






## Kantorovich-Stancu type operators including Boas-Buck type polynomials

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### Abstract

The aim of the paper is to introduce a Kantorovich-Stancu type modification of a generalization of Szász operators defined via Boas-Buck type polynomials and to obtain rates of convergence for these operators. Furthermore, we give the figures for comparing approximation properties of the operators  $\mathcal{K}_n^{(\alpha, \beta)}$  and  $\mathcal{B}_n$ .

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### 1. Introduction

The Szász–Mirakyan operators are defined by

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad (1.1)$$

where  $n \in \mathbb{N}$  [23]. We consider  $f \in C[0, \infty)$  for which the corresponding series is convergent. Up to now, various operators via special functions, especially generalizations and modifications of Szász operators, have been introduced by many authors and have been studied their approximation properties (see [1, 2, 8, 13–15, 19–22, 24, 25, 27–29]). In 1969, Jakimovski et al. [16] defined a generalization of Szász operators using Appell polynomials. In 1974, Ismail [11] introduced another generalization of Szász operators via Sheffer polynomials. Inspired by the papers [11, 16],  $f \in C[0, \infty)$  for which the corresponding series is convergent, Varma et al. [27] studied many properties of the following generalization of Szász operators defined by means of the Brenke type polynomials

$$L_n(f; x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \quad (1.2)$$

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under the assumptions

$$\begin{aligned} (i) \quad & A(1) \neq 0, \quad \frac{a_{k-r}b_r}{A(1)} \geq 0, \quad 0 \leq r \leq k, \quad k = 0, 1, 2, \dots, \\ (ii) \quad & B : [0, \infty) \rightarrow (0, \infty), \\ (iii) \quad & (1.4) \text{ and } (1.5) \text{ converge for } |t| < R, \quad (R > 1) \end{aligned} \quad (1.3)$$

where

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0 \quad \text{and} \quad B(t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \quad (r \geq 0) \quad (1.4)$$

are analytic functions and the Brenke type polynomials [5] are generated by

$$A(t) B(xt) = \sum_{k=0}^{\infty} p_k(x) t^k \quad (1.5)$$

where

$$p_k(x) = \sum_{r=0}^k a_{k-r} b_r x^r, \quad k = 0, 1, 2, \dots$$

In 2013, Aktas et al. in [1] defined the following Kantorovich-Stancu version of the operators given by (1.2) for  $n \in \mathbb{N}$ ,  $x \geq 0$  and  $f \in C[0, \infty)$  for which the corresponding series is convergent under the assumptions (1.3)

$$K_n^{(\alpha, \beta)}(f; x) := \frac{n + \beta}{A(1) B(nx)} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t) dt. \quad (1.6)$$

For  $\alpha = \beta = 0$ , this operator returns to the Kantorovich type of the operators given by (1.2) [24]

$$K_n(f; x) := \frac{n}{A(1) B(nx)} \sum_{k=0}^{\infty} p_k(nx) \int_{k/n}^{(k+1)/n} f(t) dt$$

which gives the Kantorovich version of Szász-Mirakyan operators [4] in the special case of  $B(t) = e^t$  and  $A(t) = 1$

$$K_n(f; x) := n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f(t) dt. \quad (1.7)$$

The approximation properties of the operators (1.7) can be found in [9, 18, 26, 28, 30] and the references cited therein.

In 2012, Sucu et al. [21] constructed linear positive operators by means of Boas-Buck type polynomials which give the Brenke-type polynomials, Sheffer polynomials, and Appell polynomials in the special cases. In [12], Boas-Buck-type polynomials are generated by

$$A(t) B(xH(t)) = \sum_{k=0}^{\infty} p_k(x) t^k \quad (1.8)$$

where  $A(t)$ ,  $B(t)$  and  $H(t)$  are analytic functions

$$\begin{aligned} A(t) &= \sum_{r=0}^{\infty} a_r t^r \quad (a_0 \neq 0), \quad B(t) = \sum_{r=0}^{\infty} b_r t^r \quad (b_r \neq 0) \\ H(t) &= \sum_{r=1}^{\infty} h_r t^r, \quad (h_1 \neq 0). \end{aligned} \quad (1.9)$$

These operators defined by Sucu et al. [21] are as follows for  $x \geq 0$  and  $n \in \mathbb{N}$

$$\mathcal{B}_n(f; x) := \frac{1}{A(1) B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad (1.10)$$

which satisfy

$$\begin{aligned} (i) \quad & A(1) \neq 0, \quad H'(1) = 1, \quad p_k(x) \geq 0, \quad k = 0, 1, 2, \dots, \\ (ii) \quad & B: \mathbb{R} \rightarrow (0, \infty), \\ (iii) \quad & (1.8) \text{ and } (1.9) \text{ converge for } |t| < R, \quad (R > 1). \end{aligned} \quad (1.11)$$

In the present paper, we consider a Kantorovich-Stancu version of the operators (1.10) as follows

$$\mathcal{K}_n^{(\alpha, \beta)}(f; x) := \frac{n + \beta}{A(1)B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) \int_{\frac{(k+\alpha)/(n+\beta)}{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}} f(t) dt \quad (1.12)$$

under the assumption (1.11),  $f \in C[0, \infty)$  and  $0 \leq \alpha \leq \beta$ , and we study the approximation properties of these operators. We also present special cases of these operators including Charlier polynomials and Gould-Hopper polynomials.

The case of  $H(t) = t$  in the operators (1.12) gives the Kantorovich-Stancu type operators (1.6) including Brenke-type polynomials. For  $B(t) = e^t$ , the operators (1.12) reduce to the Kantorovich-Stancu type of the operators with Sheffer polynomials defined by Ismail [11]. In the special case of  $B(t) = e^t$  and  $H(t) = t$ , we have Kantorovich-Stancu type of the operators with Appell polynomials introduced by Jakimovski et al. [16]. Also, for  $A(t) = 1$ ,  $B(t) = e^t$  and  $H(t) = t$ , it turns out the Kantorovich-Stancu type of Szász-Mirakyan operators.

## 2. Approximation properties of the operators $\mathcal{K}_n^{(\alpha, \beta)}$

First, for the operators  $\mathcal{K}_n^{(\alpha, \beta)}$  given by (1.12), we shall give some auxiliary results to prove the main theorem.

**Lemma 2.1.** *For each  $x \in [0, \infty)$ , the Kantorovich-Stancu type operators (1.12) have the following properties*

$$\mathcal{K}_n^{(\alpha, \beta)}(1; x) = 1, \quad (2.1)$$

$$\mathcal{K}_n^{(\alpha, \beta)}(s; x) = \frac{nB'(nxH(1))}{(n + \beta)B(nxH(1))}x + \frac{2A'(1) + (2\alpha + 1)A(1)}{2(n + \beta)A(1)}, \quad (2.2)$$

$$\begin{aligned} \mathcal{K}_n^{(\alpha, \beta)}(s^2; x) &= \frac{n^2B''(nxH(1))}{(n + \beta)^2B(nxH(1))}x^2 \\ &+ \frac{n[2A'(1) + (2\alpha + 2)A(1) + A(1)H''(1)]B'(nxH(1))}{(n + \beta)^2A(1)B(nxH(1))}x \\ &+ \frac{1}{3(n + \beta)^2A(1)} \left\{ 3(A''(1) + A'(1)) + 3(2\alpha + 1)A'(1) + (3\alpha^2 + 3\alpha + 1)A(1) \right\}. \end{aligned} \quad (2.3)$$

**Proof.** From the generating function of the Boas-Buck-type polynomials given by (1.8), a few calculations reveal that

$$\begin{aligned} \sum_{k=0}^{\infty} p_k(nx) &= A(1)B(nxH(1)), \\ \sum_{k=0}^{\infty} kp_k(nx) &= A'(1)B(nxH(1)) + nxA(1)B'(nxH(1)), \\ \sum_{k=0}^{\infty} k^2p_k(nx) &= n^2x^2A(1)B''(nxH(1)) + nxB'(nxH(1)) \{2A'(1) + A(1) + A(1)H''(1)\} \\ &\quad + B(nxH(1)) \{A''(1) + A'(1)\}. \end{aligned}$$

By using these equalities, we obtain the assertions of the lemma by simple calculation.  $\square$

**Lemma 2.2.** For each  $x \in [0, \infty)$ , we have

$$\begin{aligned} \mathcal{K}_n^{(\alpha, \beta)}((s-x)^2; x) &= \left\{ \frac{n^2 B''(nxH(1))}{(n+\beta)^2 B(nxH(1))} - \frac{2nB'(nxH(1))}{(n+\beta)B(nxH(1))} + 1 \right\} x^2 \\ &+ \left\{ \frac{[2A'(1) + (2\alpha+2)A(1) + H''(1)A(1)]}{(n+\beta)^2 A(1)} \frac{nB'(nxH(1))}{B(nxH(1))} \right. \\ &\quad \left. - \frac{2A'(1) + (2\alpha+1)A(1)}{(n+\beta)A(1)} \right\} x \\ &+ \frac{1}{(n+\beta)^2 A(1)} \left\{ A''(1) + (2\alpha+2)A'(1) + (\alpha^2 + \alpha + 1/3)A(1) \right\}. \end{aligned}$$

**Theorem 2.3.** Let

$$E := \left\{ f : x \in [0, \infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \rightarrow \infty \right\}$$

and

$$\lim_{y \rightarrow \infty} \frac{B'(y)}{B(y)} = 1 \quad \text{and} \quad \lim_{y \rightarrow \infty} \frac{B''(y)}{B(y)} = 1. \quad (2.4)$$

If  $f \in C[0, \infty) \cap E$ , then

$$\lim_{n \rightarrow \infty} \mathcal{K}_n^{(\alpha, \beta)}(f; x) = f(x),$$

and the operators  $\mathcal{K}_n^{(\alpha, \beta)}$  converge uniformly in each compact subset of  $[0, \infty)$ .

**Proof.** From Lemma 2.1, by considering the equality (2.4), one obtains

$$\lim_{n \rightarrow \infty} \mathcal{K}_n^{(\alpha, \beta)}(s^i; x) = x^i, \quad i = 0, 1, 2,$$

where the convergence is satisfied uniformly in each compact subset of  $[0, \infty)$ . Then, using the universal Korovkin-type property (vi) of Theorem 4.1.4 in [3] completes the proof.  $\square$

Now, we compute the rates of convergence of the operators  $\mathcal{K}_n^{(\alpha, \beta)}(f)$  to  $f$  by means of a classical approach, the second modulus of continuity and Peetre's  $K$ -functional.

Let  $f \in \tilde{C}[0, \infty)$ . Then for  $\delta > 0$ , the modulus of continuity of  $f$  which is denoted by  $w(f; \delta)$  is defined by

$$w(f; \delta) := \sup_{\substack{x, y \in [0, \infty) \\ |x-y| \leq \delta}} |f(x) - f(y)|$$

where  $\tilde{C}[0, \infty)$  denotes the space of uniformly continuous functions on  $[0, \infty)$ . Then, for any  $\delta > 0$  and each  $x \in [0, \infty)$ ,

$$|f(x) - f(y)| \leq w(f; \delta) \left( \frac{|x-y|}{\delta} + 1 \right) \quad (2.5)$$

holds.

One can estimate the rate of convergence of the sequence  $\mathcal{K}_n^{(\alpha, \beta)}(f)$  to  $f$  via the modulus of continuity as follows.

**Theorem 2.4.** If  $f \in \tilde{C}[0, \infty) \cap E$ , then we have

$$\left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| \leq 2w\left(f; \sqrt{\lambda_n(x)}\right)$$

where

$$\begin{aligned}
 \lambda &= \lambda_n(x) = \mathcal{K}_n^{(\alpha, \beta)}((s-x)^2; x) \\
 &= \left\{ \frac{n^2 B''(nxH(1))}{(n+\beta)^2 B(nxH(1))} - \frac{2nB'(nxH(1))}{(n+\beta)B(nxH(1))} + 1 \right\} x^2 \\
 &\quad + \left\{ \frac{[2A'(1) + (2\alpha+2)A(1) + H''(1)A(1)]nB'(nxH(1))}{(n+\beta)^2 A(1)} - \frac{2A'(1) + (2\alpha+1)A(1)}{(n+\beta)A(1)} \right\} x \\
 &\quad + \frac{1}{(n+\beta)^2 A(1)} \left\{ A''(1) + (2\alpha+2)A'(1) + (\alpha^2 + \alpha + 1/3)A(1) \right\}.
 \end{aligned} \tag{2.6}$$

**Proof.** Using linearity of the operators  $\mathcal{K}_n^{(\alpha, \beta)}$ , (2.1) and (2.5), we get

$$\begin{aligned}
 \left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| &\leq \frac{n+\beta}{A(1)B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |f(s) - f(x)| ds \\
 &\leq \frac{n+\beta}{A(1)B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} \left( \frac{|s-x|}{\delta} + 1 \right) w(f; \delta) ds \\
 &\leq \left\{ 1 + \frac{n+\beta}{A(1)B(nxH(1))\delta} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x| ds \right\} w(f; \delta).
 \end{aligned} \tag{2.7}$$

If we apply the Cauchy-Schwarz inequality for integration, it follows

$$\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x| ds \leq \frac{1}{\sqrt{n+\beta}} \left( \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|^2 ds \right)^{1/2},$$

which gives

$$\begin{aligned}
 \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x| ds \\
 \leq \frac{1}{\sqrt{n+\beta}} \sum_{k=0}^{\infty} p_k(nx) \left( \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|^2 ds \right)^{1/2}.
 \end{aligned} \tag{2.8}$$

Considering Cauchy-Schwarz inequality for summation on the right hand side of (2.8), one can easily obtain

$$\begin{aligned}
 \sum_{k=0}^{\infty} p_k(nx) & \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x| ds \\
 & \leq \frac{\sqrt{A(1)B(nxH(1))}}{\sqrt{n+\beta}} \left( \frac{A(1)B(nxH(1))}{n+\beta} \mathcal{K}_n^{(\alpha,\beta)}((s-x)^2; x) \right)^{1/2} \\
 & = \frac{A(1)B(nxH(1))}{n+\beta} \left( \mathcal{K}_n^{(\alpha,\beta)}((s-x)^2; x) \right)^{1/2} \\
 & = \frac{A(1)B(nxH(1))}{n+\beta} (\lambda_n(x))^{1/2}
 \end{aligned} \tag{2.9}$$

where  $\lambda_n(x)$  is given by (2.6). Taking into account this inequality in (2.7) leads to

$$\left| \mathcal{K}_n^{(\alpha,\beta)}(f; x) - f(x) \right| \leq \left\{ 1 + \frac{1}{\delta} \sqrt{\lambda_n(x)} \right\} w(f; \delta).$$

If we get  $\delta = \sqrt{\lambda_n(x)}$ , we obtain the desired. □

Now, we give the rates of convergence of the operators  $\mathcal{K}_n^{(\alpha,\beta)}$  to  $f$  by means of the second modulus of continuity and Peetre’s  $K$ -functional.

We remind that the second modulus of continuity of  $f \in C_B[0, \infty)$  is defined by

$$w_2(f; \delta) := \sup_{0 < t \leq \delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C_B}$$

where  $C_B[0, \infty)$  denotes the class of real valued functions defined on  $[0, \infty)$  that are bounded and uniformly continuous with the norm  $\|f\|_{C_B} = \sup_{x \in [0, \infty)} |f(x)|$ .

Peetre’s  $K$ -functional of the function  $f \in C_B[0, \infty)$  is defined by

$$K(f; \delta) := \inf_{g \in C_B^2[0, \infty)} \left\{ \|f - g\|_{C_B} + \delta \|g\|_{C_B^2} \right\} \tag{2.10}$$

where

$$C_B^2[0, \infty) := \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$$

and the norm  $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$  (see [7]). Also, in [6] we have the following inequality:

$$K(f; \delta) \leq M \left\{ w_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C_B} \right\} \tag{2.11}$$

for all  $\delta > 0$  where  $M$  is a constant which is independent of the function  $f$  and  $\delta$ .

**Theorem 2.5.** *Let  $f \in C_B^2[0, \infty)$ . For the operators  $\mathcal{K}_n^{(\alpha,\beta)}$  defined by (1.12), we have*

$$\left| \mathcal{K}_n^{(\alpha,\beta)}(f; x) - f(x) \right| \leq \zeta \|f\|_{C_B^2},$$

where

$$\begin{aligned} \zeta = \zeta_n(x) = & \left[ \left\{ \frac{n^2 B''(nxH(1))}{2(n+\beta)^2 B(nxH(1))} - \frac{nB'(nxH(1))}{(n+\beta)B(nxH(1))} + \frac{1}{2} \right\} x^2 \right. \\ & + \left\{ \frac{[2A'(1) + (2\alpha + 2)A(1) + H''(1)A(1)] nB'(nxH(1))}{2(n+\beta)^2 A(1)B(nxH(1))} \right. \\ & \left. - \frac{2A'(1) + (2\alpha + 1)A(1)}{2(n+\beta)A(1)} + \frac{nB'(nxH(1))}{(n+\beta)B(nxH(1))} - 1 \right\} x \\ & + \frac{1}{6(n+\beta)^2 A(1)} \left\{ 3A''(1) + (6\alpha + 6)A'(1) + (3\alpha^2 + 3\alpha + 1)A(1) \right\} \\ & \left. + \frac{2A'(1) + (2\alpha + 1)A(1)}{2(n+\beta)A(1)} \right]. \end{aligned}$$

**Proof.** From the Taylor expansion of  $f$ , the linearity of the operators  $\mathcal{K}_n^{(\alpha, \beta)}$  and the equality (2.1), we may write for  $\eta \in (x, s)$

$$\mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) = f'(x) \mathcal{K}_n^{(\alpha, \beta)}(s - x; x) + \frac{f''(\eta)}{2} \mathcal{K}_n^{(\alpha, \beta)}((s - x)^2; x). \quad (2.12)$$

Using the results in Lemma 2.1, we have

$$\mathcal{K}_n^{(\alpha, \beta)}(s - x; x) = \left\{ \frac{nB'(nxH(1))}{(n+\beta)B(nxH(1))} - 1 \right\} x + \frac{2A'(1) + (2\alpha + 1)A(1)}{2(n+\beta)A(1)} \geq 0$$

for  $s \geq x$ . Thus, by considering Lemmas 2.1 and 2.2 in (2.12), we obtain

$$\begin{aligned} \left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| & \leq \mathcal{K}_n^{(\alpha, \beta)}(s - x; x) \|f'\|_{C_B} + \frac{1}{2} \mathcal{K}_n^{(\alpha, \beta)}((s - x)^2; x) \|f''\|_{C_B} \\ & \leq \left[ \left\{ \frac{n^2 B''(nxH(1))}{2(n+\beta)^2 B(nxH(1))} - \frac{nB'(nxH(1))}{(n+\beta)B(nxH(1))} + \frac{1}{2} \right\} x^2 \right. \\ & + \left\{ \frac{[2A'(1) + (2\alpha + 2)A(1) + H''(1)A(1)] nB'(nxH(1))}{2(n+\beta)^2 A(1)B(nxH(1))} \right. \\ & \left. - \frac{2A'(1) + (2\alpha + 1)A(1)}{2(n+\beta)A(1)} + \frac{nB'(nxH(1))}{(n+\beta)B(nxH(1))} - 1 \right\} x \\ & + \frac{1}{6(n+\beta)^2 A(1)} \left\{ 3A''(1) + (6\alpha + 6)A'(1) + (3\alpha^2 + 3\alpha + 1)A(1) \right\} \\ & \left. + \frac{2A'(1) + (2\alpha + 1)A(1)}{2(n+\beta)A(1)} \right] \|f\|_{C_B^2} \end{aligned}$$

which completes the proof. □

**Theorem 2.6.** If  $f \in C_B[0, \infty)$ , then

$$\left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| \leq 2M \left\{ w_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C_B} \right\}$$

holds where

$$\delta := \delta_n(x) = \frac{1}{2} \zeta_n(x)$$

and the constant  $M > 0$  is independent of  $f$  and  $\delta$ . Also,  $\zeta_n(x)$  is given as in Theorem 2.5.

**Proof.** We assume that  $g \in C_B^2[0, \infty)$ . From Theorem 2.5, we can get

$$\begin{aligned} \left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| &\leq \left| \mathcal{K}_n^{(\alpha, \beta)}(f - g; x) \right| + \left| \mathcal{K}_n^{(\alpha, \beta)}(g; x) - g(x) \right| + |g(x) - f(x)| \\ &\leq 2 \|f - g\|_{C_B} + \zeta \|g\|_{C_B^2} \\ &= 2 \left[ \|f - g\|_{C_B} + \delta \|g\|_{C_B^2} \right]. \end{aligned} \quad (2.13)$$

Since the left-hand side of inequality (2.13) does not depend on the function  $g \in C_B^2[0, \infty)$ , it follows from Peetre's  $K$ -functional  $K(f; \delta)$  defined by (2.10)

$$\left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| \leq 2K(f; \delta).$$

By using the relation (2.11) in the last inequality, we obtain

$$\left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| \leq 2M \left\{ w_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C_B} \right\}.$$

This concludes the proof.  $\square$

We note that  $\lambda_n, \zeta_n, \delta_n \rightarrow 0$  when  $n \rightarrow \infty$  under the assumption (2.4) in Theorems 2.4-2.6.

**Remark 2.7.** For  $\alpha = \beta = 0$ , the operators (1.12) reduces to the Kantorovich type operators including Boas-Buck-type polynomials given by

$$\mathcal{K}_n(f; x) := \frac{n}{A(1)B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) \int_{k/n}^{(k+1)/n} f(t) dt.$$

For  $\alpha = \beta = 0$ , the results given above are satisfied by the Kantorovich type operators including Boas-Buck-type polynomials.

**Remark 2.8.** In the case of  $H(t) = t$ , the results obtained in the paper capture the results obtained for Kantorovich-Stancu type operators (1.6) including Brenke-type polynomials in [1].

### 3. Special cases of the operators $\mathcal{K}_n^{(\alpha, \beta)}$

**Case 1.** Gould-Hopper polynomials  $g_k^{d+1}(x, h)$  are defined through the identity

$$g_k^{d+1}(x, h) = \sum_{m=0}^{\lfloor \frac{k}{d+1} \rfloor} \frac{k!}{m!(k - (d+1)m)!} h^m x^{k - (d+1)m}$$

where, as usual,  $\lfloor \cdot \rfloor$  denotes the integer part [10], and they have generating function of the form

$$e^{ht^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x, h) \frac{t^k}{k!}. \quad (3.1)$$

Gould-Hopper polynomials are Boas-Buck-type polynomials with for the special case of  $A(t) = e^{ht^{d+1}}$ ,  $B(t) = e^t$  and  $H(t) = t$  in (1.8). From (1.12), Kantorovich-Stancu type operators including the Gould-Hopper polynomials are as follows:

$$K_n^{(\alpha, \beta)}(f; x) := (n + \beta) e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx, h)}{k!} \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t) dt$$

where  $x \in [0, \infty)$  and  $h \geq 0$  in [1].

**Case 2.** The Charlier polynomials  $C_k^{(a)}(x)$  are generated by

$$e^t \left(1 - \frac{t}{a}\right)^x = \sum_{k=0}^{\infty} C_k^{(a)}(x) \frac{t^k}{k!}, \quad |t| < a.$$



Charlier polynomials are the Boas-Buck-type polynomials for the choice  $A(t) = e^t$ ,  $B(t) = e^t$  and  $H(t) = \ln(1 - \frac{t}{a})$  in (1.8). In order to ensure the restrictions (1.11) and the assumption (2.4), we get the generating function as

$$e^t e^{-(a-1)x \ln(1-t/a)} = \sum_{k=0}^{\infty} C_k^{(a)}(- (a-1)x) \frac{t^k}{k!}, \quad |t| < a, \quad a > 1.$$

In this case, the operator (1.12) turns to

$$T_n^{(\alpha, \beta)}(f; x) := (n + \beta) e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t) dt.$$

For  $\alpha = \beta = 0$ , we have Szász-Kantorovich type operators based on Charlier polynomials in [17].

#### 4. Some graphical representations

In this section, we give the graphs to compare approximation properties of the operators  $\mathcal{K}_n^{(\alpha, \beta)}$  with  $\mathcal{B}_n$ .

Firstly, we use  $f(x) = e^{-x}$ ,  $\alpha = 1$ ,  $\beta = 2$ ,  $H(t) = t$ ,  $A(t) = 1$ ,  $B(t) = e^t$  and  $n = 10, 20, 50, 100$  for operators  $\mathcal{K}_n^{(\alpha, \beta)}$ . In Figure 1, red color line for  $n = 10$ , green color line for  $n = 20$ , brown color line for  $n = 50$ , purple color line for  $n = 100$  and black color line for  $f(x)$ .

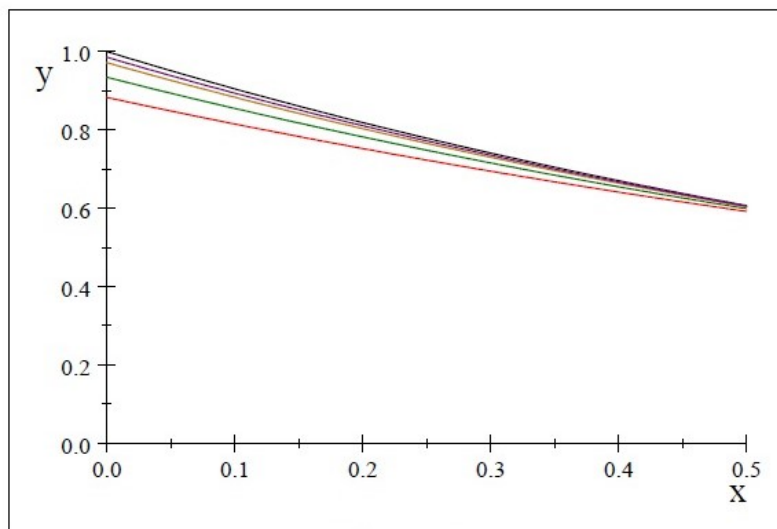


Figure 1

Now, we use  $f(x) = \sin(2\pi x)$ ,  $\alpha = 1$ ,  $\beta = 2$ ,  $H(t) = t$ ,  $A(t) = 1$ ,  $B(t) = e^t$  and  $n = 20, 50, 100, 200, 500$  for operators  $\mathcal{K}_n^{(\alpha, \beta)}$ . In Figure 2, navy color line for  $n = 20$ , light green color line for  $n = 50$ , blue color line for  $n = 100$ , pink color line for  $n = 200$ , green color line for  $n = 500$  and black color line for  $f(x)$  in  $x \in [0.5, 1]$ .

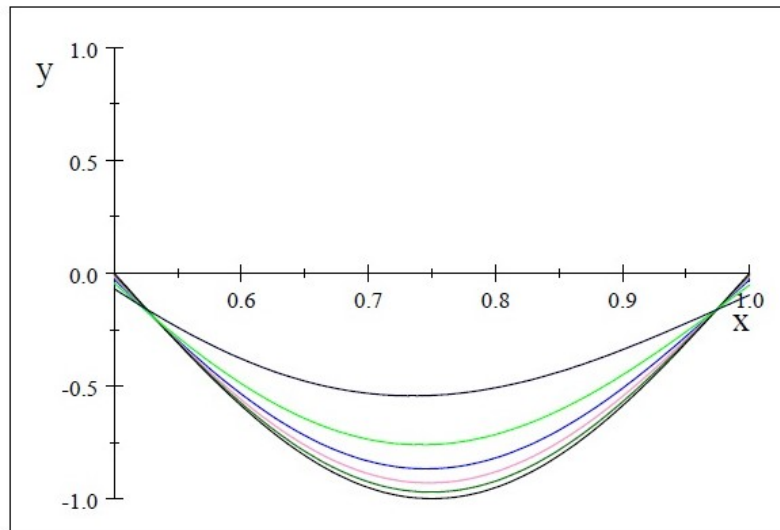


Figure 2

Finally, we compare the operators  $\mathcal{K}_n^{(\alpha,\beta)}$  with  $\mathcal{B}_n$  for  $f(x) = e^{-x}$ ,  $n = 100$ ,  $\alpha = 1$ ,  $\beta = 2$ ,  $H(t) = t$ ,  $A(t) = 1$ ,  $B(t) = e^t$ . In Figure 3, red color line for  $\mathcal{K}_n^{(\alpha,\beta)}$ , blue color line for  $\mathcal{B}_n$ , black color line for  $f(x)$ .

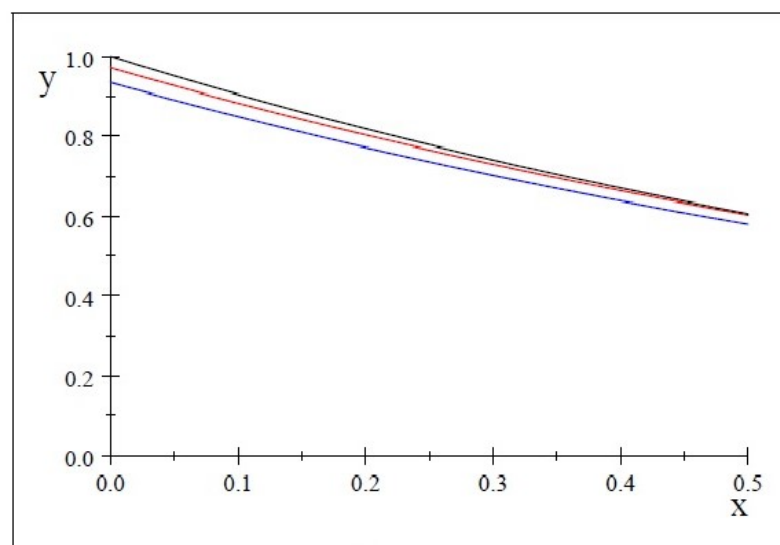


Figure 3

### References

- [1] R. Aktaş, B. Çekim and F. Taşdelen, A Kantorovich-Stancu type generalization of Szász operators including Brenke-type polynomials, *J. Func. Spaces* **2013**, 9 pages, 2013.
- [2] A. Altın, O. Dogru and F. Tasdelen, The generalization of Meyer-Konig and Zeller operators by generating functions, *J. Math. Anal. Appl.* **312** (1), 181-194, 2005.
- [3] F. Altomare and M. Campiti, Korovkin-type approximation theory and its applications. Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff, de Gruyter Studies in Mathematics, 17. Walter de Gruyter & Co., Berlin, 1994.
- [4] P.L. Butzer, On the extensions of Bernstein polynomials to the infinite interval, *Proc. Amer. Soc.* **5**, 547-553, 1954.

- [5] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
- [6] A. Ciupa, A class of integral Favard-Szász type operators, Stud. Univ. Babeş-Bolyai Math. **40** (1), 39-47, 1995.
- [7] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer-Verlag, New York, 1987.
- [8] O. Dogru, M.A. Özarşlan and F. Taşdelen, On positive operators involving a certain class of generating functions, Stud. Sci. Math. Hungar. **41** (4), 415-429, 2004.
- [9] O. Duman, M.A. Özarşlan and B. Della Vecchia, Modified  $q$ -Szász-Mirakjan-Kantorovich operators preserving linear functions, Turk. J. Math. **33** (2), 151-158, 2009.
- [10] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, Duke Math. J. **29**, 51-63, 1962.
- [11] M.E.H. Ismail, On a generalization of Szász operators, Mathematica (Cluj), **39**, 259-267, 1974.
- [12] M.E.H. Ismail, Classical and quantum orthogonal polynomial in one variable, Cambridge University Press, Cambridge, UK, 2005.
- [13] G. İçöz, F. Taşdelen Yeşildal and O. Doğru, Kantorovich Process of Linear Positive Operators Via Biorthogonal Polynomials, J. Inequal. Spec. Funct. **3**, 77-84, 2012.
- [14] G. İçöz, F. Taşdelen and S. Varma, On Linear Positive Operators Involving Biorthogonal Polynomial, Ars Combin. **105**, 319-331, 2012.
- [15] G. İçöz, S. Varma and S. Sucu, Approximation by operators including generalized Appell polynomials, Filomat **30** (2), 429-440, 2016.
- [16] A. Jakimovski and D. Leviatan, Generalized Szász operators for the approximation in the infinite interval, Mathematica (Cluj) **11**, 97-103, 1969.
- [17] A. Kajla and P. N. Agrawal, Szász-Kantorovich type operators based on Charlier polynomials, Kyungpook Math. J. **56** (3), 877-897, 2016.
- [18] G. Nowak and A. Sikorska-Nowak, Some approximation properties for modified Szász-Mirakyan-Kantorovich operators, Rev. Anal. Numér. Théor. Approx. **38** (1), 73-82, 2009.
- [19] M.A. Özarşlan,  $q$ -Laguerre type linear positive operators, Stud. Sci. Math. Hungar. **44** (1), 65-80, 2007.
- [20] M.A. Özarşlan and O. Duman, Approximation properties of Poisson integrals for orthogonal expansions, Taiwanese J. Math. **12** (5), 1147-1163, 2008.
- [21] S. Sucu, G. İçöz and S. Varma, On Some Extensions of Szász Operators Including Boas-Buck-Type Polynomials, Abstr. Appl. Anal. **2012**, 15 pages, 2012.
- [22] S Sucu and S Varma, Generalization of Jakimovski- Leviatan type Szász operators, Appl. Math. Comput. **270**, 977-983, 2015.
- [23] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Natl. Bur. Stand. **45**, 239-245, 1950.
- [24] F. Taşdelen, R. Aktaş and A. Altın, A Kantorovich Type of Szász Operators Including Brenke-Type Polynomials, Abst. Appl. Anal. **2012**, 13 pages, 2012.
- [25] F. Taşdelen and A. Erençin, The generalization of bivariate MKZ operators by multiple generating functions, J. Math. Anal. Appl. **331** (1), 727-735, 2007.
- [26] V. Totik, Approximation by Szász-Mirakjan-Kantorovich operators in  $L^p$  ( $p > 1$ ), Anal. Math. **9** (2), 147-167, 1983.
- [27] S. Varma, S. Sucu and G. İçöz, Generalization of Szász operators involving Brenke type polynomials, Comput. Math. Appl. **64** (2), 121-127, 2012.
- [28] S. Varma and F. Taşdelen, Szász type operators involving Charlier polynomials, Mathematical and Computer Modelling **56** (5-6), 118-122, 2012.
- [29] S. Varma and F. Taşdelen, On a generalization of Szász-Durrmeyer operators with some orthogonal polynomials, Stud. Univ. Babeş-Bolyai Math. **58** (2), 225-232, 2013.

- [30] Z. Walczak, On approximation by modified Szász-Mirakyan operators, *Glasnik Matematički* **37** (57), 303-319, 2002.