Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (2) (2019), 451 – 459
DOI : 10.15672/HJMS.2017.527

Research Article

Solutions of Some Diophantine Equations in terms of Generalized Fibonacci and Lucas Numbers

Bahar Demirtürk Bitim*, Refik Keskin

Department of Mathematics, Sakarya University, Sakarya, 54187, Turkey

Abstract

In this study, we present some identities involving generalized Fibonacci sequence (U_n) and generalized Lucas sequence (V_n). Then we give all solutions of the Diophantine equations $x^2 - V_nxy + (-1)^ny^2 = \pm(p^2+4)U_n^2$, $x^2 - V_nxy + (-1)^ny^2 = \pm U_n^2$, $x^2 - (p^2+4)U_nxy - (p^2+4)(-1)^ny^2 = \pm V_n^2$, $x^2 - V_nxy \pm y^2 = \pm 1$, $x^2 - (p^2+4)U_nxy - (p^2+4)(-1)^ny^2 = 1$, $x^2 - V_nxy + (-1)^ny^2 = \pm(p^2+4)$, $x^2 - V_{2n}xy + y^2 = \pm(p^2+4)V_n^2$, $x^2 - V_{2n}xy + y^2 = (p^2+4)U_n^2$ and $x^2 - V_{2n}xy + y^2 = \pm V_n^2$ in terms of the sequences (U_n) and (V_n) with $p \geq 1$ and p^2+4 squarefree.

Mathematics Subject Classification (2010). 11B37, 11B39, 11C20, 11D09

Keywords. generalized Fibonacci and Lucas sequences, Diophantine equations

1. Introduction

Let $p \geq 1$ be an integer. The generalized Fibonacci sequence $(U_n) = (U_n(p,1))$ and the generalized Lucas sequence $(V_n) = (V_n(p,1))$ are defined by

$$U_n = pU_{n-1} + U_{n-2},\ U_0 = 0, U_1 = 1$$

and

$$V_n = pV_{n-1} + V_{n-2},\ V_0 = 2, V_1 = p$$

for $n \geq 2$. The terms U_n and V_n are called the nth generalized Fibonacci and Lucas numbers, respectively. In general $U_{-n} = (-1)^{n+1}U_n$, $V_{-n} = (-1)^nV_n$ and $V_n = U_{n+1} + U_{n-1}$, for all $n \in \mathbb{N}$. Properties of these sequences are determined in [7,8,11,12] and [18].

In 1979, Kiss considered the sequence (R_n) with linear recurrence relation $R_n = AR_{n-1} - BR_{n-2}$, $R_0 = 0, R_1 = 1$ for some $n > 1$, where A, B are integers such that $A > 0$ and $B = -1$ or $A > 3$ and $B = 1$. Then he proved that for non-negative integers x, y, the equation $|x^2 - Axy + By^2| = 1$ holds if and only if x and y are consecutive terms of sequence (R_n), in [9].

In 1993, Matiyasevich mentioned that the conic $x^2 - kxy + y^2 = 1$ with $k \geq 2$ has (x, y) integer points if and only if $(x, y) = (u_n, u_{n+1})$ for some n, where $u_{n+1} = ku_n - u_{n-1}$, starting with $u_0 = 0$ and $u_1 = 1$, in [10].

*Corresponding Author.
Email addresses: demirturk@sakarya.edu.tr (B. Demirtürk Bitim), rkeskin@sakarya.edu.tr (R. Keskin)
Received: 07.04.2017; Accepted: 02.11.2017

In [12], Melham showed that the solutions of the equations $x^2 - V_m xy \pm y^2 = \pm U_m^2$ are given by $(x,y) = \pm(U_{n+m}, U_n)$ for m, $n \in \mathbb{Z}$. Moreover he showed that if m is an even integer and p^2+4 is a squarefree integer, then all solutions of the equation $y^2 - V_m xy + x^2 = \pm\left(p^2+4\right)U_m^2$ are given by $(x,y) = \mp\left(V_n, V_{n+m}\right)$ with $n \in \mathbb{Z}$. These theorems of Melham are generalized forms of the theorems given in [11], by McDaniel. In [8], Kılıç and Ömür examined more general situations of the conics that McDaniel and Melham dealt in [11] and [12], respectively.

In [1], Demirtürk and Keskin determined all solutions of the known Diophantine equations $x^2 - L_n xy - y^2 = \mp 1$, $x^2 - L_n xy + (-1)^n y^2 = \mp 5$ and new Diophantine equations; $x^2 - 5F_n xy - 5(-1)^n y^2 = \mp 1$, $x^2 - L_{2n} xy + y^2 = \mp 5F_n^2$, $x^2 - L_{2n} xy + y^2 = \mp F_n^2$, $x^2 - L_{2n} xy + y^2 = \mp L_n^2$ and $x^2 - L_{2n} xy + y^2 = \mp 5L_n^2$. Moreover in [2], the authors give solutions of generalizations of these equations.

In this paper, our main purpose is to determine all (x,y) solutions of some Diophantine equations, mentioned in the abstract.

2. Some identities concerning the sequences (U_n) and (V_n)

In this section, we obtain some identities by using special matrices including generalized Fibonacci and Lucas numbers. From [6, 13–15], the following identities are given for all $m, n \in \mathbb{Z}$ by

$$V_n^2 - pV_nV_{n-1} - V_{n-1}^2 = (-1)^n\left(p^2+4\right), \tag{2.1}$$

$$V_mU_n - U_mV_n = 2(-1)^m U_{n-m}, \tag{2.2}$$

$$V_mV_n - \left(p^2+4\right)U_mU_n = 2(-1)^n V_{m-n}, \tag{2.3}$$

$$V_mV_n + \left(p^2+4\right)U_mU_n = 2V_{n+m}, \tag{2.4}$$

$$V_mU_n + U_mV_n = 2U_{n+m}, \tag{2.5}$$

$$U_{n+1} + U_{n-1} = V_n, \tag{2.6}$$

$$V_{n+1} + V_{n-1} = \left(p^2+4\right)U_n, \tag{2.7}$$

$$V_n^2 - \left(p^2+4\right)U_n^2 = 4(-1)^n, \tag{2.8}$$

$$V_{m+1}U_n + V_mU_{n-1} = V_{n+m}. \tag{2.9}$$

Theorem 2.1.

$$V_{n+m}^2 - (p^2+4)(-1)^{n+t}U_{t-n}V_{n+m}U_{m+t} - (p^2+4)(-1)^{n+t}U_{m+t}^2 = (-1)^{m+t}V_{t-n}^2,$$

for all $m, n, t \in \mathbb{Z}$.

Proof. Assume that $A = \begin{bmatrix} V_n/2 & (p^2+4)U_n/2 \\ U_t/2 & V_t/2 \end{bmatrix}$. If we consider (2.4) and (2.5), then we have $A\begin{bmatrix} V_m \\ U_m \end{bmatrix} = \begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix}$. By using (2.3), we get

$$\begin{bmatrix} V_m \\ U_m \end{bmatrix} = A^{-1}\begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix} = \frac{2}{(-1)^n V_{t-n}}\begin{bmatrix} V_t/2 & -(p^2+4)U_n/2 \\ -U_t/2 & V_n/2 \end{bmatrix}\begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix}$$

since $\det A = \frac{V_nV_t - (p^2+4)U_nU_t}{4} = \frac{(-1)^n V_{t-n}}{2} \neq 0$. Then it follows that

$$V_m = \frac{(-1)^n\left(V_tV_{n+m} - (p^2+4)U_nU_{m+t}\right)}{V_{t-n}} \text{ and } U_m = \frac{(-1)^n\left(V_nU_{m+t} - U_tV_{n+m}\right)}{V_{t-n}}.$$

By using (2.8), we see that

$$\left(V_tV_{n+m} - (p^2+4)U_nU_{m+t}\right)^2 - (p^2+4)\left(V_nU_{m+t} - U_tV_{n+m}\right)^2 = 4(-1)^m V_{t-n}^2.$$

Hence, we obtain $\left(V_t^2-(p^2+4)U_t^2\right)V_{n+m}^2-2(p^2+4)\left(V_tU_n-V_nU_t\right)V_{n+m}U_{m+t}-(p^2+4)\left(V_n^2-(p^2+4)U_n^2\right)U_{m+t}^2=4(-1)^mV_{t-n}^2$. Thus, it is seen that

$$4(-1)^tV_{n+m}^2-4(-1)^n(p^2+4)U_{t-n}V_{n+m}U_{m+t}-4(-1)^n(p^2+4)U_{m+t}^2=4(-1)^mV_{t-n}^2$$

by (2.2) and (2.8). Then it follows that

$$V_{n+m}^2-(p^2+4)(-1)^{n+t}U_{t-n}V_{n+m}U_{m+t}-(p^2+4)(-1)^{n+t}U_{m+t}^2=(-1)^{m+t}V_{t-n}^2, \quad (2.10)$$

which concludes the proof. □

Theorem 2.2. *Let $m,n,t\in\mathbb{Z}$ and $t\neq n$. Then*

$$V_{n+m}^2-(-1)^{n+t}V_{t-n}V_{n+m}V_{m+t}+(-1)^{n+t}V_{m+t}^2=(-1)^{m+t+1}(p^2+4)U_{t-n}^2.$$

Proof. Assume that $B=\begin{bmatrix} V_n/2 & (p^2+4)U_n/2\\ V_t/2 & (p^2+4)U_t/2\end{bmatrix}$. By using (2.4), we can write the matrix multiplication $B\begin{bmatrix} V_m\\ U_m\end{bmatrix}=\begin{bmatrix} V_{n+m}\\ V_{m+t}\end{bmatrix}$. Since $t\neq n$, we get $\det B=\frac{(p^2+4)(-1)^nU_{t-n}}{2}\neq 0$ by (2.2). Hence it is seen that

$$\begin{bmatrix} V_m\\ U_m\end{bmatrix}=B^{-1}\begin{bmatrix} V_{n+m}\\ V_{m+t}\end{bmatrix}=\frac{2(-1)^n}{(p^2+4)\,U_{t-n}}\begin{bmatrix} (p^2+4)U_t/2 & -(p^2+4)U_n/2\\ -V_t/2 & V_n/2\end{bmatrix}\begin{bmatrix} V_{n+m}\\ V_{m+t}\end{bmatrix}.$$

Thus, it follows that

$$V_m=\frac{(-1)^n(U_tV_{n+m}-U_nV_{m+t})}{U_{t-n}} \text{ and } U_m=\frac{(-1)^n(V_nV_{m+t}-V_tV_{n+m})}{(p^2+4)U_{t-n}}.$$

Since $V_m^2-(p^2+4)U_m^2=4\,(-1)^m$ by (2.8), we get

$$(p^2+4)\left(U_tV_{n+m}-U_nV_{m+t}\right)^2-\left(V_nV_{m+t}-V_tV_{n+m}\right)^2=4\,(-1)^m\,(p^2+4)U_{t-n}^2.$$

Hence, it is seen that

$$V_{n+m}^2-(-1)^{n+t}V_{t-n}V_{n+m}V_{m+t}+(-1)^{n+t}V_{m+t}^2=(-1)^{m+t+1}(p^2+4)U_{t-n}^2 \quad (2.11)$$

by (2.3) and (2.8). □

Using (2.5) and the matrix multiplication

$$\begin{bmatrix} U_n/2 & V_n/2\\ U_t/2 & V_t/2\end{bmatrix}\begin{bmatrix} V_m\\ U_m\end{bmatrix}=\begin{bmatrix} U_{n+m}\\ U_{m+t}\end{bmatrix},$$

we can give the following theorem.

Theorem 2.3. *Let $m,\ n,\ t\in\mathbb{Z}$ and $t\neq n$. Then*

$$U_{n+m}^2-V_{t-n}U_{n+m}U_{m+t}+(-1)^{n+t}U_{m+t}^2=(-1)^{m+t}U_{t-n}^2. \quad (2.12)$$

In this section, we also recall divisibility rules of the sequences (U_n) and (V_n). We omit their proofs, since they are proved in [3–5, 16, 17].

Theorem 2.4. *Let $m,\ n\in\mathbb{N}$. $V_n|U_m$ iff $m=2kn$ for some $k\in\mathbb{N}$.*

Theorem 2.5. *Let $m,\ n\in\mathbb{N}$ and $U_n\neq 1$. $U_n|U_m$ iff $m=kn$ for some $k\in\mathbb{N}$.*

Theorem 2.6. *Let $m,\ n\in\mathbb{N}$ and $V_n\neq 1$. $V_n|V_m$ iff $m=(2k+1)n$ for some $k\in\mathbb{N}$.*

Theorem 2.7. *Let $m,\ n\in\mathbb{N}$ and $n>1$. $U_n|V_m$ iff $n=2$ and m is an odd integer, where $p\geq 3$.*

3. Solutions of some Diophantine equations

In this section, firstly we remind some Diophantine equations with their solutions. These equations are studied in [7,11,18]. We use these equations for determining all solutions of more general Diophantine equations. Throughout this paper, unless otherwise stated, we will take $p \geq 1$ and p^2+4 will be a squarefree integer.

Theorem 3.1. *All solutions of the equation* $x^2 - pxy - y^2 = \pm 1$ *are given by* $(x, y) = \mp(U_{m+1}, U_m)$ *with* $m \in \mathbb{Z}$.

Corollary 3.2. *All solutions of the equations* $x^2 - pxy - y^2 = -1$ *and* $x^2 - pxy - y^2 = 1$ *are given by* $(x, y) = \mp(U_{2m}, U_{2m-1})$ *and* $(x, y) = \mp(U_{2m+1}, U_{2m})$ *with* $m \in \mathbb{Z}$*, respectively.*

Theorem 3.3. *All solutions of the equation* $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = -V_n^2$ *and* $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = V_n^2$ *are given by* $(x, y) = \mp(V_{n+m}, U_m)$ *with* m *odd and* m *even, respectively.*

Proof. Suppose that $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = -V_n^2$ for some integers x and y. By using (2.8) in this equation, we get $\left(2x - (p^2+4)U_n y\right)^2 - (p^2+4)V_n^2 y^2 = -4V_n^2$. Hence it is seen that $V_n | 2x - (p^2+4)U_n y$. Then taking

$$u = \frac{\left(\dfrac{(2x - (p^2+4)U_n y)}{V_n} + py\right)}{2} \quad \text{and} \quad v = y,$$

we obtain $u = (x - V_{n-1}y)/V_n$ by (2.7). Thus it follows that

$$u^2 - puv - v^2 = \left(x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2\right)/V_n^2 = -V_n^2/V_n^2 = -1,$$

by (2.1) and (2.7). From Corollary 3.2, it is seen that $(u, v) = \mp(U_{m+1}, U_m)$ for some odd m. Hence

$$(x - V_{n-1}y)/V_n = \mp U_{m+1} \text{ and } y = \mp U_m.$$

Now using (2.9), we obtain

$$(x, y) = \mp(V_{n+m}, U_m)$$

for some odd m. Conversely, if $(x, y) = \mp(V_{n+m}, U_m)$ for some odd m, then it can be seen that $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = -V_n^2$ by (2.10).

Now assume that $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = V_n^2$ for some integers x and y. Then taking $u = (x - V_{n-1}y)/V_n$ and $v = y$, we obtain

$$u^2 - puv - v^2 = 1$$

by (2.1) and (2.7). From Corollary 3.2, we get $(u, v) = \mp(U_{m+1}, U_m)$ for some even m. Thus, it follows that $(x, y) = \mp(V_{n+m}, U_m)$ by (2.9), where m is even. Conversely, if $(x, y) = \mp(V_{n+m}, U_m)$ for some even m, then it can be seen that $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = V_n^2$ by (2.10). □

Theorem 3.4. *All solutions of the equation* $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = 1$ *are given by* $(x, y) = \mp\left(V_{(2t+1)n}/V_n, U_{2tn}/V_n\right)$ *with* $t \in \mathbb{Z}$.

Proof. Assume that $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = 1$ for some integers x and y. Multiplying both sides of this equation by V_n^2, we get

$$(V_n x)^2 - (p^2+4)U_n (V_n x)(V_n y) - (p^2+4)(-1)^n (V_n y)^2 = V_n^2.$$

Thus, it follows that $V_n x = \mp V_{n+m}$ and $V_n y = \mp U_m$ for some integer m by Theorem 3.3. Hence, we get $(x, y) = \mp(V_{n+m}/V_n, U_m/V_n)$. From Theorem 2.4 and Theorem 2.6, it can be seen that $m = 2tn$ for some $t \in \mathbb{Z}$. Therefore, we obtain $(x, y) = \mp\left(V_{(2t+1)n}/V_n, U_{2tn}/V_n\right)$.

Conversely, if $(x,y) = \mp\left(V_{(2t+1)\,n}/V_n, U_{2tn}/V_n\right)$ for some $t \in \mathbb{Z}$, then it is easy to verify that $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = 1$ by (2.10). □

The following corollary can be given from Theorems 3.3, 2.4 and 2.6.

Corollary 3.5. *The equation $x^2 - (p^2+4)U_n xy - (p^2+4)(-1)^n y^2 = -1$ has no solution.*

Now we will prove Theorem 3.6, which is stated by Melham in [12].

Theorem 3.6. *All solutions of the equation $x^2 - V_n xy + (-1)^n y^2 = -(p^2+4)U_n^2$ and $x^2 - V_n xy + (-1)^n y^2 = (p^2+4)U_n^2$are given by $(x,y) = \mp(V_{n+m}, V_m)$ with m even and m odd, respectively.*

Proof. Suppose that $x^2 - V_n xy + (-1)^n y^2 = -(p^2+4)U_n^2$ for some integers x and y. Then using (2.8), we get $(2x - V_n y)^2 - (p^2+4)U_n^2 y^2 = -4(p^2+4)U_n^2$. Thus, it follows that $U_n | 2x - V_n y$. Therefore, there is an integer z such that $2x - V_n y = U_n z$. Hence we can write $z^2 - (p^2+4)y^2 = -4(p^2+4)$. This implies that $(p^2+4)|z$ since p^2+4 is square free. Then there is an integer a such that $z = (p^2+4)a$ and we have $2x - V_n y = (p^2+4)U_n a$. Thus, it follows that

$$y^2 - p^2 a^2 = 4 + 4a^2.$$

Hence $y^2 - p^2 a^2$ is even. Then we can see that y and pa have the same parity. Taking $u = (y + pa)/2$ and $v = a$, we obtain

$$u = \frac{y + p\left(\dfrac{2x - V_n y}{(p^2+4)U_n}\right)}{2} = \frac{px + V_{n-1}y}{(p^2+4)U_n}$$

and

$$v = \frac{2x - V_n y}{(p^2+4)U_n}.$$

Thus, we get

$$u^2 - puv + v^2 = -(p^2+4)\left(x^2 - V_n xy + (-1)^n y^2\right)/(p^2+4)^2 U_n^2 = 1.$$

Therefore it follows that $(u,v) = \mp(U_{m+1}, U_m)$ for some even m by Corollary 3.2. Thus, we obtain

$$(px + V_{n-1}y)/(p^2+4)U_n = \mp U_{m+1} \text{ and } (2x - V_n y)/(p^2+4)U_n = \mp U_m.$$

This together with (2.4), (2.6) and (2.7) yields $(x,y) = \mp(V_{n+m}, V_m)$ for some even m.

Conversely, if $(x,y) = \mp(V_{n+m}, V_m)$ for some even m, then it follows that $x^2 - V_n xy + (-1)^n y^2 = -(p^2+4)U_n^2$ by (2.11).

Following the similar steps, we obtain the expected solutions of the equation $x^2 - V_n xy + (-1)^n y^2 = (p^2+4)U_n^2$. □

Theorem 3.7. *If n is even, then all solutions of the equation $x^2 - V_{2n}xy + y^2 = -(p^2+4)U_n^2$ are given by $(x,y) = \mp\left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)$ with $t \in \mathbb{Z}$. If n is odd, then all solutions of the equation $x^2 - V_{2n}xy + y^2 = (p^2+4)U_n^2$ are given by $(x,y) = \mp\left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)$ with $t \in \mathbb{Z}$.*

Proof. Assume that n is even and $x^2 - V_{2n}xy + y^2 = -(p^2+4)U_n^2$ for some integers x and y. Multiplying both sides of this equation by V_n^2 and considering the fact that $U_{2n} = U_n V_n$, we get

$$(V_n x)^2 - V_{2n}(V_n x)(V_n y) + (V_n y)^2 = -(p^2+4)U_{2n}^2.$$

From Theorem 3.6, it follows that $(x,y) = \mp(V_{2n+m}/V_n, V_m/V_n)$ for some even m. Moreover, since x and y are integers, there is an integer t such that $m = (2t+1)\,n$ by Theorem 2.6. Therefore we obtain $(x,y) = \mp\left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)$.

Conversely, if n is even and $(x, y) = \mp \left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)$ for some $t \in \mathbb{Z}$, then it follows that $x^2 - V_{2n}xy + y^2 = -(p^2+4)U_n^2$ by (2.11).

Similarly it can be easily seen that, if n is odd, then all solutions of the equation $x^2 - V_{2n}xy + y^2 = (p^2+4)U_n^2$ are given by $(x, y) = \mp \left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)$ with $t \in \mathbb{Z}$ by Theorem 3.6, Theorem 2.6 and Equation (2.11). □

By using Theorems 3.7, 3.6, and 2.6, the following corollary can be proved. So, we omit its proof.

Corollary 3.8. *If n is odd, then the equation $x^2 - V_{2n}xy + y^2 = -(p^2+4)U_n^2$ and if n is even, then the equation $x^2 - V_{2n}xy + y^2 = (p^2+4)U_n^2$ has no solution.*

Theorem 3.9. *All solutions of the equation $x^2 - V_nxy + (-1)^n y^2 = -(p^2+4)$ are given by $(x, y) = \mp (V_{n+m}/U_n, V_m/U_n)$ with m even and $U_n|V_m$.*

Proof. Assume that $x^2 - V_nxy + (-1)^n y^2 = -(p^2+4)$ for some integers x and y. Multiplying both sides of the equation by U_n^2, we get

$$(U_nx)^2 - V_n (U_nx)(U_ny) + (-1)^n (U_ny)^2 = -(p^2+4)U_n^2.$$

Hence using Theorem 3.6, we obtain the expected result.

Conversely, if m is even and $(x, y) = \mp (V_{n+m}/U_n, V_m/U_n)$, then it follows that $x^2 - V_nxy + (-1)^n y^2 = -(p^2+4)$ by (2.11). □

The following corollaries can be given from Theorem 3.9 and Theorem 2.7.

Corollary 3.10. *All solutions of the equation $x^2 - pxy - y^2 = -(p^2+4)$ are given by $(x, y) = \mp (V_{2t+1}, V_{2t})$ with $t \in \mathbb{Z}$.*

Corollary 3.11. *If $p \geq 3$, then the equation $x^2 - (p^2+2)xy + y^2 = -(p^2+4)$ has no solution.*

Theorem 3.12. *All solutions of the equation $x^2 - V_nxy + (-1)^n y^2 = p^2+4$ are given by $(x, y) = \mp (V_{n+m}/U_n, V_m/U_n)$ with m odd and $U_n|V_m$.*

Proof. Assume that $x^2 - V_nxy + (-1)^n y^2 = p^2+4$ for some integers x and y. Multiplying both sides of the equation by U_n^2, we get

$$(U_nx)^2 - V_n (U_nx)(U_ny) + (-1)^n (U_ny)^2 = (p^2+4)U_n^2.$$

Hence using Theorem 3.6, we have $(x, y) = \mp (V_{n+m}/U_n, V_m/U_n)$ for some odd m with $U_n|V_m$.

If m is odd and $(x, y) = \mp (V_{n+m}/U_n, V_m/U_n)$, then by using (2.11), it is seen that $x^2 - V_nxy + (-1)^n y^2 = p^2+4$. □

The following corollaries can be given from Theorem 3.12 and Theorem 2.7.

Corollary 3.13. *All solutions of the equation $x^2 - pxy - y^2 = p^2+4$ are given by $(x, y) = \mp (V_{2t+2}, V_{2t+1})$ with $t \in \mathbb{Z}$.*

Corollary 3.14. *All solutions of the equation $x^2 - (p^2+2)xy + y^2 = p^2+4$ are given by $(x, y) = \mp (V_{2t+3}/p, V_{2t+1}/p)$ with $t \in \mathbb{Z}$.*

Moreover, the following corollary can be proven easily.

Corollary 3.15. *All solutions of the equation $x^2 - 6xy + y^2 = 8$ are given by $(x, y) = \mp (V_{2t+3}/2, V_{2t+1}/2)$ with $t \in \mathbb{Z}$.*

Now we give the following theorem without proof, since it can be proved in the same manner with the proof of Theorem 3.12.

Theorem 3.16. *All solutions of the equation* $x^2 - V_{2n}xy + y^2 = -(p^2+4)V_n^2$ *are given by* $(x,y) = \mp(V_{2n+m}/U_n, V_m/U_n)$ *with* m *even and* $U_n|V_m$.

The following corollaries can be given by using Theorem 3.16 and Theorem 2.7.

Corollary 3.17. *All solutions of the equation* $x^2 - (p^2+2)xy + y^2 = -p^2(p^2+4)$ *are given by* $(x,y) = \mp(V_{2t+2}, V_{2t})$ *with* $t \in \mathbb{Z}$.

Corollary 3.18. *If* $p \geq 3$, *then the equation* $x^2 - [p^2(p^2+4)+2]xy + y^2 = -(p^2+4)(p^2+2)^2$ *has no solutions.*

Theorem 3.19. *All solutions of the equation* $x^2 - V_{2n}xy + y^2 = (p^2+4)V_n^2$ *are given by* $(x,y) = \mp(V_{2n+m}/U_n, V_m/U_n)$ *with* m *odd and* $U_n|V_m$.

Proof. Assume that $x^2 - V_{2n}xy + y^2 = (p^2+4)V_n^2$ for some integers x and y. Multiplying both sides of this equation by U_n^2, we have

$$(U_nx)^2 - V_{2n}(U_nx)(U_ny) + (U_ny)^2 = (p^2+4)U_{2n}^2.$$

Then it follows that $(x,y) = \mp(V_{2n+m}/U_n, V_m/U_n)$ for some odd m with $U_n|V_m$ by Theorem 3.6.

Conversely, if m is odd and $(x,y) = \mp(V_{2n+m}/U_n, V_m/U_n)$, then we get $x^2 - V_{2n}xy + y^2 = (p^2+4)V_n^2$ by (2.11). □

The following corollaries can be given by using Theorem 2.7 and Theorem 3.19.

Corollary 3.20. *All solutions of the equation* $x^2 - (p^2+2)xy + y^2 = p^2(p^2+4)$ *are given by* $(x,y) = \mp(V_{2t+3}, V_{2t+1})$ *with* $t \in \mathbb{Z}$.

Corollary 3.21. *If* $p \geq 2$, *then all solutions of the equation* $x^2 - [p^2(p^2+4)+2]xy + y^2 = (p^2+4)(p^2+2)^2$ *are given by* $(x,y) = \mp\left(V_{(2t+5)}/p, V_{(2t+1)}/p\right)$ *with* $t \in \mathbb{Z}$.

Now we give the following theorem which is stated by Kılıç and Ömür in [8].

Theorem 3.22. *All solutions of the equation* $x^2 - V_nxy + (-1)^n y^2 = -U_n^2$ *and* $x^2 - V_nxy + (-1)^n y^2 = U_n^2$ *are given by* $(x,y) = \mp(U_{n+m}, U_m)$ *with* m *odd and* m *even, respectively.*

Proof. Suppose that $x^2 - V_nxy + (-1)^n y^2 = -U_n^2$ for some integers x and y. Completing the square gives $(2x - V_ny)^2 - (p^2+4)U_n^2y^2 = -4U_n^2$, and it is seen that $U_n|2x - V_ny$. Thus, it follows that

$$((2x - V_ny)/U_n)^2 - (p^2+4)y^2 = -4.$$

Taking $u = (((2x - V_ny)/U_n) + py)/2 = (x - U_{n-1}y)/U_n$ and $v = y$, we have $u^2 - puv - v^2 = -1$. Therefore, from Corollary 3.2, we get $(u,v) = \mp(U_{m+1}, U_m)$ for some odd m. By using the fact that $U_{m+1}U_n + U_mU_{n-1} = U_{n+m}$, we get $(x,y) = \mp(U_{n+m}, U_m)$.

Conversely, if $(x,y) = \mp(U_{n+m}, U_m)$ for some odd m, then it can be seen that $x^2 - V_nxy + (-1)^n y^2 = -U_n^2$ by (2.12).

Following the similar steps, we obtain the expected solutions of the equation $x^2 - V_nxy + (-1)^n y^2 = U_n^2$. □

Theorem 3.23. *All solutions of the equation* $x^2 - V_{2n}xy + y^2 = U_n^2$ *are given by* $(x,y) = \mp\left(U_{(2t+2)n}/V_n, U_{2tn}/V_n\right)$ *with* $t \in \mathbb{Z}$.

Proof. Assume that $x^2 - V_{2n}xy + y^2 = U_n^2$ for some integers x and y. Multiplying both sides of this equation by V_n^2, we get

$$(V_nx)^2 - V_{2n}(V_nx)(V_ny) + (V_ny)^2 = U_{2n}^2.$$

Then from Theorem 3.22, it follows that $(x,y) = \mp(U_{2n+m}/V_n, U_m/V_n)$ for some even m. Hence, using Theorem 2.4, it is seen that $m = 2tn$ for some $t \in \mathbb{Z}$. Therefore, $(x,y) = \mp\left(U_{(2t+2)n}/V_n, U_{2tn}/V_n\right)$.

Conversely, if $(x,y) = \mp\left(U_{(2t+2)n}/V_n, U_{2tn}/V_n\right)$ for some $t \in \mathbb{Z}$, then it is seen that $x^2 - V_{2n}xy + y^2 = U_n^2$ by (2.12). □

Theorem 3.24. *The equation $x^2 - V_{2n}xy + y^2 = -U_n^2$ has no solution.*

Proof. Assume that $x^2 - V_{2n}xy + y^2 = -U_n^2$ for some integers x and y. Multiplying both sides of this equation by V_n^2, we get

$$(V_n x)^2 - V_{2n}(V_n x)(V_n y) + (V_n y)^2 = -U_{2n}^2.$$

From Theorem 3.22, it follows that $(x,y) = \mp(U_{2n+m}/V_n, U_m/V_n)$ for some odd m. This together with Theorem 2.4 yields the result. □

Theorem 3.25. *If n is odd, then all solutions of the equation $x^2 - V_{2n}xy + y^2 = -V_n^2$ are given by $(x,y) = \mp\left(U_{(2t+3)n}/U_n, U_{(2t+1)n}/U_n\right)$ with $t \in \mathbb{Z}$.*

Proof. Assume that $x^2 - V_{2n}xy + y^2 = -V_n^2$ for some integers x and y. Multiplying both sides of this equation by U_n^2, we get

$$(U_n x)^2 - V_{2n}(U_n x)(U_n y) + (U_n y)^2 = -U_{2n}^2.$$

Then from Theorem 3.22, it follows that $(x,y) = \mp(U_{2n+m}/U_n, U_m/U_n)$ for some odd $m \in \mathbb{Z}$. Hence, using Theorem 2.5 it is seen that $n|m$. Since n and m are odd, we have $m = (2t+1)n$ for some $t \in \mathbb{Z}$. Therefore, $(x,y) = \mp\left(U_{(2t+3)n}/U_n, U_{(2t+1)n}/U_n\right)$.

Conversely, if n is odd and $(x,y) = \mp\left(U_{(2t+3)n}/U_n, U_{(2t+1)n}/U_n\right)$ for some $t \in \mathbb{Z}$, then from (2.12), it follows that $x^2 - V_{2n}xy + y^2 = -V_n^2$. □

Now we can give the following corollaries by using Theorem 3.22, Theorem 2.5, and Equation (2.12).

Corollary 3.26. *If n is even, then the equation $x^2 - V_{2n}xy + y^2 = -V_n^2$ has no solutions.*

Corollary 3.27. *If n is even, then all solutions of the equation $x^2 - V_{2n}xy + y^2 = V_n^2$ are given by $(x,y) = \mp\left(U_{(t+2)n}/U_n, U_{tn}/U_n\right)$ with $t \in \mathbb{Z}$. If n is odd, then all solutions of the equation $x^2 - V_{2n}xy + y^2 = V_n^2$ are given by $(x,y) = \mp\left(U_{(2t+2)n}/U_n, U_{2tn}/U_n\right)$ with $t \in \mathbb{Z}$.*

Theorem 3.28. *If n is odd, then all solutions of the equations $x^2 - V_n xy - y^2 = -1$ and $x^2 - V_n xy - y^2 = 1$ are given by $(x,y) = \mp\left(U_{(2t+2)n}/U_n, U_{(2t+1)n}/U_n\right)$ and $(x,y) = \mp\left(U_{(2t+1)n}/U_n, U_{2tn}/U_n\right)$ with $t \in \mathbb{Z}$, respectively. If n is even, then all solutions of the equation $x^2 - V_n xy + y^2 = 1$ are given by $(x,y) = \mp\left(U_{(t+1)n}/U_n, U_{tn}/U_n\right)$ with $t \in \mathbb{Z}$.*

Proof. Assume that n is odd and $x^2 - V_n xy - y^2 = -1$ for some integers x and y. Multiplying both sides of this equation by U_n^2, we get

$$(U_n x)^2 - V_n(U_n x)(U_n y) - (U_n y)^2 = -U_n^2.$$

From Theorem 3.22, it follows that $x = \mp U_{n+m}/U_n$ and $y = \mp U_m/U_n$ for some odd m. Since x and y are integers, it is clear that $m = (2t+1)n$ for some $t \in \mathbb{Z}$ by Theorem 2.5. Then we obtain

$$(x,y) = \mp\left(U_{(2t+2)n}/U_n, U_{(2t+1)n}/U_n\right).$$

Conversely, if $n \geq 3$ is odd and $(x,y) = \mp\left(U_{(2t+2)n}/U_n, U_{(2t+1)n}/U_n\right)$ for some $t \in \mathbb{Z}$, then it follows that $x^2 - V_n xy - y^2 = -1$ by (2.12).

If n is odd, then in a similar way, it is easy to see that all solutions of the equation $x^2 - V_n xy - y^2 = 1$ are given by $(x,y) = \mp\left(U_{(2t+1)n}/U_n, U_{2tn}/U_n\right)$ with $t \in \mathbb{Z}$.

Now assume that n is even and $x^2-V_nxy+y^2=1$ for some integers x and y. Multiplying both sides of this equation by U_n^2 and using Theorem 3.22, it is seen that $x=\mp U_{n+m}/U_n$ and $y=\mp U_m/U_n$, for some even m. Since x and y are integers, it is clear that $m=tn$ for some $t\in\mathbb{Z}$ by Theorem 2.5. Then we obtain

$$(x,y)=\mp\left(U_{(t+1)n}/U_n,U_{tn}/U_n\right).$$

Moreover, if n is even and $(x,y)=\mp\left(U_{(t+1)n}/U_n,U_{tn}/U_n\right)$ with $t\in\mathbb{Z}$, then it follows that $x^2-V_nxy+y^2=1$ by (2.12). □

Multiplying both sides of the equation $x^2-V_nxy+y^2=-1$ by U_n^2 and using Theorem 2.5 and Theorem 3.22, the following corollary can be given.

Corollary 3.29. *If n is even, then the equation $x^2-V_nxy+y^2=-1$ has no solution.*

References

[1] B. Demirtürk and R. Keskin, *Integer Solutions of Some Diophantine Equations via Fibonacci and Lucas Numbers*, J. Integer Seq. **12**, 1-14, 2009.

[2] B. Demirtürk Bitim and R. Keskin, *On Some Diophantine Equations*, J. Inequal. Appl. **162**, 1-12, 2013.

[3] P. Hilton and J. Pedersen, *On generalized Fibonaccian and Lucasian numbers*, Math. Gaz. **90** (518), 215-222, 2006.

[4] P. Hilton, J. Pedersen and L. Somer, *On Lucasian numbers*, Fibonacci Quart. **35**, 43-47, 1997.

[5] D.E. Hinkel, *An investigation of Lucas Sequences*, Master thesis, Arizona University, 35 pages, 2007.

[6] A.F. Horadam, *Basic properties of certain generalized sequences of numbers*, Fibonacci Quart. **3** (3), 161-176, 1965.

[7] R. Keskin and B. Demirtürk, *Solutions of Some Diophantine Equations Using Generalized Fibonacci and Lucas Sequences*, Ars Combin. **111**, 161-179, 2013.

[8] E. Kılıç and N. Ömür, *Conics characterizing the generalized Fibonacci and Lucas sequences with indices in arithmetic progressions*, Ars Combin. **94**, 459-464, 2010.

[9] P. Kiss, *Diophantine representations of generalized Fibonacci numbers*, Elem. Math. **34**, 129-132, 1979.

[10] Y.V. Matiyasevich, *Hilbert's Tenth Problem*, MIT Press, Cambridge, MA, 1993.

[11] W.L. McDaniel, *Diophantine Representation of Lucas Sequences*, Fibonacci Quart. **33**, 58-63, 1995.

[12] R. Melham, *Conics Which Characterize Certain Lucas Sequences*, Fibonacci Quart. **35**, 248-251, 1997.

[13] S. Rabinowitz, *Algorithmic Manipulation of Fibonacci Identities*, Applications of Fibonacci Numbers, **6** (edited by G. E. Bergum, et al.), Kluwer Academic Pub., Dordrect, The Netherlands, 389-408, 1996.

[14] P. Ribenboim, *Square classes of Fibonacci and Lucas numbers*, Port. Math. **46** (2), 159-175, 1989.

[15] P. Ribenboim, *The Little book of big primes*, Springer-Verlag, New York, 1991.

[16] P. Ribenboim, *An Algorithm to Determine the Points with Integral Coordinates in Certain Elliptic Curves*, J. Number Theory **74**, 19-38, 1999.

[17] P. Ribenboim, *My numbers, My friends*, Springer-Verlag Inc., New York, 2000.

[18] S. Zhiwei, *Singlefold Diophantine representation of the sequence $U_0=0$, $U_1=1$ and $U_{n+2}=mU_{n+1}+U_n$*, Pure and Applied Logic, Beijing Univ. Press, Beijing, 97-101, 1992.