lemma 1 and using well known Hölder inequality, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)a)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)b)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{\left|f'(x)\right|^{\frac{p}{p-1}},\left|f'(a)\right|^{\frac{p}{p-1}}\right\}dt\right)^{\frac{p-1}{p}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{\left|f'(x)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

$$=\frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^{\frac{p}{p-1}},\left|f'(a)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}.$$

which completes the proof.

Remark 2: *In Theorem 7, if choose* $x=\frac{a+b}{2}$*, we obtain (1.2) inequality.*

Theorem 8: : Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°(the interior of I) such that $f'\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f'|^q$ is quasi-convex on [a,b], $q\geq 1$, then the following inequality holds:

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq\frac{(x-a)^2}{2(b-a)}\left(\max\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{2(b-a)}\left(\max\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Proof: From Lemma 1 and using the well known power mean inequality, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq\frac{(x-a)^2}{b-a}\int_0^1(1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left|f'(tx+(1-t)b)\right|dt$$

$$\leq\frac{(x-a)^2}{b-a}\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

Since $|f'|^q$ is quasi-convex we have

$$\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|^q dt \le \frac{1}{2}\max\left\{\left|f'(x)\right|^q, \left|f'(a)\right|^q\right\}$$

and

$$\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|^q dt \le \frac{1}{2}\max\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}$$

Therefore, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\le \frac{(x-a)^2}{2(b-a)}\left(\max\left\{\left|f'(x)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{2(b-a)}\left(\max\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Remark 3: *In Theorem 8, if choose* $x = \dfrac{a+b}{2}$*, we obtain (1.3) inequality.*

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