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Research Article

On generalized weakly symmetric $(LCS)_n$-manifolds

Kanak Kanti Baishya

Department of Mathematics, Kurseong College, Kurseong, W. Bengal- 734 203, India

Abstract

The object of the present paper is to study generalized weakly symmetric and weakly Ricci symmetric $(LCS)_n$-manifolds. Our aim is to bring out different type of curvature restrictions for which $(LCS)_n$-manifolds are sometimes Einstein and some other time remain η-Einstein. Finally, the existence of such manifold is ensured by a non-trivial example.

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Keywords. $(LCS)_n$-manifold, generalized weakly symmetric manifold, Einstein space, η-Einstein space

1. Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) has been initiated by Shaikh [25]. Thereafter, a lot of study has been carried out. For details we refer [5, 12, 19, 27–30, 33] and the references therein.

The notion of weakly symmetric Riemannian manifold have been introduced by Tamássy and Binh [34]. Thereafter, a lot research has been carried out in this topic. For details, we refer to see [1, 2, 10, 13, 14, 21–24, 26, 31, 32] and the references there in.

In the spirit of Tamássy and Binh [34], a Riemannian manifold $(M^n, g)(n > 2)$, is said to be a weakly symmetric manifold, if its curvature tensor $\bar{R}$ of type $(0, 4)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,\ W) &= A_1(X)\bar{R}(Y,U,V,\ W)\\ &+ B_1(Y)\bar{R}(X,U,V,\ W) + B_1(U)\bar{R}(Y,X,V,\ W)\\ &+ D_1(V)\bar{R}(Y,U,X,\ W) + D_1(W)\bar{R}(Y,U,V,\ X) \qquad (1.1)\end{aligned}$$

where A_1, B_1 & D_1 are non-zero 1-forms defined by $A_1(X\ ) = g(X, \sigma_1)$, $B_1(X) = g(X, \varrho_1)$ and $D_1(X) = g(X, \pi_1)$, for all X and $\bar{R}(Y,U,V,\ W) = g(R(Y,U)V,\ W)$, ∇ being the operator of the covariant differentiation with respect to the metric tensor g. An n-dimensional Riemannian manifold of this kind is denoted by $(WS)_n$-manifold.

Email address: kanakkanti.kc@gmail.com
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Keeping in tune with Dubey [11], the author have introduced the notion of a generalized weakly symmetric Riemannian manifold (which is abbreviated hereafter as $(GWS)_n$-manifold). An n-dimensional Riemannian manifold is said to be generalized weakly symmetric if it admits the equation

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,W) &= A_1(X)\bar{R}(Y,U,V,W) + B_1(Y)\bar{R}(X,U,V,W)\\ &+B_1(U)\bar{R}(Y,X,V,W) + D_1(V)\bar{R}(Y,U,X,W)\\ &+D_1(W)\bar{R}(Y,U,V,X) + A_2(X)\bar{G}(Y,U,V,W)\\ &+B_2(Y)\bar{G}(X,U,V,W) + B_2(U)\bar{G}(Y,X,V,W)\\ &+D_2(V)\bar{G}(Y,U,X,W) + D_2(W)\bar{G}(Y,U,V,X) \qquad (1.2)\end{aligned}$$

where

$$\bar{G}(Y,U,V,W) = [g(U,V)g(Y,W) - g(Y,V)g(U,W)] \qquad (1.3)$$

and A_i, B_i & D_i are non-zero 1-forms defined by $A_i(X) = g(X,\sigma_i)$, $B_i(X) = g(X,\varrho_i)$, and $D_i(X) = g(X,\pi_i)$, for $i = 1,2$. The beauty of such $(GWS)_n$-manifold is that it has the flavour of

(i) locally symmetric space [7] (for $A_i = B_i = D_i = 0$),
(ii) locally recurrent space [36] (for $A_1 \neq 0$, $A_2 = B_i = D_i = 0$),
(iii) generalized recurrent space [11] (for $A_i \neq 0$. $B_i = D_i = 0$),
(iv) pseudo symmetric space [8] ($\frac{A_1}{2} = B_1 = D_1 = H_1 \neq 0$, $A_2 = B_2 = D_2 = 0$),
(v) generalized pseudo symmetric space [3] (for $\frac{A_i}{2} = B_i = D_i = H_i \neq 0$),
(vi) semi-pseudo symmetric space [35] ($A_i = B_2 = D_2 = 0, B_1 = D_1 \neq 0$),
(vii) generalized semi-pseudo symmetric space [4] ($A_i = 0, B_i = D_i \neq 0$),
(viii) almost pseudo symmetric space [9] (for $A_1 = H_1 + K_1, B_1 = D_1 = H_1 \neq 0$ and $A_2 = B_2 = D_2 = 0$),
(ix) almost generalized pseudo symmetric space [6] ($A_i = H_i + K_i, B_i = D_i = H_i \neq 0$),
(x) weakly symmetric space [34] (for A_1, B_1, $D_i \neq 0$, $A_2 = B_2 = D_2 = 0$).

Analogously, we have introduced generalized weakly Ricci symmetric $(LCS)_n$-manifold which is defined as follows

An n-dimensional Riemannian manifold is said to be generalized weakly Ricci symmetric if it admits the equation

$$\begin{aligned}(\nabla_X S)(Y,Z) &= A_1(X)S(Y,Z) + B_1(Y)S(X,Z) + D_1(Z)S(Y,X)\\ &+A_2(X)\bar{g}(Y,Z) + B_2(Y)g(X,Z) + D_2(Z)g(Y,X) \qquad (1.4)\end{aligned}$$

where and A_i, B_i & D_i are non-zero 1-forms defined by $A_i(X) = g(X,\sigma_i)$, $B_i(X) = g(X,\varrho_i)$, and $D_i(X) = g(X,\pi_i)$, for $i = 1,2$. The beauty of generalized weakly Ricci symmetric manifold is that it has the flavour of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo Ricci symmetric, generalized pseudo Ricci symmetric, semi-pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric, almost generalized pseudo Ricci symmetric and weakly Ricci symmetric space as special cases.

Now, if the vectors associated to the 1-forms A_1, B_1 & D_1 are respectively co-directional with that of A_2, B_2 & D_2 that is $A_1(X) = \phi A_2(X)$, $B_1(X) = \phi B_2(X)$ & $D_1(X) = \phi D_2(X)\ \forall\ X$, where ϕ being a non-zero constant function, then the relation (1.4) turnes into

$$(\nabla_X Z)(Y,U) = A_1(X)Z(Y,U) + B_1(Y)Z(X,U) + D_1(U)Z(X,U)$$

where $Z(X,Y) = S(X,Y) + \phi\, g(X,Y)$ is well known Z-tensor introduced in ([15, 18]). This leads to the following

Proposition 1.1. *Every generalized weakly Ricci symmetric manifold is a weakly Z-symmetric manifold provided the vector fields associated to the 1-forms A_1, B_1 & D_1 are co-directional with that of A_2, B_2 & D_2 respectively.*

Our work is structured as follows. Section 2 is concerned with $(LCS)_n$-manifolds and some known results. In section 3, we have investigated a generalized weakly symmetric $(LCS)_n$-manifold and it is observed that such a space is an η-Einstein manifold provided $B^*(\xi) \neq -\alpha$. We also tabled different type of curvature restrictions for which $(LCS)_n$-manifolds are sometimes Einstein and some other time remain η-Einstein. Section 4, is concerned with a generalized weakly Ricci-symmetric $(LCS)_n$-manifold which is found to be an η-Einstein space. Finally, we have constructed an example of a generalized weakly symmetric $(LCS)_n$-manifold.

2. $(LCS)_n$-manifolds and some known results

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,\ldots,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(U,U) < 0$ (resp, ≤ 0, $= 0$, > 0), [20]. The category to which a given vector falls is called its causal character.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vecotor field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X) \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0) \tag{2.3}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5), we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor. Thus the Lorentzian manifold M^ntogether with the unit timelike concircular vector field ξ, its associated 1-form η and (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [5]. In a $(LCS)_n$-manifold, the following relations hold [25]:

$$\eta(\xi) = -1, \quad \phi \circ \xi = 0, \tag{2.7}$$

$$\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X) \tag{2.11}$$

for any vector fields X, Y, Z.

Lemma 2.1. *Let* (M^n, g) *be a* $(LCS)_n$*-manifold. Then for any X; Y;Z the following relation holds:*

$$(\nabla_W S)(X,\xi) = (n-1)[\alpha(\alpha^2-\rho)g(X,W) + (2\alpha\rho-\beta)\eta(W)\eta(X)] - \alpha S(X,W) \tag{2.12}$$

In this connection we would like to mention that equation (2.3) is the defining property of concircular or unit time-like torse-forming vector field. In ([16], Theorem 2.1), the authors proved that a Lorentzian manifold is twisted, i.e. the metric is written in the form

$$ds^2 = -dt^2 + f(t, x^\gamma)^2 \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta,$$

if and only if it admits a unit time-like torse-forming vector field. Moreover eq (2.4) and the consequent integrability relations (2.10) and (2.11) in [16] ensure that the unit time-like vector is an eigen vector of the Ricci tensor. Also, Proposition 3.7 of [17] ensures that the space-time is a generalized Robertson-Walker space-time, i.e. the metric is written in the form

$$ds^2 = -dt^2 + f(t)^2 \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta,$$

$\tilde{g}$ being the metric tensor of a $n-1$ dimensional Riemannian manifold.

3. Generalized weakly symmetric $(LCS)_n$-manifold

A non-flat n-dimensional $(LCS)_n$-manifold $(M^n; g)$ $(n > 2)$, is termed as generalized weakly symmetric manifold, if its Riemannian curvature tensor $\bar{R}$ of type $(0; 4)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,W) = {} & A^*(X)\bar{R}(Y,U,V,W) + B^*(Y)\bar{R}(X,U,V,W) \\ & + B^*(U)\bar{R}(Y,X,V,W) + D^*(V)\bar{R}(Y,U,X,W) \\ & + D^*(W)\bar{R}(Y,U,V,X) + \alpha^*(X)G(Y,U,V,W) \\ & + \beta^*(Y)G(X,U,V,W) + \beta^*(U)G(Y,X,V,W) \\ & + \gamma^*(V)G(Y,U,X,W) + \gamma^*(W)G(Y,U,V,X)\end{aligned} \tag{3.1}$$

where

$$G(Y,U,V,W) = [g(U,V)g(Y,W) - g(Y,V)g(U,W)] \tag{3.2}$$

and A^*, B^*, D^*, α^*, β^* & γ^* are non-zero 1-forms which are defined as $A^*(X) = g(X,\theta_1)$, $B^*(X) = g(X,\phi_1)$, $D^*(X) = g(X,\pi_1)$, $\alpha^*(X) = g(X,\theta_2)$, $\beta^*(X) = g(X,\phi_2)$ and $\gamma^*(X) = g(X,\pi_2)$.

Now, contracting U over V in both sides of (3.1) we find

$$\begin{aligned}(\nabla_X S)(Y,W) = {} & A^*(X)S(Y,W) + B^*(Y)S(X,W) + D^*(W)S(Y,X) \\ & - B^*(R(Y,X)W) + D^*(R(X,W)Y) + (n-1)[\alpha^*(X) \\ & g(Y,W) + \beta^*(Y)g(X,W) + \gamma^*(W)g(Y,X)] - \beta^*(Y)g(X,W) \\ & + [\beta^*(X) + \gamma^*(X)]g(Y,W) - \gamma^*(W)g(X,Y)\end{aligned} \tag{3.3}$$

which yields

$$\begin{aligned}&(n-1)[\alpha(\alpha^2-\rho)g(X,W) + (2\alpha\rho-\beta)\eta(W)\eta(X)] - \alpha S(X,W) \\ &= (\alpha^2-\rho)[(n-1)\{A^*(X)\eta(W) + D^*(W)\eta(X)\} + \eta(W)B^*(X) \\ &\quad - g(X,W)B^*(\xi) + \eta(W)D^*(X) - \eta(X)D^*(W)] + B^*(\xi)S(X,W) \\ &\quad + (n-1)[\alpha^*(X)\eta(W) + \beta^*(\xi)g(X,W) + \gamma^*(W)\eta(X)] \\ &\quad - \beta^*(\xi)\, g(X,W) + [\beta^*(X) + \gamma^*(X)]\,\eta(W) - \gamma^*(W)\eta(X)\end{aligned} \tag{3.4}$$

for $Y = \xi$. Setting $X = W = \xi$ in the foregoing equation, we obtain

$$\begin{aligned} -(2\alpha\rho - \beta) &= (\alpha^2 - \rho)[A^*(\xi) + B^*(\xi) + D^*(\xi)] \\ &\quad +[\alpha^*(\xi) + \beta^*(\xi) + \gamma^*(\xi)]. \end{aligned} \tag{3.5}$$

In a weakly symmetric $(LCS)_n$-manifold we have the relation (3.4). Setting $X = \xi$ in (3.4) we get

$$\begin{aligned} (n-2)[(\alpha^2 - \rho)D^*(W) + \gamma^*(W)] = {} & [(n-1)\{(2\alpha\rho - \beta) + (\alpha^2 - \rho)\{A^*(\xi) + B^*(\xi)\}\} \\ & + (\alpha^2 - \rho)D^*(\xi)]\eta(W) + [(n-1)\{\alpha^*(\xi) + \beta^*(\xi)\} \\ & + \gamma^*(\xi)]\eta(W). \end{aligned} \tag{3.6}$$

In view of (3.5), the relation (3.6) reduces to

$$[(\alpha^2 - \rho)D^*(W) + \gamma^*(W)] = -[(\alpha^2 - \rho)D^*(\xi) + \gamma^*(\xi)]\eta(W). \tag{3.7}$$

Again, contracting over Y and W in (3.1) we get

$$\begin{aligned} (\nabla_X S)(U, V) &= A^*(X)S(U, V) + B^*(R(X, U)V) + B^*(U)S(X, V) \\ &\quad +D^*(V)S(U, X) + D^*(R(X, V)U) + (n-1)[\{\alpha^*(X)g(U, V) \\ &\quad +\beta^*(U)g(X, V) + \gamma^*(V)g(X, U)\}] + [\gamma^*(X)g(U, V) \\ &\quad -\gamma^*(V)g(U, X) + \beta^*(X)g(U, V) - \beta^*(U)g(X, V). \end{aligned} \tag{3.8}$$

Setting $V = \xi$ in (3.8) and using (2.12), (2.11), we get

$$\begin{aligned} & (n-1)[\alpha(\alpha^2 - \rho)g(X, U) + (2\alpha\rho - \beta)\eta(U)\eta(X)] - \alpha S(X, U) \\ & \quad = (\alpha^2 - \rho)[(n-1)\{A^*(X)\eta(U) + B^*(U)\eta(X)\} + B^*(X)\eta(U) - B^*(U)\eta(X) \\ & \qquad + D^*(X)\eta(U) - +D^*(\xi)g(X, U)] + D^*(\xi)S(U, X) + (n-1)[\{\alpha^*(X)\eta(U) \\ & \qquad + \beta^*(U)\eta(X) + \gamma^*(\xi)g(X, U)\}] + [\gamma^*(X)\eta(U) \\ & \qquad - \gamma^*(\xi)g(U, X) + \beta^*(X)\eta(U) - \beta^*(U)\eta(X), \end{aligned} \tag{3.9}$$

which turns into

$$[(\alpha^2 - \rho)B(U) + \beta(U)] = -[(\alpha^2 - \rho)B(\xi) + \beta(\xi)]\eta(U) \tag{3.10}$$

for $X = \xi$ and

$$[(\alpha^2 - \rho)A^*(X) + \alpha^*(X)] = -[(\alpha^2 - \rho)A^*(\xi) + \alpha^*(\xi)]\eta(X) \tag{3.11}$$

for $U = \xi$. In view of (3.5), (3.7), (3.10) and (3.11) we have

$$\begin{aligned} (2\alpha\rho - \beta)\eta(X) &= (\alpha^2 - \rho)[A^*(X) + B^*(X) + D^*(X)] \\ &\quad +[\alpha^*(X) + \beta^*(X) + \gamma^*(X)]. \end{aligned} \tag{3.12}$$

Now, making use of (3.10)-(3.12) in (3.4), we find that

$$\begin{aligned} -[\alpha + B^*(\xi)]S(X, W) = {} & [(n-2)\beta^*(\xi) - (\alpha^2 - \rho)\{(n-1)\alpha + B^*(\xi)\}]g(X, W) \\ & - (n-2)[(2\alpha\rho - \beta)\eta(W)\eta(X) + (\alpha^2 - \rho)\{A^*(X)\eta(W) \\ & + D^*(W)\eta(X)\} + \{\alpha^*(X)\eta(W) + \gamma^*(W)\eta(X)\}] \end{aligned} \tag{3.13}$$

which leaves

$$\begin{aligned} S(X, W) &= \left[(\alpha^2 - \rho) + (n-2)\left(\frac{(\alpha^2 - \rho)\alpha - \beta^*(\xi)}{\alpha + B^*(\xi)}\right)\right] g(X, W) \\ &\quad - \frac{(n-2)[(\alpha^2 - \rho)B^*(\xi) + \gamma^*(\xi)]}{[\alpha + B^*(\xi)]}\eta(W)\eta(X) \end{aligned} \tag{3.14}$$

after a straight forward calculation. Approaching in a different manner, we can also have

$$S(X,W) = \left[(\alpha^2-\rho)+(n-2)\left(\frac{(\alpha^2-\rho)\alpha-\gamma^*(\xi)}{\alpha+D^*(\xi)}\right)\right]g(X,W) - \frac{(n-2)[(\alpha^2-\rho)D^*(\xi)+\beta^*(\xi)]}{[\alpha+D^*(\xi)]}\eta(W)\eta(X). \tag{3.15}$$

This leads to the followings.

Theorem 3.1. *A generalized weakly symmetric $(LCS)_n$-manifold $M^n(\phi,\xi,\eta,g)(n>2)$ is an η-Einstein provided that $B^*(\xi)\neq -\alpha$.*

Theorem 3.2. *In an $(LCS)_n$-manifold the following table hold good*

Type of curvature restriction	*Nature of the space corresponding to curvature restriction*
locally symmetric space	*Einstein space*
locally recurrent space	*Einstein space*
generalized recurrent space	*Einstein space*
pseudo symmetric space	*η-Einstein space*
generalized pseudo symmetric space	*η-Einstein space*
semi-pseudo symmetric space	*η-Einstein space*
generalized semi-pseudo symmetric space	*η-Einstein space*
almost pseudo symmetric space	*η-Einstein space*
almost generalized pseudo symmetric space	*η-Einstein space*
weakly symmetric space	*η-Einstein space*

Note that if a manifold is locally recurrent, then it is Ricci recurrent, i.e. $\nabla_k R_{jl} = \beta_k R_{jl}$, for a non-null one form β_k which leaves after transvection $\nabla_k R = \beta_k R$. Consequently, the manifold is Ricci flat as it is known that the scalar curvature of an Einstein manifold is constant. Thus we can state the following corollary.

Corollary 3.3. *Every locally recurrent $(LCS)_n$ manifold is Ricci flat.*

4. Generalized weakly Ricci symmetric $(LCS)_n$-manifold

A non-flat n-dimensional $(LCS)_n$-manifold $(M^n;g)$ $(n>2)$, is said to be a generalized weakly Ricci symmetric manifold, if its Ricci tensor S of type $(0,2)$ is not identically zero and admits the identity

$$(\nabla_X S)(Y,Z) = A_1^*(X)S(Y,Z)+B_1^*(Y)S(X,Z)+D_1^*(Z)S(Y,X) + A_2^*(X)g(Y,Z)+B_2^*(Y)g(X,Z)+D_2^*(Z)g(Y,X) \tag{4.1}$$

where A_i^*, B_i^* & D_i^* are non-zero 1-forms which are defined as $A_i^*(X)=g(X,\theta_i)$, $B_i^*(X)=g(X,\phi_i)$, $D_i^*(X)=g(X,\pi_i)$ for $i=1,2$. Setting, $Y=\xi$ in (4.1) and then making use of (2.12), we have

$$(n-1)[\alpha(\alpha^2-\rho)g(X,Z)+(2\alpha\rho-\beta)\eta(Z)\eta(X)]-\alpha S(X,Z) = (\alpha^2-\rho)(n-1)[A_1^*(X)\eta(Z)+D_1^*(Z)\eta(X)]+B_1^*(\xi)S(X,Z) + A_2^*(X)\eta(Z)+B_2^*(\xi)g(X,Z)+D_2^*(Z)\,\eta(X) \tag{4.2}$$

which yields

$$(\alpha^2-\rho)(n-1)[A_1^*(\xi)+B_1^*(\xi)+D_1^*(\xi)]+[A_2^*(\xi)+B_2^*(\xi)+D_2^*(\xi)\ ] \\ =-(n-1)(2\alpha\rho-\beta), \tag{4.3}$$

for $X=Z=\xi$.

Setting $Z=\xi$ in (4.2) we obtain

$$(n-1)(\alpha^2-\rho)[A_1^*(X)+A_1^*(\xi)]=-[A_2^*(X)+A_2^*(\xi)\eta(X)]. \tag{4.4}$$

Proceeding in a similar manner we can find

$$(\alpha^2-\rho)(n-1)[B_1^*(X)+B_1^*(\xi)]=-[B_2^*(X)+B_2^*(\xi)\eta(X)], \tag{4.5}$$

$$(\alpha^2-\rho)(n-1)[D_1^*(X)+D_1^*(\xi)]=-[D_2^*(\xi)+D_2^*(X)\eta(X)]. \tag{4.6}$$

Theorem 4.1. *In a generalized weakly Ricci symmetric $(LCS)_n$-manifold $M^n(\phi,\xi,\eta,g)(n>2)$ the 1-forms are related by*

$$(\alpha^2-\rho)(n-1)[A_1^*(X)+B_1^*(X)+D_1^*(X)]+[A_2^*(X)+B_2^*(X)+D_2^*(X)] \\ =(n-1)(2\alpha\rho-\beta)\eta(X). \tag{4.7}$$

Proof. Adding (4.4), (4.5) & (4.6) and then making use of (4.3) in the resultant, one can easily obtain (4.7). □

Now, making use of (4.3)-(4.7)in (4.2), we find that

$$S(X,Z) = \left[\frac{(n-1)\alpha(\alpha^2-\rho)-B_2^*(\xi)}{\alpha+B_1^*(\xi)}\right]g(X,Z) - \left[\frac{(\alpha^2-\rho)(n-1)B_1^*(\xi)+B_2^*(\xi)}{\alpha+B_1^*(\xi)}\right]\eta(X)\eta(Z) \tag{4.8}$$

This leads to the followings

Theorem 4.2. *A generalized weakly Ricci symmetric $(LCS)_n$-manifold $M^n(\phi,\xi,\eta,g)$ is an η-Einstein provided that $B_1^*(\xi)\neq-\alpha$.*

Theorem 4.3. *In an $(LCS)_n$-manifold the following table holds good*

Type of curvature restriction	*Nature of the space corresponding to curvature restriction*
Ricci symmetric space	*Einstein space*
Ricci recurrent space	*Einstein space*
generalized Ricci-recurrent space	*Einstein space*
pseudo Ricci-symmetric space	*η-Einstein space*
generalized pseudo Ricci-symmetric space	*η-Einstein space*
semi-pseudo Ricci-symmetric space	*η-Einstein space*
generalized semi-pseudo Ricci-symmetric space	*η-Einstein space*
almost pseudo Ricci-symmetric space	*η-Einstein space*
almost generalized pseudo Ricci-symmetric space	*η-Einstein space*
weakly Ricci-symmetric space	*η-Einstein space*

Note that if a manifold is Ricci recurrent, i.e. $\nabla_k R_{jl} = \beta_k R_{jl}$, for a non-null one form β_k which leaves after transvection $\nabla_k R = \beta_k R$. Consequently, the manifold is Ricci flat as it is known that the scalar curvature of an Einstein manifold is constant. Thus we can state the following corollary.

Corollary 4.4. *Every locally Ricci recurrent $(LCS)_n$ manifold is Ricci flat.*

5. Existence of generalized weakly symmetric $(LCS)_3$-manifold

Example 5.1. Let $M^3(\phi, \xi, \eta, g)$ be an $(LCS)_n$-manifold (M^3, g) with a ϕ-basis

$$e_1 = e^z \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right), e_2 = \phi e_1 = e^z \frac{\partial}{\partial y}, e_3 = \xi = e^{2z} \frac{\partial}{\partial z}.$$

Then from Koszul's formula for Lorentzian metric g, we can obtain the Levi-Civita connection as follows

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e^{2z} e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e^{2z} e_3, \\ \nabla_{e_2} e_3 = -e^{2z} e_2, & \nabla_{e_2} e_2 = -e^{2z} e_3 - e^z e_1, & \nabla_{e_2} e_1 = -e^{2z} e_2, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)^3$ structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is an $(LCS)^3$-manifold with $\alpha = -e^{2z} \neq= 0$ and $\rho = 2e^{4z}$. Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\begin{aligned} \bar{R}(e_1, e_2, e_1, e_2) &= (1 - e^{2z}) e^{2z} \\ \bar{R}(e_1, e_3, e_1, e_3) &= -e^{4z} = \bar{R}(e_2, e_3, e_2, e_3). \end{aligned}$$

Since $\{e_1,\ e_2, e_3\}$ forms a basis, any vector field $X,\ Y,\ U,\ V \in \chi(M)$ can be written as

$$X = \sum_1^3 a_i e_i,\ Y = \sum_1^3 b_i e_i,\ U = \sum_1^3 c_i e_i,\ V = \sum_1^3 d_i e_i,$$

Then

$$\begin{aligned} \bar{R}(X, Y, U, V) &= [(a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)](1 - e^{2z}) e^{2z} \\ &\quad -[(a_1 b_3 - a_3 b_1)(c_1 d_3 - c_3 d_1)] e^{4z} \\ &\quad -[(a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2)] e^{4z} \\ &= T_1 \text{ (say)}, \\ \bar{R}(e_1, Y, U, V) &= -b_3(c_1 d_3 - c_3 d_1) e^{4z} + b_2(c_1 d_2 - c_2 d_1(1 - e^{2z}) e^{2z} \\ &= \lambda_1 \text{ (say)}, \\ \bar{R}(e_2, Y, U, V) &= -b_3(c_2 d_3 - c_3 d_2) e^{4z} - b_1(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\ &= \lambda_2 \text{ (say)}, \\ \bar{R}(e_3, Y, U, V) &= b_1(c_1 d_3 - c_3 d_1) e^{4z} + b_2(c_2 d_3 - c_3 d_2) e^{4z} = \lambda_3 \text{ (say)}, \\ \bar{R}(X, e_1, U, V) &= a_3(c_1 d_3 - c_3 d_1) e^{4z} - a_2(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\ &= \lambda_4 \text{ (say)}, \\ \bar{R}(X, e_2, U, V) &= a_3(c_2 d_3 - c_3 d_2) e^{4z} + a_1(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\ &= \lambda_5 \text{ (say)}, \\ \bar{R}(X, e_3, U, V) &= -a_1(c_1 d_3 - c_3 d_1) e^{4z} - a_2(c_2 d_3 - c_3 d_2) e^{4z} = \lambda_6 \text{ (say)}, \\ \bar{R}(X, Y, e_1, V) &= -d_3(a_1 b_3 - a_3 b_1) e^{4z} + d_2(a_1 b_2 - a_2 b_1)(1 - e^{2z}) e^{2z} \\ &= \lambda_7 \text{ (say)}, \end{aligned}$$

$$
\begin{aligned}
\bar{R}(X,Y,e_2,V) &= -d_3(a_2b_3-a_3b_2)e^{4z}-d_1(a_1b_2-a_2b_1)(1-e^{2z})e^{2z}\\
&= \lambda_8 \text{ (say)},\\
\bar{R}(X,Y,e_3,V) &= d_1(a_1b_3-a_3b_1)e^{4z}+d_2(a_2b_3-a_3b_2)=\lambda_9 \text{ (say)},\\
\bar{R}(X,Y,U,e_1) &= c_3(a_1b_3-a_3b_1)e^{4z}-c_2(a_1b_2-a_2b_1)(1-e^{2z})e^{2z}\\
&= \lambda_{10} \text{ (say)},\\
\bar{R}(X,Y,U,e_2) &= c_3(a_2b_3-a_3b_2)e^{4z}+c_1(a_1b_2-a_2b_1)(1-e^{2z})e^{2z}\\
&= \lambda_{11} \text{ (say)},\\
\bar{R}(X,Y,U,e_3) &= -c_1(a_1b_3-a_3b_1)e^{4z}-c_2(a_2b_3-a_3b_2)e^{4z}=\lambda_{12} \text{ (say)},\\
\bar{G}(X,Y,U,V) &= (b_1c_1+b_2c_2-b_3c_3)(a_1d_1+a_2d_2-a_3d_3)\\
&\quad -(a_1c_1+a_2c_2-a_3c_3)(b_1d_1+b_2d_2-b_3d_3)=T_2 \text{ (say)},\\
\bar{G}(e_1,Y,U,V) &= (b_2c_2-b_3c_3)d_1-(b_2d_2-b_3d_3)c_1=\omega_1 \text{ (say)},\\
\bar{G}(e_2,Y,U,V) &= (b_1c_1-b_3c_3)d_2-(b_1d_1-b_3d_3)c_2=\omega_2 \text{ (say)},\\
\bar{G}(e_3,Y,U,V) &= -(b_1c_1+b_2c_2)d_3+(b_1d_1+b_2d_2)c_3=\omega_3 \text{ (say)},\\
\bar{G}(X,e_1,U,V) &= (a_2d_2-a_3d_3)c_1-(a_2c_2-a_3c_3)d_1=\omega_4 \text{(say)},\\
\bar{G}(X,e_2,U,V) &= (a_1d_1-a_3d_3)c_2-(a_1c_1-a_3c_3)d_2=\omega_5 \text{(say)},\\
\bar{G}(X,e_3,U,V) &= -(a_1d_1+a_2d_2)c_3+(a_1c_1+a_2c_2)d_3=\omega_6 \text{ (say)},\\
\bar{G}(X,Y,e_1,V) &= (a_2d_2-a_3d_3)b_1-(b_2d_2-b_3d_3)a_1=\omega_7 \text{ (say)},\\
\bar{G}(X,Y,e_2,V) &= (a_1d_1-a_3d_3)b_2-(b_1d_1-b_3d_3)a_2=\omega_8 \text{ (say)},\\
\bar{G}(X,Y,e_3,V) &= -(a_1d_1+a_2d_2)b_3+(b_1d_1+b_2d_2)a_3=\omega_9 \text{ (say)},\\
\bar{G}(X,Y,U,e_1) &= (b_2c_2-b_3c_3)a_1-(a_2c_2-a_3c_3)b_1=\omega_{10} \text{ (say)},\\
\bar{G}(X,Y,U,e_2) &= (b_1c_1-b_3c_3)a_2-(a_1c_1-a_3c_3)b_2=\omega_{11} \text{ (say)},\\
\bar{G}(X,Y,U,e_3) &= -(b_1c_1+b_2c_2)a_3+(a_1c_1+a_2c_2)b_3=\omega_{12} \text{ (say)},
\end{aligned}
$$

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

$$
\begin{aligned}
(\nabla_{e_1}\bar{R})(X,Y,U,V) &= e^{2z}[a_1\lambda_3+a_3\lambda_1+b_1\lambda_6+b_3\lambda_4\\
&\quad +c_1\lambda_9+c_3\lambda_7+d_1\lambda_{12}+b_3\lambda_{10}],
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{e_2}\bar{R})(X,Y,U,V) &= e^{2z}[(a_1+a_3)\lambda_2+a_2\lambda_3+(b_1+b_3)\lambda_5+b_2\lambda_6\\
&\quad +(c_1+c_3)\lambda_8+c_2\lambda_9+(d_1+d_3)\lambda_{11}+d_2\lambda_{12}]\\
&\quad +e^{z}[a_2\lambda_1+b_2\lambda_4+c_2\lambda_7+d_2\lambda_{10},
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{e_3}\bar{R})(X,Y,U,V) &= 2[(a_1b_2-a_2b_1)(c_1d_2-c_2d_1)](1-2e^{2z})e^{4z}\\
&\quad -4[(a_1b_3-a_3b_1)(c_1d_3-c_3d_1)]e^{6z}\\
&\quad -4[(a_2b_3-a_3b_2)(c_2d_3-c_3d_2)]e^{6z}.
\end{aligned}
$$

For the following choice of the the 1-forms

$$
\begin{aligned}
A_1^*(e_1) &= \frac{e^{2z}[a_1\lambda_3+a_3\lambda_1+b_1\lambda_6+b_3\lambda_4}{T_1},\\
A_2^*(e_1) &= \frac{c_1\lambda_9+c_3\lambda_7+d_1\lambda_{12}+b_3\lambda_{10}}{T_2},\\
A_1(e_2) &= -\frac{e^{2z}\{(a_1+a_3)\lambda_2+a_2\lambda_3+(b_1+b_3)\lambda_5+b_2\lambda_6(c_1+c_3)\lambda_8+c_2\lambda_9+d_1\}}{T_1},
\end{aligned}
$$

$$
\begin{aligned}
A_2^*(e_2) &= -\frac{e^{2z}\{d_3)\lambda_{11}+d_2\lambda_{12}\}+e^z\{a_2\lambda_1+b_2\lambda_4+c_2\lambda_7+d_2\lambda_{10}\}}{T_2},\\
A_1^*(e_3) &= -4,\\
B_1^*(e_3) &= \frac{1}{a_3\lambda_3+b_3\lambda_6},\\
B_2^*(e_3) &= \frac{1}{a_3\omega_3+b_3\omega_6},\\
D_1^*(e_3) &= -\frac{1}{c_3\lambda_9+d_3\lambda_{12}},\\
D_2^*(e_3) &= -\frac{1}{c_3\omega_9+d_3\omega_{12}},\\
A_2^*(e_3) &= -2\frac{(a_1b_2-a_2b_1)(c_1d_2-c_2d_1)e^{2z}}{T_2},
\end{aligned}
$$

one can easily verify the relations

$$
\begin{aligned}
(\nabla_{e_i}\bar{R})(X,Y,U,V) &= A_1^*(e_i)\bar{R}(X,Y,U,V)\\
&+B_1^*(X)\bar{R}(e_i,Y,U,V)+B_1^*(Y)\bar{R}(X,e_i,U,V)\\
&+D_1^*(U)\bar{R}(X,Y,e_i,V)+D_1^*(V)\bar{R}(X,Y,U,e_i)\\
&+A_2^*(e_i)\bar{G}(X,Y,U,V)\\
&+B_2^*(X)\bar{G}(e_i,Y,U,V)+B_2^*(Y)\bar{G}(X,e_i,U,V)\\
&+D_2^*(U)\bar{G}(X,Y,e_i,V)+D_2^*(V)\bar{G}(X,Y,U,e_i)
\end{aligned}
$$

for $i = 1,2,3$.

From the above, we can state the following theorem.

Theorem 5.2. *There exists an $(LCS)_3$-manifold (M^3, g) which is a generalized weakly symmetric.*

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References

[1] H. Bağdatlı Yılmaz, *On decomposable almost pseudo conharmonically symmetric manifolds*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **51** (1), 111-124, 2012.

[2] H. Bağdatlı Yılmaz, *On Almost Pseudo Quasi-Conformally Symmetric Manifolds*, pre-print.

[3] K.K. Baishya, *On generalized weakly symmetric manifolds*, Bull. Transilv. Univ. Braȳsov Ser. III **10** (59), 31-38, 2017.

[4] K.K. Baishya, *On generalized semi-pseudo symmetric manifold*, submitted.

[5] K.K. Baishya and P.R. Chowdhury, *η-Ricci solitons in $(LCS)_n$-manifolds*, Bull. Transilv. Univ. Braȳsov Ser. III **9** (58), 1-12, 2016.

[6] K.K. Baishya, P.R. Chowdhury, M. Josef and P. Peska, *On almost generalized weakly symmetric Kenmotsu manifolds*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **55** (2), 5-15, 2016.

[7] E. Cartan, *Sur une classes remarquable d'espaces de Riemannian*, Bull. Soc. Math. France **54**, 214-264, 1926.

[8] M.C. Chaki, *On pseudo symmetric manifolds*, Analele Ştiinţifice Ale Univerşiţatii "AL I.Cuza'" Din Iaşi **33**, 53-58, 1987.

[9] M.C. Chaki and T. Kawaguchi, *On almost pseudo Ricci symmetric manifolds*, Tensor **68** (1), 10-14, 2007.

[10] U.C. De and S. Bandyopadhyay, *On weakly symmetric spaces*, Acta Math. Hung. **83**, 205-212, 2000.

[11] R.S.D. Dubey, *Generalized recurrent spaces*, Indian J. Pure Appl. Math. **10** (12), 1508-1513, 1979.

[12] S.K. Hui and M. Atceken, *Contact warped productsemi-slant submanifolds of $(LCS)_n$-manifolds*, Acta Univ. Sapientiae Mathematica **3** (2), 212-224, 2011.

[13] J.P. Jaiswal and R.H. Ojha, *On weakly pseudo-projectively symmetric manifolds*, Differential Geometry - Dynamical Systems **12**, 83-94, 2010.

[14] F. Malek and M. Samawaki, *On weakly symmetric Riemannian manifolds*, Differential Geometry - Dynamical Systems, **10**, 215-220, 2008.

[15] C.A. Mantica and L.G. Molinari, *Weakly Z-symmetric manifolds*, Acta Math. Hungar. **135**, 80-96, 2012.

[16] C.A. Mantica and L.G. Molinari, *Twisted Lorentzian manifolds: a characterization with torse-forming time-like unit vectors*, Gen. Relativ. Gravit. **49**:51, 2017.

[17] C.A. Mantica and L.G. Molinari, *Generalized Robertson-Walker space-times*, a survey, Int. J. Geom. Meth. Mod.Phys. **14** (3), 1730001, 2017.

[18] C.A. Mantica and Y.J. Suh, *Pseudo Z-symmetric Riemannian manifolds with harmonic curvature tensors*, Int. J. Geom. Meth. Mod. Phys. **9**, 1250004, 2012.

[19] D. Narain and S. Yadav, *On weak concircular symmetries of $(LCS)_{2n+1}$-manifolds*, Global Journal of Science Frontier Research **12**, 85-94, 2012.

[20] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, Inc, New York, 1983.

[21] F. Özen and S. Altay, *On weakly and pseudo symmetric Riemannian spaces*, Indian J. Pure Appl. Math. **33** (10), 1477-1488, 2001.

[22] F. Özen and S. Altay, *On weakly and pseudo concircular symmetric structures on a Riemannian manifold*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **47**, 129-138, 2008.

[23] M. Prvanovic, *On weakly symmetric riemannian manifolds*, Pub. Math. Debrecen **46**, 19-25, 1995.

[24] M. Prvanovic, *On totally umbilical submanifolds immersed in a weakly symmetric riemannian manifolds*, Pub. Math. Debrecen **6**, 54-64, 1998.

[25] A.A. Shaikh, *On Lorentzian almost para contact manifolds with a structure of the concircular type*, Kyungpook Math. J. **43**, 305-314 ,2003.

[26] A.A. Shaikh and K.K. Baishya, *On weakly quasi-conformally symmetric manifolds*, Soochow J. Math. **31** (4), 581-595, 2005.

[27] A.A. Shaikh and K.K. Baishya, *On concircular structure spacetimes*, J. Math. Stat. **1**, 129-132, 2005.

[28] A.A. Shaikh and K.K. Baishya, *On concircular structurespacetimes II*, American J. Appl. Sci. **3**, 1790-1794, 2006.

[29] A.A. Shaikh and T.Q. Binh, *On weakly symmetric $(LCS)_n$-manifolds*, J. Adv. Math. Studies **2**, 75-90, 2009.

[30] A.A. Shaikh and S.K. Hui, *On generalized ϕ-recurrent $(LCS)_n$-manifolds*, AIP Conference Proceedings **1309**, 419-429, 2010.

[31] A.A. Shaikh and S.K. Jana, *On weakly symmetric manifolds*, Publ. Math. Debrecen **71** (1-2), 2007.

[32] A.A. Shaikh, I. Roy and S.K. Hui, *On totally umbilical hypersurfaces of weakly conharmonically symmetric spaces*, Global Journal of Science Frontier Research **10** (4), 28-31, 2010.

[33] G.T. Sreenivasa, Venkatesha and C.S. Bagewadi, *Some results on $(LCS)_{2n+1}$-manifolds*, Bull. Math. Anal. Appl. **3** (1), 64-70, 2009.

[34] L. Tamássy and T.Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Colloq. Math. Soc. János Bolyai **56**, 663-670, 1989.

[35] M. Tarafdar and M.A.A. Jawarneh, *Semi-Pseudo Ricci Symmetric manifold*, J. Indian. Inst. Sci. **73**, 591-596, 1993.

[36] A.G. Walker, *On Ruse's space of recurrent curvature*, Proc. London Math. Soc. **52**, 36-54, 1950.

[37] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics **3**, World Scientific Publishing, 1985.