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Research Article

Projective coordinate spaces over modules

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Abstract

In this paper, we investigate some properties of the (right) modules constructed over the local ring and also construct a projective coordinate space over the (right) module. Finally, in a 3-dimensional projective coordinate space, the incidence matrix for a line that combines the certain two points and also all points of a line given with the incidence matrix are found by the help of Maple programme.

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1. Introduction

Jukl, in [5], introduced the real plural algebra of order m and so investigated the linear forms on a free finite dimensional module M, especially their kernel. Jukl continued to study on free finite dimensional modules in [6]. In [3], Erdoğan et. al. investigated some properties of the (left) modules constructed over the real plural algebra and later, in [2], Çiftçi and Erdoğan obtained an n-dimensional projective coordinate space over $(n+1)$-dimensional (left) module constructed by the help of this real plural algebra. For more detailed information on modules, see [8]. For the algebraic and linear algebraic notions that will be used throughout this paper, we refer to [4] and [9].

In this paper we will study by the algebra $\boldsymbol{A} := F\eta_0 + F\eta_1 + F\eta_2 + ... + F\eta_{n-1}$ with a basis $\{1, \eta_1, \eta_2, \eta_3, ..., \eta_{n-1}\}$ such that $\eta_i\eta_j = 0$ for $\eta_i \notin F$ (where F is a field). We immediately state that this algebra is not isomorphic to the real plural algebra of order m. For this reason, by taking this algebra instead of the real plural algebra of order m, we will reconsider almost all of the results that are obtained in [2, 3]. So, we will be able to investigate some properties of the (right) modules constructed over the algebra and also to construct an $(m-1)$- dimensional projective coordinate space over the m- dimensional (right) module.

The remainder of the paper is organized as follows. In Section 2, there are some basic definitions and results from the literature. In Section 3, we investigate some properties of the (right) modules constructed over $\boldsymbol{A}$. In Section 4, we construct an projective coordinate space over the (right) module. Finally, in a 3-dimensional projective coordinate space, the incidence matrix for a line that combines the certain two points and also all points of a line given with the incidence matrix are found by the help of Maple programme.

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2. Preliminaries

In this section, first of all, we will start by recalling some definitions and results from [5].

Definition 2.1 ([5, Def. 1.1])**.** The real plural algebra of order n is every linear algebra $\mathcal{A}$ on $\mathbb{R}$ having as a vector space over $\mathbb{R}$ a basis $\{1, \eta, \eta^2, \cdots, , \eta^{n-1}\}$ where $\eta^n = 0$ for $\eta \notin \mathbb{R}$.

By Definition 2.1, we see that an element x of $\mathcal{A}$ is of the form $x = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{n-1}\eta^{n-1}$ where $a_i \in \mathbb{R}$ for $0 \leq i \leq n-1$.

A ring with identity element is called local if the set of its non-units form an ideal.

Now we can state the following two results without proof.

Proposition 2.2 ([5, Prop. 1.3])**.** *An element $x = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{n-1}\eta^{n-1} \in \mathcal{A}$ is a unit if and only if $a_0 \neq 0$.*

Proposition 2.3 ([5, Prop. 1.5])**.** *$\mathcal{A}$ is a local ring with the maximal ideal $\eta\mathcal{A}$. The ideals $\eta^j\mathcal{A}$, $1 \leq j \leq n$, are all ideals in $\mathcal{A}$.*

In [5, Prop. 1.7], it is stated that $\mathcal{A}$ is isomorphic to the linear algebra of matrix $M_{nn}(\mathbb{R})$ of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \\ 0 & b_0 & b_1 & \ddots & \ddots & b_{n-2} \\ \vdots & 0 & b_0 & \ddots & b_2 & \ddots \\ \vdots & \vdots & 0 & \ddots & b_1 & b_2 \\ \vdots & \vdots & \vdots & \ddots & b_0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_0 \end{pmatrix},$$

where $b_i \in \mathbb{R}$ for $0 \leq i \leq n-1$ (for the detailed proof of this fact, see [3]).

A module that is constructed over a local ring $\boldsymbol{A}$ is called an $\boldsymbol{A}$-module. So, we can give the following definition.

Definition 2.4. Let $\boldsymbol{A}$ be a local ring. Let M be a finitely generated $\boldsymbol{A}$-module. Then M is an $\boldsymbol{A}$-space of finite dimension if there exists $E_1, E_2, ..., E_n$ in M with

i) $M = E_1\boldsymbol{A} \oplus E_2\boldsymbol{A} \oplus ... \oplus E_n\boldsymbol{A}$

ii) the map $\boldsymbol{A} \to E_i\boldsymbol{A}$ *defined by* $x \to E_i x$ *is an isomorphism for* $1 \leq i \leq n$.

Let F be a field. Consider $\boldsymbol{A} := F\eta_0 + F\eta_1 + F\eta_2 + ... + F\eta_{n-1}$ with componentwise addition and multiplication as follows:

$$\begin{aligned} a \cdot b &= (a_0 + a_1\eta_1 + a_2\eta_2 + ... + a_{n-1}\eta_{n-1}) \cdot (b_0 + b_1\eta_1 + b_2\eta_2 + ... + b_{n-1}\eta_{n-1}) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)\,\eta_1 + (a_0b_2 + a_2b_0)\,\eta_2 + ... + (a_0b_{n-1} + a_{n-1}b_0)\,\eta_{n-1}, \end{aligned}$$

where $\eta_i\eta_j = 0$ for $1 \leq i, j \leq n-1$ and the set $\{1, \eta_1, \eta_2, \eta_3, ..., \eta_{n-1}\}$ is a basis of $\boldsymbol{A}$. Then, $\boldsymbol{A}$ is a unital, commutative and associative local ring with the maximal ideal $\mathbf{I} = \eta\ \boldsymbol{A} = \{a_1\eta_1 + a_2\eta_2 + ... + a_{n-1}\eta_{n-1}\,|a_i \in F,\ \ 1 \leq i \leq n-1\}$. So, we can reach the result that an element $\alpha = a_0 + a_1\eta_1 + a_2\eta_2 + ... + a_{n-1}\eta_{n-1} \in \boldsymbol{A}$ is a unit if and only if $a_0 \neq 0$. In that case, note that $\alpha^{-1} = a_0^{-1} - a_0^{-1}a_1a_0^{-1}\eta_1 - a_0^{-1}a_2a_0^{-1}\eta_2 - \cdots - a_0^{-1}a_{n-1}a_0^{-1}\eta_{n-1}$. Moreover, the local ring we will study on is considered as the vector space $F(n) := F \times F^{n-1} = \{(a_0, v) \mid v = (a_1, a_2, ..., a_{n-1}) \in F^{n-1}\}$ with with componentwise addition and multiplication as follows:

$$\begin{aligned} a \cdot b &= (a_0, v) \cdot (b_0, w) \\ &= (a_0b_0, a_0w + vb_0) \\ &= (a_0b_0, (a_0b_1 + a_1b_0, a_0b_2 + a_2b_0, ..., a_0b_{n-1} + a_{n-1}b_0))\,, \end{aligned}$$

where $v = (a_1, a_2, \cdots a_{n-1})$, $w = (b_1, b_2, \cdots, b_{n-1}) \in F^{n-1}$. In this case, $F(n)$ is local with $\mathbf{I} = \{0\} \times F^{n-1}$ as ideal of non-units. For more detailed information on $F(n)$, see [1].

Hence, it is clear that the local ring $\boldsymbol{A}$ is not isomorphic to the real plural algebra of order n. But, it is isomorphic to $F(n)$. Throughout this paper we restrict ourselves to the local ring $\boldsymbol{A}$.

3. $\boldsymbol{A}$-Modules

In this section, we investigate some properties of the (right) modules constructed over $\boldsymbol{A}$. First, we give the following result, the analogue of Theorem 6 in [3]

Proposition 3.1. *None of the units of $\boldsymbol{A}$ are zero divisors, namely for every $\alpha, \beta \in \boldsymbol{A}$; $\alpha = a_0 + a_1\eta_1 + a_2\eta_2 + ... + a_{n-1}\eta_{n-1}$, $a_0 \neq 0$ and $\beta = b_0 + b_1\eta_1 + b_2\eta_2 + ... + b_{n-1}\eta_{n-1}$ if $\alpha \cdot \beta = 0$ or $\beta \cdot \alpha = 0$, then $\beta = 0$. Also for $1 \leq k \leq n-1$ and $\alpha = a_k\eta_k + a_{k+1}\eta_{k+1} + ... + a_{n-1}\eta_{n-1}$, $a_k \neq 0$ if $\alpha \cdot \beta = 0$ or $\beta \cdot \alpha = 0$ then $\beta = b_1\eta_1 + b_2\eta_2 + ... + b_{n-1}\eta_{n-1}$.*

Proof. If α is a unit, then there is an inverse element α^{-1} and since $\boldsymbol{A}$ is associative; $\alpha \cdot \beta = 0 \Rightarrow \alpha^{-1}(\alpha \cdot \beta) = \alpha^{-1} \cdot 0 \Rightarrow \beta = 0$. For $\beta \cdot \alpha = 0$, it is easily seen that $\beta = 0$ by similar calculations.

Now let $\alpha = a_k\eta_k + a_{k+1}\eta_{k+1} + ... + a_{n-1}\eta_{n-1}$, $a_k \neq 0$ for $1 \leq k \leq n-1$, we have $\alpha \cdot \beta = (a_k\eta_k + a_{k+1}\eta_{k+1} + ... + a_{n-1}\eta_{n-1})(b_0 + b_1\eta_1 + b_2\eta_2 + ... + b_{n-1}\eta_{n-1}) = 0$. Then $\alpha \cdot \beta = 0 \Rightarrow (a_k b_0)\,\eta_k + (a_{k+1} b_0)\,\eta_{k+1} + ... + (a_{n-1} b_0)\,\eta_{n-1} = 0\eta_k + 0\eta_{k+1} + ... + 0\eta_{n-1}$

Forcing the coefficient of η_k to be zero, we obtain $a_k b_0 = 0$, and since $a_k \neq 0$, we find $b_0 = 0$. Thus we have $\beta = b_1\eta_1 + b_2\eta_2 + ... + b_{n-1}\eta_{n-1}$, $b_i \in F$, $1 \leq i \leq n-1$. □

Now, we can state the following result without proof, the analogue of Proposition 7 in [3].

Proposition 3.2. *Let $\mathbf{K} = M_{nn}(F)$ be the (linear) algebra of matrix of the form*

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_0 \end{pmatrix}_{n \times n},$$

where $a_i \in F$ for $0 \leq i \leq n-1$. Then the map $f : \boldsymbol{A} \to \mathbf{K} = M_{nn}(F)$ which is given as

$$f(\alpha) = (a_{ij}) = \begin{cases} a_{ii} = a_0 \ , & 1 \leq i \leq n \\ a_{1j} = a_{j-1} \ , & 2 \leq j \leq n \\ a_{ij} = 0 & otherwise \end{cases}$$

for every $\alpha = a_0 + a_1\eta_1 + a_2\eta_2 + ... + a_{n-1}\eta_{n-1} \in \boldsymbol{A}$ is an isomorphism.

Now we would like to find a basis of $\mathbf{K} = M_{nn}(F)$. Let us take any element of $\mathbf{K}$ such that

$$\mathbf{a} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ 0 & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_0 \end{pmatrix} \in \mathbf{K}.$$

Then, the element can be written in the following form:

$$\mathbf{a} = a_0 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} +$$

$$a_2 \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus we have $\mathbf{a} = a_0 I_n + a_1\eta_1 + a_2\eta_2 + ... + a_{n-1}\eta_{n-1}$. Moreover, the set $\{\eta_0 = I_n, \eta_1, \eta_2, ..., \eta_{n-1}\}$ is a basis of $\mathbf{K}$. We can express any element of this set in general as follows: for $1 \leq k \leq n-1$, $\eta_k = (a_{ij})_{n\times n}$, where

$$(a_{ij}) = \begin{cases} a_{1j} = 1 & j = k+1, \\ a_{ij} = 0 & \textit{otherwise}. \end{cases}$$

Now, we will construct a (right) module M over the algebra $\boldsymbol{A}$, by the following proposition, although a (left) module is obtained in Proposition 8 of [3]. Thanks to this, we will obtain a basis of M.

Proposition 3.3. *$M = F_n^m$ is a right module over the linear algebra of matrix $\mathbf{K} = M_{nn}(F)$. Then the following set as a basis of $\mathbf{K}$-(right)module M.*

$$\left\{ E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ldots, \right.$$

$$\left. E_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

Proof. Linear independence of this set is obvious. Moreover for every $X \in M$, X can be written as follows:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix}_{m\times n}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_{11} \end{pmatrix}_{n\times n}$$

$$+\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ 0 & x_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_{21} \end{pmatrix}_{n\times n}$$

$$+\cdots+\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \\ 0 & x_{m1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_{m1} \end{pmatrix}_{n\times n}.$$

Thus $[E_1, E_2, \cdots, E_m] = M$. Consequently, the set $\{E_1, E_2, \cdots, E_m\}$ is a basis of $\boldsymbol{K}$-module M. □

Now, from [7], we give a definiton, will be used in the next section.

Definition 3.4. Let R be a local ring, R_0 be the maximal ideal of R and M be a free module with unity over R. Let S be a non-empty subset of the module M. Let M_0 be a submodule of M constructed over R_0. For $x_1, x_2, \cdots, x_k \in S$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in R$, if

$$\sum_{i=1}^{k} \alpha_i x_i \in M_0 \Rightarrow \alpha_i \in R_0 \text{ for every } i$$

holds, then S is called $R-$independent. Otherwise, S is called an $R-$dependent subset.

Finally, we would like to complete this section by giving two results, without proof, on $\boldsymbol{A}$-spaces. They are the analogues of Theorem 9 and Proposition 10 in [3], respectively.

Proposition 3.5. *Let* $M = \boldsymbol{A}^n$. *Then, for* $u_1, u_2, ..., u_k \in \boldsymbol{A} \setminus \mathbf{I}$ *and* $x_{ij} \in \mathbf{I}$, *there are linearly independent vectors such that* $\alpha_1 = (u_1, x_{21}, x_{31}, ..., x_{n1})$, $\alpha_2 = (x_{12}, u_2, x_{32}, ..., x_{n2})$, $\alpha_3 = (x_{13}, x_{23}, u_3, ..., x_{n3})$,...,$\alpha_k = (x_{1k}, x_{2k}, x_{3k}, ..., u_k)$. *For* $k = n$, *the set* $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *is a basis for* M.

Proposition 3.6. *An* $\mathbf{A}$*-module* M *over a local ring* $\mathbf{A}$ *is an* $\mathbf{A}$*-space if and only if it is a free finitely dimensional module.*

4. Construction of a projective coordinate space

In this section, an $(m-1)$-dimensional projective coordinate space over the right module obtained in the previous section will be constructed with the help of equivalence classes, by following the similar method given in [2]. So, the points and lines of this space are determined and the points are classified.

We know from the previous section that, the set $M = F_n^m$ is a m-dimensional right-module over the local ring $\mathbf{K} = M_{nn}(F)$ and the set $\{E_1, E_2, \ldots, E_m\}$ is a basis of M. Each element of a $\mathbf{K}$-module M can be expressed uniquely as a linear combination of $E_1, E_2, \ldots, E_m$. Furthermore a maximal ideal of $\mathbf{K}$ is denoted by

$$\mathbf{I} = \left\{ \begin{pmatrix} 0 & a_1 & \cdots & \cdots & a_{n-1} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; a_i \in F, 1 \le i \le n-1 \right\}.$$

Now let us define the set

$$M_0 = \left\{ \sum_{i=1}^{m} E_i A_i \,|A_i \in \mathbf{I},\ 1 \leq i \leq m \right\}.$$

Then, we get

$$M_0 = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & x_{m2} & \cdots & x_{mn} \end{pmatrix} \mid x_{ij} \in F \right\}.$$

Now, we consider equivalence relation on the elements of

$$M^* = M \setminus M_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} |1 \leq i \leq m,\ \exists\ x_{i1} \neq 0 \right\},$$

whose equivalence classes are the one-dimensional right submodules of M with the set M_0 deleted. Thus, if $X, Y \in M^*$, then X is equivalent to Y if $Y = X\lambda$ for $\lambda \in \mathbf{K}^* = \mathbf{K} \backslash \mathbf{I}$. The set of equivalence classes is denoted by $P(M)$. Then $P(M)$ is called an $(m-1)$-dimensional projective coordinate space and the elements of $P(M)$ are called points; the equivalence class of vector X is the point $\overline{X}$. Consequently, X is called a coordinate vector for $\overline{X}$ or that X is a vector representing of $\overline{X}$. In this case, $X\lambda$ with $\lambda \in \mathbf{K}^*$ also represents $\overline{X}$; that is, by $\overline{X\lambda} = \overline{X}$. Thus, $\overline{X}$ can be expressed as follows:

$$\begin{aligned} \overline{X} &= \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ 0 & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_0 \end{pmatrix}_{n\times n} \\ &= \begin{pmatrix} x_{11}a_0 & x_{11}a_1 + x_{12}a_0 & \cdots & x_{11}a_{n-1} + x_{1n}a_0 \\ x_{21}a_0 & x_{21}a_1 + x_{22}a_0 & \cdots & x_{21}a_{n-1} + x_{2n}a_0 \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1}a_0 & x_{m1}a_1 + x_{m2}a_0 & \cdots & x_{m1}a_{n-1} + x_{mn}a_0 \end{pmatrix}_{m\times n}, \end{aligned}$$

where $a_0 \neq 0 \ \wedge\ 1 \leq i \leq m, \quad \exists\ x_{i1} \neq 0$.

Let $\overline{X}, \overline{Y}, \cdots$ be $p+1$ points such that any two of them are $\mathbf{K}$-independent. Then the set $\Pi_p = Sp\{\overline{X}, \overline{Y}, \cdots\} \backslash M_0$ is called a subspace of dimension p or p-space.

In $P(M)$, a point is a subspace of dimension 0 and a line is a subspace of dimension 1.

For $X \in M^*$, the set $\overline{X} = \{X\lambda \,|\lambda \in \mathbf{K}^*\}$ is a 0-dimensional subspace of $P(M)$. So, $\overline{X}$ is a point of $P(M)$.

Now, we investigate the condition of being $\mathbf{K}$-independent for two different points $\overline{X}$ and $\overline{Y}$ of $P(M)$.

Firstly, let us denote the coordinate vectors for the points $\overline{X}$ and $\overline{Y}$ by X and Y, respectively. We form a linear combination as

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ 0 & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_0 \end{pmatrix} +$$

$$\begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & \cdots & b_{n-1} \\ 0 & b_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & b_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & b_0 \end{pmatrix} =$$

$$\begin{pmatrix} (x_{11}a_0+y_{11}b_0) & (x_{11}a_1+x_{12}a_0)+(y_{11}b_1+y_{12}b_0) & \cdots & (x_{11}a_{n-1}+x_{1n}a_0)+(y_{11}b_{n-1}+y_{1n}b_0) \\ (x_{21}a_0+y_{21}b_0) & (x_{21}a_1+x_{22}a_0)+(y_{21}b_1+y_{22}b_0) & \cdots & (x_{21}a_{n-1}+x_{2n}a_0)+(y_{21}b_{n-1}+y_{2n}b_0) \\ \vdots & \vdots & \vdots & \vdots \\ (x_{m1}a_0+y_{m1}b_0) & (x_{m1}a_1+x_{m2}a_0)+(y_{m1}b_1+y_{m2}b_0) & \cdots & (x_{m1}a_{n-1}+x_{mn}a_0)+(y_{m1}b_{n-1}+y_{mn}b_0) \end{pmatrix}.$$

If this matrix is an element of M_0 then we can write

$$\begin{aligned} x_{11}a_0 + y_{11}b_0 &= 0, \\ x_{21}a_0 + y_{21}b_0 &= 0, \\ &\vdots \\ x_{m1}a_0 + y_{m1}b_0 &= 0. \end{aligned} \tag{4.1}$$

Let us denote the coefficient matrix of (4.1) by

$$A = \begin{pmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \\ \vdots & \vdots \\ x_{m1} & y_{m1} \end{pmatrix}.$$

If $rankA = 2$, then we get $a_0 = b_0 = 0$. So this shows that

$$\begin{pmatrix} 0 & a_1 & \cdots & a_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_1 & \cdots & b_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbf{I}.$$

In that case, the coordinate vectors X and Y for the points $\overline{X}$ and $\overline{Y}$, respectively, are **K**-independent if and only if the rank of the coefficient matrix is equal to 2. That is, first columns of the coordinate vectors X and Y are linearly independent vectors.

Let the set $Sp\{\overline{X}, \overline{Y}\} = \{X\lambda + Y\gamma \,|\, \exists\ \lambda, \gamma \in \mathbf{K}^*\}$ be a 1-dimensional subspace of $P(M)$ such that $\overline{X}$ and $\overline{Y}$ are **K**-independent elements. Then $Sp\{\overline{X}, \overline{Y}\}$ is a line of $P(M)$. It is denoted by

$$Sp\{\overline{X},\overline{Y}\} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ 0 & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_0 \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & \cdots & b_{n-1} \\ 0 & b_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & b_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & b_0 \end{pmatrix},$$

where $a_0 \neq 0 \wedge 1 \leq i \leq m$, $\exists\, x_{i1} \neq 0$ or $b_0 \neq 0 \wedge 1 \leq i \leq m$, $\exists y_{i1} \neq 0$.

We know that for every coordinate vector $X \in M^*$ of the point $\overline{X} \in P(M)$, X can be written uniquely as a linear combination of the vectors $E_1, E_2, \cdots, E_m$. So the matrix X is expressed as $X = \sum_{i=1}^{m} E_i X_i$ or as

$$X = (X_1, X_2, \cdots, X_m) \in \mathbf{K}^m,$$

where

$$X_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{11} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & x_{11} \end{pmatrix},\ X_2 = \begin{pmatrix} x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ 0 & x_{21} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & x_{21} \end{pmatrix},$$

$$\cdots,\ X_m = \begin{pmatrix} x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \\ 0 & x_{m1} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & x_{m1} \end{pmatrix}.$$

There are two cases:

Case 1: For the first component of the coordinate vector X of the point $\overline{X}$, if $x_{11} \neq 0$, then $X_1 \notin \mathbf{I}$ and

$$X_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{11} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & x_{11} \end{pmatrix}$$

is a unit element so there is an inverse of X_1. If we multiply both sides of the equation with the inverse matrix X_1^{-1}, we get

$$X=(I_n,X_2,\cdots,X_m)=\begin{pmatrix}1&0&\cdots&0\\x_{21}&x_{22}&\cdots&x_{2n}\\\vdots&\vdots&\vdots&\vdots\\x_{m1}&x_{m2}&\cdots&x_{mn}\end{pmatrix}.$$

Thus, this type of points are called proper points.

Case 2: For the first component of the coordinate vector X of the point $\overline{X}$, if $x_{11}=0$, then $X_1\in\mathbf{I}$. So, the inverse of the matrix X_1 does not exist. Thus we call the points of $P(M)$ whose coordinate vectors are in the form

$$\begin{pmatrix}0&x_{12}&\cdots&x_{1n}\\x_{21}&x_{22}&\cdots&x_{2n}\\\vdots&\vdots&\vdots&\vdots\\x_{m1}&x_{m2}&\cdots&x_{mn}\end{pmatrix}$$

as ideal points.

Now, by giving a definition we will handle a special example related to the definition.

Definition 4.1. An s-space is the set of points whose representing vectors

$$\begin{pmatrix}x_{11}&x_{12}&\cdots&x_{1n}\\x_{21}&x_{22}&\cdots&x_{n}\\\vdots&\vdots&\vdots&\vdots\\x_{m1}&x_{m2}&\cdots&x_{mn}\end{pmatrix}$$

of the points $\overline{X}$ satisfy the equations $XA=0$, where A is an $m\times((m-1)-s)$ matrix of rank $(m-1)-s$ with coefficients in $\boldsymbol{K}$.

Now let us take $m=4$ and $n=2$, so we study an example of a 3-dimensional projective coordinate space $P(M)$. For the 3-dimensional projective coordinate space, first we will determine all points of a line whose incidence matrix is given and then we will determine the incidence matrix of a line that goes through the given points.

Example 4.2. In the 3-dimensional projective coordinate space $P(M)$, any line, 1-dimensional subspace Π_1 is the set of points whose representing vectors $\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\\x_{31}&x_{32}\\x_{41}&x_{42}\end{pmatrix}$ of the points $\overline{X}$ satisfy the equations $XA=0$, where A is a 4×2 matrix of rank 2 with coefficients in $\boldsymbol{K}$. Thus $\Pi_1=\left\{\overline{X}\ \middle|XA=0, A\in\boldsymbol{K}_2^4\diagdown I_2^4\right\}$ is obtained. First, we identify all points of a line whose incidence matrix is

$$\begin{bmatrix}a&e\\b&f\\c&g\\d&h\end{bmatrix}=\begin{bmatrix}\begin{pmatrix}a_0&a_1\\0&a_0\end{pmatrix}&\begin{pmatrix}e_0&e_1\\0&e_0\end{pmatrix}\\\begin{pmatrix}b_0&b_1\\0&b_0\end{pmatrix}&\begin{pmatrix}f_0&f_1\\0&f_0\end{pmatrix}\\\begin{pmatrix}c_0&c_1\\0&c_0\end{pmatrix}&\begin{pmatrix}g_0&g_1\\0&g_0\end{pmatrix}\\\begin{pmatrix}d_0&d_1\\0&d_0\end{pmatrix}&\begin{pmatrix}h_0&h_1\\0&h_0\end{pmatrix}\end{bmatrix}\in\boldsymbol{K}_2^4\diagdown I_2^4.$$

As a consequence of the incidence matrix, it is trivial to see that $\exists a_0,b_0,c_0,d_0,e_0,f_0,g_0,h_0\neq 0$.

For $XA=0$, we have the following cases:

Case 1: For the coordinate vector X of the point $\overline{X}$, if $x_{11} \neq 0$, then $X = (I_2, X_2, X_3, X_4) \in \boldsymbol{K}^4$. Thus we obtain the following equations from $XA = 0$:

$$\begin{aligned} a_0 + x_{21}b_0 + x_{31}c_0 + x_{41}d_0 &= 0, \\ a_1 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\ e_0 + x_{21}f_0 + x_{31}g_0 + x_{41}h_0 &= 0, \\ e_1 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 &= 0. \end{aligned} \tag{4.2}$$

If we solve (4.2) by using the Maple programme, we get the following solutions:

$$\begin{aligned} x_{21} &= -\frac{(a_0g_0 - c_0e_0) + (d_0g_0 - c_0h_0)x_{41}}{b_0g_0 - f_0c_0}, \\ x_{22} &= \frac{a'}{c_0^2f_0^2 + b_0^2g_0^2 - 2b_0g_0f_0c_0} \\ x_{31} &= \frac{(-b_0e_0 + f_0a_0) + (f_0d_0 - b_0h_0)x_{41}}{b_0g_0 - f_0c_0} \\ x_{32} &= -\frac{b'}{c_0^2f_0^2 + b_0^2g_0^2 - 2b_0g_0f_0c_0} \end{aligned}$$

$$x_{41} = x_{41},\ x_{42} = x_{42},$$

where

$$a' = \left(\begin{array}{c} \left(\begin{array}{c} g_0^2b_1a_0 - e_1f_0c_0^2 + c_0g_1f_0a_0 + g_0a_1f_0c_0 - g_0c_0f_1a_0 \\ -g_0c_1f_0a_0 - g_0b_1c_0e_0 + b_0g_0c_0e_1 + b_0g_0c_1e_0 - \\ b_0c_0g_1e_0 + f_1c_0^2e_0 - b_0g_0^2a_1 \end{array} \right) + \\ \left(\begin{array}{c} -h_1f_0c_0^2 + f_1c_0^2h_0 - b_0g_0^2d_1 + c_0g_1f_0d_0 + g_0^2b_1d_0 \\ +g_0d_1f_0c_0 - g_0c_1f_0d_0 - g_0f_1c_0d_0 - g_0b_1c_0h_0 + \\ b_0g_0c_0h_1 + b_0g_0c_1h_0 - b_0c_0g_1h_0 \end{array} \right) x_{41} + \\ \left(-h_0f_0c_0^2 - b_0g_0^2d_0 + g_0d_0f_0c_0 + b_0g_0c_0h_0\right) x_{42} \end{array} \right)$$

$$b' = \left(\begin{array}{c} \left(\begin{array}{c} b_0^2e_1g_0 - b_0^2g_1e_0 - c_1f_0^2a_0 + a_1f_0^2c_0 - b_0e_1f_0c_0 - b_0e_1f_0c_0 \\ -f_1a_0b_0g_0 + b_0g_1f_0a_0 + f_1c_0b_0e_0 - f_0b_0a_1g_0 + f_0b_0c_1e_0 + \\ f_0b_1a_0g_0 - f_0b_1c_0e_0 \end{array} \right) + \\ \left(\begin{array}{c} b_0^2h_1g_0 + b_0^2g_1h_0 - b_0h_1f_0c_0 - f_1d_0b_0g_0 + b_0g_1f_0d_0 + f_1c_0b_0h_0 \\ -c_1f_0^2d_0 + d_1f_0^2c_0 + f_0b_0c_1h_0 - f_0b_0d_1g_0 - f_0b_1c_0h_0 + f_0b_1d_0g_0 \end{array} \right) x_{41} \\ +(b_0^2h_0g_0 - b_0h_0f_0c_0 + d_0f_0^2c_0 - f_0b_0d_0g_0)x_{42} \end{array} \right),$$

Case 2: For the coordinate vector X of the point $\overline{X}$, if $x_{11} = 0$, then $\overline{X}$ is an ideal point of the form

$$\left(\begin{array}{c} X_1 = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} x_{21} & x_{22} \\ 0 & x_{21} \end{pmatrix}, \\ X_3 = \begin{pmatrix} x_{31} & x_{32} \\ 0 & x_{31} \end{pmatrix},\ X_4 = \begin{pmatrix} x_{41} & x_{42} \\ 0 & x_{41} \end{pmatrix} \end{array} \right).$$

Here, we know that $\exists\ x_{22}, x_{23}, x_{24} \neq 0$. Thus we obtain the following equations from $XA = 0$:

$$\begin{aligned} x_{21}b_0 + x_{31}c_0 + x_{41}d_0 &= 0, \\ x_{12}a_0 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\ x_{21}f_0 + x_{31}g_0 + x_{41}h_0 &= 0, \\ x_{12}e_0 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 &= 0. \end{aligned} \tag{4.3}$$

If we solve (4.3) by using the Maple programme, we get the following solutions:

$$\begin{aligned}
x_{21} &= \frac{(-d_0 g_0 + h_0 c_0)\, x_{41}}{-f_0 c_0 + b_0 g_0}, \\
x_{22} &= \frac{a''}{g_0^2 {b_0}^2 + f_0^2 c_0^2 - 2 f_0 c_0 b_0 g_0}, \\
x_{31} &= -\frac{(-f_0 d_0 + b_0 h_0)\, x_{41}}{-f_0 c_0 + b_0 g_0}, \\
x_{32} &= -\frac{b''}{g_0^2 {b_0}^2 + f_0^2 c_0^2 - 2 f_0 c_0 b_0 g_0} \\
x_{41} &= x_{41},\ x_{42} = x_{42},\ x_{12} = x_{12}
\end{aligned}$$

where

$$a'' = \left(\begin{array}{c} (-b_0 g_0^2 a_0 + b_0 g_0 e_0 c_0 + g_0 f_0 a_0 c_0 - f_0 c_0^2 e_0)\, x_{12} + \\ \left(\begin{array}{c} -b_0 g_0^2 d_1 + b_0 g_0 h_1 c_0 + b_0 g_0 c_1 h_0 - b_0 c_0 g_1 h_0 + b_1 d_0 g_0^2 \\ + g_0 f_0 d_1 c_0 - g_0 b_1 h_0 c_0 - g_0 c_0 f_1 d_0 - g_0 f_0 c_1 d_0 + f_1 h_0 c_0^2 \\ + c_0 f_0 g_1 d_0 - f_0 c_0^2 h_1) \end{array} \right) x_{41} \\ + (-b_0 g_0^2 d_0 + b_0 g_0 h_0 c_0 + g_0 f_0 d_0 c_0 - f_0 c_0^2 h_0) x_{42} \end{array} \right),$$

$$b'' = \left(\begin{array}{c} (e_0 b_0^2 g_0 - b_0 f_0 e_0 c_0 + f_0^2 a_0 c_0 - f_0 a_0 b_0 g_0)\, x_{12} \\ \left(\begin{array}{c} -b_0 f_0 h_1 c_0 + b_0 f_0 g_1 d_0 - b_0 f_1 g_0 d_0 + b_0 f_1 h_0 c_0 + h_1 b_0^2 g_0 - g_1 b_0^2 h_0 + \\ + f_0^2 d_1 c_0 + b_1 g_0 f_0 d_0 - b_1 h_0 f_0 c_0 + f_0 c_1 b_0 h_0 - f_0^2 c_1 d_0 - f_0 d_1 b_0 g_0 \end{array} \right) x_{41} \\ + (f_0^2 d_0 c_0 + h_0 b_0^2 g_0 - b_0 f_0 h_0 c_0 - f_0 d_0 b_0 g_0) x_{42} \end{array} \right).$$

Now conversely, we have a new situation. We determine the incidence matrix of a line whose points are given. This also has two cases:

Case 1: Let us take the coordinate vectors

$$X = \begin{pmatrix} 1 & 0 \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ y_{21} & y_{22} \\ y_{31} & y_{32} \\ y_{41} & y_{42} \end{pmatrix}$$

of proper points $\overline{X}$ and $\overline{Y}$, respectively. Then we search the incidence matrix of the form

$$A = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} & \begin{pmatrix} e_0 & e_1 \\ 0 & e_0 \end{pmatrix} \\ \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} & \begin{pmatrix} f_0 & f_1 \\ 0 & f_0 \end{pmatrix} \\ \begin{pmatrix} c_0 & c_1 \\ 0 & c_0 \end{pmatrix} & \begin{pmatrix} g_0 & g_1 \\ 0 & g_0 \end{pmatrix} \\ \begin{pmatrix} d_0 & d_1 \\ 0 & d_0 \end{pmatrix} & \begin{pmatrix} h_0 & h_1 \\ 0 & h_0 \end{pmatrix} \end{bmatrix} \in \boldsymbol{K}_2^4 \diagdown I_2^4.$$

we know that the coordinate vectors of these points are as follows

$$X = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_{21} & x_{22} \\ 0 & x_{21} \end{pmatrix}, \begin{pmatrix} x_{31} & x_{32} \\ 0 & x_{31} \end{pmatrix}, \begin{pmatrix} x_{41} & x_{42} \\ 0 & x_{41} \end{pmatrix} \right)$$

and

$$Y = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y_{21} & y_{22} \\ 0 & y_{21} \end{pmatrix}, \begin{pmatrix} y_{31} & y_{32} \\ 0 & y_{31} \end{pmatrix}, \begin{pmatrix} y_{41} & y_{42} \\ 0 & y_{41} \end{pmatrix} \right).$$

Thus we obtain the following equations from $XA = 0$ and $YA = 0$:

$$\begin{aligned}
a_0 + x_{21}b_0 + x_{31}c_0 + x_{41}d_0 &= 0, \\
a_1 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\
e_0 + x_{21}f_0 + x_{31}g_0 + x_{41}h_0 &= 0, \\
e_1 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 &= 0, \\
a_0 + y_{21}b_0 + y_{31}c_0 + y_{41}d_0 &= 0, \\
a_1 + y_{21}b_1 + y_{22}b_0 + y_{31}c_1 + y_{32}c_0 + y_{41}d_1 + y_{42}d_0 &= 0, \\
e_0 + y_{21}f_0 + y_{31}g_0 + y_{41}h_0 &= 0, \\
e_1 + y_{21}f_1 + y_{22}f_0 + y_{31}g_1 + y_{32}g_0 + y_{41}h_1 + y_{42}h_0 &= 0.
\end{aligned} \tag{4.4}$$

If we solve (4.4) by using the Maple programme, then we get the following solutions:

$$\begin{aligned}
&a_0 = -\frac{(x_{21}y_{31} - y_{21}x_{31})c_0 + (x_{21}y_{41} - y_{21}x_{41})d_0}{-y_{21} + x_{21}}, \quad a_1 = -\frac{a_1'}{y_{21}^2 - 2x_{21}y_{21} + x_{21}^2}, \\
&b_0 = -\frac{(x_{31} - y_{31})c_0 + (x_{41} - y_{41})d_0}{-y_{21} + x_{21}}, \quad b_1 = -\frac{b_1'}{y_{21}^2 - 2x_{21}y_{21} + x_{21}^2}, \\
&e_0 = -\frac{(x_{21}y_{31} - y_{21}x_{31})g_0 + (x_{21}y_{41} - y_{21}x_{41})h_0}{-y_{21} + x_{21}}, \quad e_1 = -\frac{e_1'}{y_{21}^2 - 2x_{21}y_{21} + x_{21}^2}, \\
&f_0 = -\frac{(x_{31} - y_{31})g_0 + (x_{41} - y_{41})h_0}{-y_{21} + x_{21}}, \quad f_1 = -\frac{f_1'}{y_{21}^2 - 2x_{21}y_{21} + x_{21}^2}, \\
&c_0 = c_0,\ c_1 = c_1,\ d_0 = d_0,\ d_1 = d_1,\ g_0 = g_0,\ g_1 = g_1,\ h_0 = h_0,\ h_1 = h_1,
\end{aligned}$$

where

$$a_1' = \begin{pmatrix} \begin{pmatrix} y_{42}x_{21}^2 - x_{21}y_{22}x_{41} - y_{21}x_{42}x_{21} + x_{21}y_{22}y_{41} - y_{21}y_{42}x_{21} - \\ y_{21}x_{22}y_{41} + y_{21}x_{22}x_{41} + x_{42}y_{21}^2 \end{pmatrix} d_0 \\ +(y_{41}x_{21}^2 - y_{21}y_{41}x_{21} - y_{21}x_{41}x_{21} + x_{41}y_{21}^2)d_1 \\ + \begin{pmatrix} y_{32}x_{21}^2 - y_{21}x_{32}x_{21} + x_{21}y_{22}y_{31} - x_{21}y_{22}x_{31} - y_{21}y_{32}x_{21} + \\ x_{32}y_{21}^2 + y_{21}x_{22}x_{31} - y_{21}x_{22}y_{31} \end{pmatrix} c_0 \\ +(y_{31}x_{21}^2 - y_{21}x_{31}x_{21} - y_{21}y_{31}x_{21} + x_{31}y_{21}^2)c_1 \end{pmatrix},$$

$$b_1' = \begin{pmatrix} (-x_{22}x_{31} - y_{22}y_{31} + y_{22}x_{31} - y_{32}x_{21} + x_{22}y_{31} + x_{32}x_{21} + y_{32}y_{21} - x_{32}y_{21})c_0 \\ +(x_{31}x_{21} - x_{31}y_{21} - y_{31}x_{21} + y_{31}y_{21})c_1 \\ +(-y_{42}x_{21} + y_{42}y_{21} - x_{42}y_{21} + y_{22}x_{41} + x_{22}y_{41} - y_{22}y_{41} - x_{22}x_{41} + x_{42}x_{21})d_0 \\ +(x_{41}x_{21} + y_{41}y_{21} - y_{41}x_{21} - x_{41}y_{21})d_1 \end{pmatrix},$$

$$e_1' = \begin{pmatrix} \begin{pmatrix} y_{32}x_{21}^2 - x_{21}y_{22}x_{31} - y_{21}x_{32}x_{21} - y_{21}y_{32}x_{21} + \\ x_{21}y_{22}y_{31} + x_{32}y_{21}^2 + y_{21}x_{22}x_{31} - y_{21}x_{22}y_{31} \end{pmatrix} g_0 + \\ (y_{31}x_{21}^2 - y_{21}x_{31}x_{21} - y_{21}y_{31}x_{21} + x_{31}y_{21}^2)g_1 + \\ \begin{pmatrix} y_{42}x_{21}^2 - x_{21}y_{22}x_{41} - y_{21}x_{42}x_{21} + x_{21}y_{22}y_{41} - y_{21}y_{42}x_{21} \\ +x_{42}y_{21}^2 - y_{21}x_{22}y_{41} + y_{21}x_{22}x_{41} \end{pmatrix} h_0 \\ +(y_{41}x_{21}^2 - y_{21}y_{41}x_{21} - y_{21}x_{41}x_{21} + x_{41}y_{21}^2)h_1 \end{pmatrix},$$

$$f_1' = \begin{pmatrix} (-y_{42}x_{21} + y_{42}y_{21} - x_{42}y_{21} + y_{22}x_{41} + x_{22}y_{41} - y_{22}y_{41} - x_{22}x_{41} + x_{42}x_{21})h_0 \\ +(-y_{41}x_{21} - x_{41}y_{21} + y_{41}y_{21} + x_{41}x_{21})h_1 \\ +(-x_{22}x_{31} - y_{22}y_{31} + y_{22}x_{31} - y_{32}x_{21} + x_{22}y_{31} + x_{32}x_{21} + y_{32}y_{21} - x_{32}y_{21})g_0 \\ +(x_{31}x_{21} - x_{31}y_{21} - y_{31}x_{21} + y_{31}y_{21})g_1 \end{pmatrix}.$$

Case 2: Let us take the coordinate vectors

$$X = \begin{pmatrix} 1 & 0 \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \\ y_{41} & y_{42} \end{pmatrix}$$

of proper and ideal points $\overline{X}$ and $\overline{Y}$, respectively. Here for the point $\overline{Y}$, we know that $\exists$ y_{21}, y_{31}, $y_{41} \neq 0$. The coordinate vectors of these points as follows:

$$X = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_{21} & x_2 \\ 0 & x_{21} \end{pmatrix}, \begin{pmatrix} x_{31} & x_{32} \\ 0 & x_{31} \end{pmatrix}, \begin{pmatrix} x_{41} & x_{42} \\ 0 & x_{41} \end{pmatrix} \right)$$

and

$$Y = \left(\begin{pmatrix} 0 & y_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y_{21} & y_{22} \\ 0 & y_{21} \end{pmatrix}, \begin{pmatrix} y_{31} & y_{32} \\ 0 & y_{31} \end{pmatrix}, \begin{pmatrix} y_{41} & y_{42} \\ 0 & y_{41} \end{pmatrix} \right)$$

Then we obtain the following equations from $XA = 0$ and $YA = 0:$

$$\begin{aligned} a_0 + x_{21}b_0 + x_{31}c_0 + x_{41}d_0 &= 0, \\ a_1 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\ e_0 + x_{21}f_0 + x_{31}g_0 + x_{41}h_0 &= 0, \\ e_1 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 &= 0, \\ y_{21}b_0 + y_{31}c_0 + y_{41}d_0 &= 0, \\ y_{12}a_0 + y_{21}b_1 + y_{22}b_0 + y_{31}c_1 + y_{32}c_0 + y_{41}d_1 + y_{42}d_0 &= 0, \\ y_{21}f_0 + y_{31}g_0 + y_{41}h_0 &= 0, \\ y_{12}e_0 + y_{21}f_1 + y_{22}f_0 + y_{31}g_1 + y_{32}g_0 + y_{41}h_1 + y_{42}h_0 &= 0. \end{aligned} \tag{4.5}$$

If we solve (4.5) by using the Maple programme, then we get the following solutions:

$$\begin{aligned} & a_0 = \frac{(x_{41}y_{21} - y_{41}x_{21})\, b_0 + (-y_{41}x_{31} + x_{41}y_{31})\, c_0}{y_{41}},\ a_1 = \frac{a_1''}{y_{41}^2}, \\ & b_0 = b_0,\ \ b_1 = b_1,\ \ c_0 = c_0,\ \ c_1 = c_1, \\ & d_0 = -\frac{y_{21}b_0 + y_{31}c_0}{y_{41}},\ \ d_1 = -\frac{d_1''}{y_{41}^2}, \\ & e_0 = \frac{(-y_{41}x_{31} + x_{41}y_{31})\, g_0 + (-y_{41}x_{21} + x_{41}y_{21})\, f_0}{y_{41}},\ \ e_1 = \frac{e_1''}{y_{41}^2}, \\ & h_0 = -\frac{y_{21}f_0 + y_{31}g_0}{y_{41}},\ \ h_1 = -\frac{h_1''}{y_{41}^2}, \\ & f_0 = f_0,\ \ f_1 = f_1,\ \ g_0 = g_0,\ \ g_1 = g_1, \end{aligned}$$

where

$$a_1'' = \begin{pmatrix} (x_{41}y_{41}y_{22} - x_{41}y_{42}y_{21} - x_{41}y_{12}y_{41}x_{21} + y_{12}x_{41}^2y_{21} + y_{41}x_{42}y_{21} - y_{41}^2x_{22})b_0 \\ +(x_{41}y_{41}y_{21} - y_{41}^2x_{21})b_1 \\ +(x_{41}y_{41}y_{32} + y_{12}x_{41}^2y_{31} - x_{41}y_{42}y_{31} - x_{41}y_{12}y_{41}x_{31} - y_{41}^2x_{32} + y_{41}x_{42}y_{31})c_0 + \\ (x_{41}y_{41}y_{31} - y_{41}^2x_{31})c_1 \end{pmatrix},$$

$$\begin{aligned} d_1'' = {} & (y_{41}y_{22} - y_{42}y_{21} - y_{12}y_{41}x_{21} + y_{12}x_{41}y_{21})b_0 + (y_{41}y_{21})\, b_1 + \\ & (y_{41}y_{32} + y_{12}x_{41}y_{31} - y_{42}y_{31} - y_{12}y_{41}x_{31})c_0 + (x_{41}y_{41}y_{31})\, c_1 \end{aligned}$$

$$e_1'' = \begin{pmatrix} (y_{41}x_{42}y_{21} - y_{41}^2x_{22} + y_{12}x_{41}^2y_{21} + x_{41}y_{41}y_{22} - x_{41}y_{12}y_{41}x_{21} - x_{41}y_{42}y_{21})f_0 \\ +(-y_{41}^2x_{21} + x_{41}y_{41}y_{21})f_1 \\ +(-y_{41}^2x_{32} + y_{41}x_{42}y_{31} - x_{41}y_{42}y_{31} + y_{12}x_{41}^2y_{31} - x_{41}y_{12}y_{41}x_{31} + x_{41}y_{41}y_{32})g_0 \\ +(x_{41}y_{41}y_{31} - y_{41}^2x_{31})g_1 \end{pmatrix},$$

$$\begin{aligned} h_1'' = {} & (y_{12}x_{41}y_{21} + y_{41}y_{22} - y_{12}y_{41}x_{21} - y_{42}y_{21})f_0 + (y_{41}y_{21})\, f_1 \\ & + (-y_{42}y_{31} + y_{12}x_{41}y_{31} - y_{12}y_{41}x_{31} + y_{41}y_{32})g_0 + (y_{41}y_{31})g_1. \end{aligned}$$

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