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Research Article

n-Hopfian and n-co-Hopfian Abelian Groups

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Abstract

For any natural number n we define and study the two notions of n-Hopfian and n-co-Hopfian abelian groups. These groups form proper subclasses of the classes of Hopfian and co-Hopfian groups, respectively, and some of their exotic properties are established as well. We also consider and investigate ω-Hopfian and ω-co-Hopfian modules over the formal matrix ring.

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1. Introduction and background

All groups into consideration in this paper, unless specified something else, are assumed to be abelian. The used notions and notations are classical as the unexplained ones follow those from [4], [5] and [10]. For instance, for a group G, the symbol $t(G)$ denotes its torsion part.

Recall that a group G is said to be *Hopfian* if each epimorphism $G \to G$ is an automorphism. Also, it is well known that a group is Hopfian if, and only if, it does not have proper isomorphic quotient groups.

Some obvious examples of such groups are these:

- Finite groups.
- All torsion-free groups of finite rank.
- Every group G with endomorphism ring $E(G) \cong \mathbb{Z}$; in particular, the group of integers $\mathbb{Z}$.

In [4, Problem 75] was asked to explore Hopfian groups. Our strategy here is devoted to the comprehensive investigation of Hopfian groups with torsion automorphism group. In regard to that, a new way to sharp somewhat the concept of *Hopficity* is the following one, in which there is some part of novelty:

Definition 1.1. A group G is called *n-Hopfian* if there exists a natural n such that for each epimorphism φ of G the equality $\varphi^n = 1$ holds. If, however, for every epimorphism φ there is a positive integer $n(\varphi)$ with $\varphi^{n(\varphi)} = 1$, then G is said to be *ω-Hopfian*.

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It is self-evident that n-Hopfian groups are ω-Hopfian for any $n \in \mathbb{N}$, but as it will be shown in the sequel the converse is manifestly untrue. Moreover, a Hopfian group is ω-Hopfian (respectively, n-Hopfian for some $n \in \mathbb{N}$) if, and only if, its automorphism group is torsion (respectively, bounded by n).

Some obvious examples of such groups are these:

- Finite groups are n-Hopfian, where n is the LCM of the orders of its automorphisms.
- Any group G with $E(G) \cong \mathbb{Z}$ is 2-Hopfian. In particular, $\mathbb{Z}$ is 2-Hopfian.

In fact, each endomorphism acts as a multiple of some integer n. If $nG = G$, then $E(G)$ will also be divisible by n. Thus all epimorphisms are multiple of ± 1. Hence G is 2-Hopfian. Notice also that for a torsion-free group G of rank 1 it is true that $E(G) \cong \mathbb{Z}$ if, and only if, $G \neq pG$ for each prime p.

Some other non-trivial constructions of n-Hopfian groups will be given below.

Recall that a group G is said to be *co-Hopfian* if each monomorphism $G \to G$ is an automorphism. Also, it is well known that a group is co-Hopfian if, and only if, it does not have proper isomorphic subgroups to itself.

Some obvious examples of such groups are these:

- Finite groups.
- The quasi-cyclic (Prüfer group) group $\mathbb{Z}(p^\infty)$.
- Every group whose non-zero endomorphisms are epimorphisms.

A new way to sharp somewhat the concept of *co-Hopficity* is the following one:

Definition 1.2. A group G is called *n-co-Hopfian* if there exists a natural n such that for each monomorphism φ of G the equality $\varphi^n = 1$ holds. If, however, for every monomorphism φ there is a positive integer $n(\varphi)$ with $\varphi^{n(\varphi)} = 1$, then G is said to be *ω-co-Hopfian*.

It is self-evident that n-co-Hopfian groups are ω-co-Hopfian for all $n \in \mathbb{N}$, but as it will be shown in the sequel the converse is manifestly untrue. However, a co-Hopfian group is ω-co-Hopfian (respectively, n-co-Hopfian for some $n \in \mathbb{N}$) if, and only if, its automorphism group is torsion (respectively, bounded by n).

Some obvious examples of such groups are these:

- Finite groups are n-co-Hopfian, where n is the LCM of the orders of its automorphisms.
- Every reduced group whose non-zero endomorphisms are epimorphisms, that is, $\mathbb{Z}(p)$.

Some other non-trivial constructions of n-co-Hopfian groups will be given below.

The leitmotif of the present article is to explore in all details the two new notions of n-Hopficity and n-co-Hopficity and to compare the obtained results with these principally known for Hopfian and co-Hopfian groups, respectively, by giving up their discrepancies. It is worthwhile noticing that some common generalizations to both Hopficity and co-Hopficity are presented in [2] and [3], respectively.

2. Examples

Example 2.1. A torsion-free group of rank 1 is ω-Hopfian if, and only if, it is 2-Hopfian. In particular, $\mathbb{Q}$ is *not* ω-Hopfian.

Proof. One way being elementary, let A be such a group and assuming $pA = A$ for any prime p, then $p \cdot 1_A$ is an automorphism of A with $(p \cdot 1_A)^n = p^n \cdot 1_A \neq 1_A$ for all n. Hence $A \neq pA$ and, as we already commented, $E(A) \cong \mathbb{Z}$. But as we have seen in the fifth bullet above, A is 2-Hopfian, as asserted. □

Example 2.2. Let $k \in \mathbb{N}$ and p a prime. Then the group $\mathbb{Z}(p^k)$, where p is an odd prime, is both $(p^k - p^{k-1})$-Hopfian and $(p^k - p^{k-1})$-co-Hopfian. The group $\mathbb{Z}(2^k)$ is both 2^{k-2}-Hopfian and 2^{k-2}-co-Hopfian whenever $k \geq 3$, whereas $\mathbb{Z}(2)$ is both 1-Hopfian and 1-co-Hopfian, and $\mathbb{Z}(4)$ is both 2-Hopfian and 2-co-Hopfian.

Proof. Knowing with the aid of [10] that $\mathrm{Aut}(\mathbb{Z}(p^k)) \cong U(\mathbb{Z}_{p^k})$, then we differ the subsequent cases:

Case 1: *p is odd.* As it is well-known $U(\mathbb{Z}_{p^k})$ is a cyclic group of order $p^k - p^{k-1}$, which implies that $\mathbb{Z}(p^k)$ is simultaneously $(p^k - p^{k-1})$-Hopfian and $(p^k - p^{k-1})$-co-Hopfian.

Case 2: $p = 2$. If $k = 1$, then $U(\mathbb{Z}_2)$ is cyclic of order 1, whence $\mathbb{Z}(2)$ is simultaneously 1-Hopfian and 1-co-Hopfian.

If $k = 2$, then $U(\mathbb{Z}_4)$ is cyclic of order 2, whence $\mathbb{Z}(4)$ is simultaneously 2-Hopfian and 2-co-Hopfian.

If $k \geq 3$, then $U(\mathbb{Z}_{2^k})$ is the direct product of a cyclic group of order 2^{k-2} and a cyclic group of order 2, whence $\mathbb{Z}(2^k)$ is simultaneously 2^{k-2}-Hopfian and 2^{k-2}-co-Hopfian. □

Various other examples could be exhibited taking into account the basic results alluded to below (cf. Corollary 3.17, Proposition 3.20, Corollary 3.21, etc.).

3. Main results

3.1. n-Hopfian groups

Subgroups of n-Hopfian (respectively, ω-Hopfian) groups need *not* be again the same. However, some subgroups inherit this property. Specifically, the following is valid:

Proposition 3.1. *If G is an n-Hopfian (an ω-Hopfian) group, then kG is an n-Hopfian (an ω-Hopfian) group for any $k \in \mathbb{N}$.*

Proof. If $f : kG \to kG$ is an epimorphism of kG, then in view of [4, Proposition 113.3] there exists an epimorphism φ of G whose restriction $\varphi \,|\, kG = f$. Since $\varphi^n = 1_G$, we conclude that $\varphi^n \,|\, kG = f^n \,|\, kG = 1_{kG}$, as required. □

It is worthwhile noticing that the converse implication is not, however, true: Indeed, any infinite k-bounded group is not necessarily ω-Hopfian.

The next result completely settles when a torsion group is ω-Hopfian. What can be offered is the following one:

Theorem 3.2. *Every torsion ω-Hopfian group is finite.*

Proof. Suppose first that G is a p-torsion ω-Hopfian group. Utilizing Proposition 3.6 and Corollary 3.13, G should be reduced. If we assume in a way of contradiction that G is infinite, then it has an unbounded basic subgroup. Therefore, appealing to Example 2.2, for any prime $q \neq p$, the automorphism $q \cdot 1_G$ has an infinite order and thus G is manifestly not ω-Hopfian. That is why, G must be finite. Furthermore, in the general case, since an ω-Hopfian group cannot have an infinite number of non-zero p-primary components, we are done. □

Remark. Another approach for proving up the last statement could be as follows: If G is an ω-Hopfian p-group, then it is Hopfian and hence both reduced and semi-standard. Supposing G is unbounded, we may unambiguously say that its automorphism group has center isomorphic to the group of units of the ring $\mathbb{J}_p$ of p-adic integers, which group is known to be not torsion – a contradiction. So, G has to be simultaneously bounded and semi-standard, whence it is finite, as pursued.

On the other vein, as it is well-known (see, for instance, [6]), there exist unbounded Hopfian separable p-groups of cardinality not exceeding $2^{\aleph_0}$ and finite Ulm-Kaplansky invariants. This group is, certainly, not torsion-complete as the next assertion illustrates.

Theorem 3.3. *Any Hopfian direct sum of torsion-complete p-groups is finite.*

Proof. Suppose first that G is a Hopfian torsion-complete p-group. Assuming contradictiously that the basic subgroup B of such a group G is infinite, then would exist an epimorphism $f : B \to B$ which is not an automorphism (otherwise B will be a Hopfian direct sum of cyclic groups and thus by [7, Theorem 2] it must be finite, a contradiction). But it is well known that $G = \overline{B}$, where $\overline{B}$ is the torsion completion of B in the p-adic topology (see [5]). The map f is then extendible to an epimorphism $\overline{f} : G \to G$ like this $\overline{f}(G) = \overline{f}(\overline{B}) = \overline{f(B)} = \overline{B} = G$, thus contradicting Hopficity of G because $\overline{f}$ is an epimorphism of G which is not automorphism (as the restriction f is not so). Finally, B is finite guaranteeing that G is bounded. Since we have seen above that bounded Hopfian p-groups have to be finite, we conclude that so is G, as expected.

Turning out to the general case, suppose now that G is a Hopfian group which is a direct sum of torsion-complete groups. If we assume that this sum is infinite, we can separate a cyclic direct summand for each torsion-complete direct summand, so that we will obtain an infinite Hopfian direct sum of cyclic p-groups, which is hardly true. Therefore, it must be that the direct sum of torsion-complete groups is finite, and thus it is torsion-complete. We henceforth apply the preceding case to get the pursued claim after all. □

Since any separable p-group can be embedded as a pure and dense subgroup in a torsion-complete p-group (e.g., vol. II of [4]), the last statement also shows that there is no abundance of Hopfian separable p-groups.

In the case of torsion-free groups, the n-torsion exponent can be calculated explicitly like this:

Proposition 3.4. *If G is a torsion-free group with torsion commutative group $Aut(G)$ and n is the order of some its automorphism, then $n \in \{2, 3, 4, 6, 12\}$.*

Proof. It follows from the corresponding properties of Section 116 in [4]. □

Proposition 3.5. *If $A = B \oplus C$, where B and C are fully invariant n-Hopfian and m-Hopfian groups, respectively, then A is an $[m, n]$-Hopfian group, where $[m, n]$ is the $LCM(m, n)$.*

Proof. It follows immediately from the fact that in this situation the ring isomorphism $E(A) \cong E(B) \times E(C)$ holds. □

It is worth noticing that, however, there exists a 2-Hopfian group, say $\mathbb{Z}$, such that $\mathbb{Z} \oplus \mathbb{Z}$ is Hopfian but even **not** ω-Hopfian. Indeed, the multiplicative group (i.e., the group of units) of the ring $E(\mathbb{Z} \oplus \mathbb{Z})$ is not torsion, as required.

Proposition 3.6. *A direct summand of an ω-Hopfian group (respectively, of an n-Hopfian group) is again ω-Hopfian (respectively, n-Hopfian).*

Proof. Let $G = H \oplus K$ be ω-Hopfian. We claim that H is ω-Hopfian too. To that purpose, assuming φ is an epimorphism of H, we then have that $\varphi + 1_K$ is obviously an epimorphism of G. Therefore, $(\varphi + 1_K)^n = \varphi^n + 1_K = 1_G$ which enables us that $\varphi^n = 1_H$, as required.

The same idea is workable for ω-Hopfian groups such that the automorphism φ becomes $n(\varphi)$-torsion, as required. Certainly, it may occur that G is ω-Hopfian but H is n-Hopfian. □

In the torsion-free case we can say even a little more:

Lemma 3.7. *Each direct summand of an ω-Hopfian torsion-free group is fully invariant.*

Proof. Otherwise this group will have nilpotent endomorphisms, but this contradicts property a) of Section 116 from [4]. □

Proposition 3.8. *If $A = B \oplus C$ is a group, where B is fully invariant in A, B, C are ω-Hopfian and the group $Hom(C, B)$ is torsion, then A is ω-Hopfian.*

Proof. Since both B and C are Hopfian, then A is of necessity also Hopfian. Moreover, one observes that the semi-direct product $G = H \leftthreetimes K$ of the torsion groups H and K remains a torsion group. To that goal, if $g \in G$, then $g = xy$, where $x \in H$ and $y \in K$. Furthermore, it follows that $g^n = x^n z$ for some $z \in K$. If now $x^n = 1$, then $g^{mn} = 1$, where $z^m = 1$, which argues the claim. Using this, periodicity of the group $\mathrm{Aut}(A)$ now follows from the formula

$$\mathrm{Aut}(A) = [\mathrm{Hom}(C, B)] \leftthreetimes [\mathrm{Aut}(B) \times \mathrm{Aut}(C)],$$

accomplished with [4, § 113]. □

As an immediate consequence, we yield:

Corollary 3.9. *If $A = B \oplus C$ is a group for which B is an ω-Hopfian torsion-free subgroup and C is a finite subgroup, then A is an ω-Hopfian group.*

Corollary 3.10. *A free group is ω-Hopfian if, and only if, it is 2-Hopfian if, and only if, its rank is equal to 1, that is, the group is isomorphic to $\mathbb{Z}$.*

Proof. It follows directly from Lemma 3.7. □

Proposition 3.11. *A direct sum of cyclic groups A is ω-Hopfian if, and only if, $A = A_0 \oplus (\bigoplus_{i=1}^{k} A_{p_i})$, where either $A_0 = \{0\}$ or $A_0 \cong \mathbb{Z}$ and A_{p_i} are finite p_i-groups for some different primes p_i; $1 \leq i \leq k$. In particular, A is n-Hopfian for some suitable natural n.*

Proof. It follows by a combination of Lemma 3.7 and Theorem 3.2. □

Proposition 3.12. *A torsion-free group G of rank 1 is ω-Hopfian if, and only if, $G \neq pG$ for all primes p.*

Proof. If we assume that $G = pG$ for any prime p, then it is not too hard that $p \cdot 1_G$ is an automorphism of G with $p^n \cdot 1_G \neq 1_G$, as required. □

Corollary 3.13. *A non-zero divisible group is not ω-Hopfian.*

Proof. Knowing that a divisible group is Hopfian if it is torsion-free of finite rank, we need apply Lemma 3.7 in combination with Example 2.1 to get the claim. □

Corollary 3.14. *Torsion-free ω-Hopfian groups are reduced.*

Proof. This follows from Proposition 3.6 accomplished with Corollary 3.13. □

Corollary 3.15. *Non-zero algebraically compact torsion-free groups are not ω-Hopfian.*

Proof. This follows from the fact that the automorphism group of such a group is not torsion. □

The last statement can be extended to the following.

Theorem 3.16. *A direct sum of algebraically compact groups is ω-Hopfian if, and only if, it is finite.*

Proof. Since finite groups are always n-Hopfian for some appropriate positive integer n, and thus ω-Hopfian, the sufficiency follows.

To treat the necessity, suppose we first deal with an algebraically compact ω-Hopfian group. We will use the complete description of algebraically compact groups from Section 40 in [4] as well as from Theorem 3.2 of Chapter 6 in [5]. With the aid of Proposition 3.6, each component in the direct decomposition of an algebraically compact group must be ω-Hopfian. So, with Corollary 3.13 at hand, we now know that any algebraically compact

ω-Hopfian group is reduced. Moreover, any p-adic algebraically compact group is the direct sum of a torsion-free group and an adjusted algebraically compact group (see [4, Theorem 55.5]). However, by Corollary 3.15, the torsion-free part must be zero. Finally, [8, Theorem 1] tells us that a Hopfian p-adic adjusted algebraically compact group has to be finite, and in view of Example 2.2 the number of these p-adic components is also finite, thus giving the desired assertion.

To turn out to the general case, suppose we now have an ω-Hopfian group which is an arbitrary direct sum of algebraically compact groups. In accordance with Corollary 3.15, each direct summand has to be an adjusted algebraically compact group in which we separate a cyclic direct summand. But owing to Proposition 3.11, the direct sum is necessarily finite and hence an algebraically compact group. We hereafter employ the previous case to conclude the wanted claim after all. □

Corollary 3.17. *A separable torsion-free group or a vector torsion-free group are ω-Hopfian if, and only if, their endomorphism ring is commutative, and every rank one direct summand is* 2*-Hopfian. In particular, these groups are* 2*-Hopfian, too.*

Proof. We again need to combine Lemma 3.7 with Example 2.1 in order to infer the claim. □

Every rank 1 torsion-free group of type which is equal to $(k_1, k_2, \cdots)$, where all k_i are finite ($i \in \mathbb{N}$), has automorphism group isomorphic to $\mathbb{Z}(2)$. Hence, for each cardinal satisfying $0 < \alpha \leq 2^{\aleph_0}$, there is a decomposable group having group of automorphisms isomorphic to an elementary 2-group of power α.

Recall that a *sp-group* A is a reduced mixed group with an infinite number of non-zero p-components A_p such that the natural embedding $\bigoplus_p A_p \to A$ can be extended to a pure embedding $A \to \prod_p A_p$. In [1] was established the following criterion for a group to be a sp-group. Specifically, the following is valid:

Theorem 3.18. *The following three conditions are equivalent for a reduced mixed group A with an infinite number of non-zero p-components A_p:*

(1) A is a sp-group, i.e., the pure embeddings $\oplus_p A_p \subset A \subseteq \prod_p A_p$ hold;

(2) The embeddings $\oplus_p A_p \subset A \subseteq \prod_p A_p$ hold and $A/(\oplus_p A_p)$ is a divisible torsion-free group;

(3) For each prime p there is a group B_p such that $A = A_p \oplus B_p$ with $pB_p = B_p$.

We now arrive at the following result.

Theorem 3.19. *Any sp-group is not ω-Hopfian.*

Proof. Every epimorphism ϕ of such a group A can be written as $\phi = (..., \phi_p, ...)$, where ϕ_p is an epimorphism of the p-component A_p. Since the number of these A_p is infinite, for each natural n there exists a prime p with the property that if $\phi_p^{n_p} = 1$ for some $n_p \in \mathbb{N}$, then $n_p > n$. Certainly, $\phi^n \neq 1$ for every $n \in \mathbb{N}$, which substantiates our claim. □

The following considers certain (homological) extensions of ω-Hopficity.

Proposition 3.20. *Let $0 \to H \to G \to K \to 0$ be an exact sequence. If H, K are both ω-Hopfian groups and if H is invariant under each surjection $\varphi : G \to G$, then G is ω-Hopfian. In particular, extensions of ω-Hopfian torsion groups by torsion-free ω-Hopfian groups are again ω-Hopfian.*

Proof. Letting $\varphi : G \to G$ be a surjection, then by assumption, $\varphi(H) \subseteq H$ and so we get an induced map $\overline{\varphi} : G/H \to G/H$ giving the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \varphi\,|\,H} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \overline{\varphi}} & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

Since φ is onto, $\overline{\varphi}$ is also onto and so K, being Hopfian, assures that $\overline{\varphi}$ is an automorphism. If we show that $(\varphi\,|\,H) : H \to H$ is onto, then as H is Hopfian, $\varphi\,|\,H$ will also be an automorphism and the result will follow by an appeal to the well-known "Five Lemma". However, the fact that $\varphi\,|\,H$ is onto follows immediately from the commutativity of the first square of the diagram above. If now $(\varphi\,|\,H)^n = 1$ and $(\overline{\varphi})^m = 1$, then $\varphi^k = 1$, where $k = [n, m]$. □

The next consequence somewhat characterizes mixed ω-Hopfian groups.

Corollary 3.21. *A group G is ω-Hopfian if both $t(G)$ and $G/t(G)$ are ω-Hopfian groups. In addition, if G splits, then the converse is also true.*

Proof. The "if" part follows directly from Proposition 3.20.

As for the other part, since $G \cong t(G) \oplus G/t(G)$, we just employ Proposition 3.6 to conclude the claim. □

3.2. ω-Hopfian modules over the formal matrix ring

We will here consider some results concerning ω-Hopfian modules over the ring of formal matrix extending somewhat the corresponding assertions from [9]. Before doing that, we need some background material from [11].

To that aim, suppose that R, S are associative unital rings and M, N are R-S-bimodules. Suppose also that are given the bimodule homomorphisms $\varphi : M \otimes_S N \to R$ and $\psi : N \otimes_S M \to S$, which satisfy the conditions: $(mn)m' = m(nm')$ and $(nm)n' = n(mn')$ for all $m, m' \in M$ and $n, n' \in N$. So, $mn = \varphi(m \otimes n)$ and $nm = \psi(n \otimes m)$. The set of all matrix of the kind $\begin{pmatrix} r & m \\ n & s \end{pmatrix}$, where $r \in R$, $s \in S$, $m \in M$, $n \in N$, endowed with the usual matrix operations, is called the *ring of formal matrix* (or *the formal matrix ring*). The so-defined ring will be denoted by $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$.

If I and J are the images of the homomorphisms φ and ψ respectively, one may write $I = MN$, $J = NM$, where MN (respectively NM) means the set of all finite sums of elements of the sort mn (respectively nm). The ideals I and J are said to be *trace ideals* for the ring K. In the case when $I = 0 = J$, we will say that K is a ring with *zero trace ideals*.

Now, let X and Y be left R-module and S-module, respectively. Let also exist the homomorphisms of R-module $f : M \otimes_S Y \to X$ and S-module $g : N \otimes_R X \to Y$, respectively, which satisfy the equalities $m(nx) = (mn)x$, $n(my) = (nm)y$ for all $m \in M$, $n \in N$, $x \in X$, $y \in Y$. Here the element nx is identified by $g(n \otimes x)$, and the element my by $f(m \otimes y)$. The vector-column group $\begin{pmatrix} X & Y \end{pmatrix}$ forms a left K-module under the standard multiplication of matrix columns. The converse is also valid. Every left K-module is a naturally isomorphic to some column module. For simplicity, every module of the type $\begin{pmatrix} X & Y \end{pmatrix}$, along with its elements, will be hereafter written as rows. Right K-module means a vector-row module, in which the module multiplication is defined as a production of rows and matrices. Homomorphisms f and g are also often called *homomorphisms of module multiplication*. Let MY (respectively NX) denotes the set of all finite sums of elements of the sort my (respectively nx). Certainly, $MY = \operatorname{Im} f$ and $NX = \operatorname{Im} g$. The map $\Phi : (X, Y) \to (X_1, Y_1)$ will be a K-homomorphism if and only if there are an R-homomorphism $\alpha : X \to X_1$, an S-homomorphism $\beta : Y \to Y_1$ with the properties

$\alpha(my) = m\beta(y)$, $\beta(nx) = n\alpha(x)$ and $\Phi(x, y) = (\alpha(x), \beta(y))$ for all $m \in M$, $n \in N$, $x \in X$, $y \in Y$. With this at hand, the K-module homomorphisms will be henceforth written as the para (α, β).

As in the case of groups, a K-module V is called *ω-Hopfian* provided each its epimorphism φ is n-torsion for some natural $n \geq 1$ depending on φ.

We thus come to

Proposition 3.22. *Suppose $V = (A, B)$ is a K-module. If both A and B are ω-Hopfian modules, then V is an ω-Hopfian module.*

Proof. Given an epimorphism $\Phi : V \to V$, we can write that $\Phi = (\alpha, \beta)$, where α is an endomorphism of the module A, β is an endomorphism of the module B and $\Phi(a, b) = (\alpha(a), \beta(b))$ for $(a, b) \in V$. It is clear that both α and β are epimorphisms and hence automorphisms. Therefore, Φ is also an automorphism. If now $\alpha^n = 1$ and $\beta^m = 1$, then $\Phi^k = 1$ for some $k = LCM(n, m)$. □

Furthermore, for any R-module X and S-module Y one can define K-modules $(X, T(X))$ and $(T(Y), Y)$, where $T(X) = N \otimes_R X$ and $T(Y) = M \otimes_S Y$.

The following five consequences are helpful.

Corollary 3.23. *The K-module $(X, T(X))$ is ω-Hopfian if, and only if, X is an ω-Hopfian R-module. Similarly for the K-module $(T(Y), Y)$.*

Proof. The claim follows by a combination of the next four crucial facts: Firstly, all homomorphisms of the K-modules act coordinate-wise. Secondly, any endomorphism of the module $(X, T(X))$ equals to $(\alpha, 1_N \otimes \alpha)$ for the unique endomorphism α of the module X (see [11, Lemma 2.2]). Thirdly, one sees that if α is an epimorphism, then $1_N \otimes \alpha$ is also an epimorphism. Fourthly, if $\alpha^n = 1_X$, then $(1_N \otimes \alpha)^n = 1_N \otimes \alpha^n = 1_{T(X)}$. □

Corollary 3.24. *If the ring K has trace ideals I, J satisfying the equalities $I = R$, $J = S$, then the ω-Hopficity of the K-module (A, B) is equivalent to the ω-Hopficity of the R-module A and is equivalent to the ω-Hopficity of the S-module B.*

Proof. Utilizing [11, Corollary 8.2] there are isomorphisms of K-modules $(A, B) \cong (A, T(A)) \cong (T(B), B)$. We next just employ Corollary 3.23. □

Corollary 3.25. *Suppose K is a ring with zero trace ideals, i.e., $I = 0 = J$. Then the following two items hold:*

(1) If all indecomposable projective R-modules and S-modules are ω-Hopfian, then any indecomposable projective K-module is also ω-Hopfian.

(2) The assertion in (1) remains true replacing "projective" by "flat".

Proof. Assume that (A, B) is a projective K-module. Appealing to [11, Theorem 7.3] there exist projective R-module X and projective S-module Y for which the isomorphism $(A, B) \cong (X, T(X)) \oplus (T(Y), Y)$ is fulfilled. If the module (A, B) is indecomposable, then it is isomorphic to either module $(X, T(X))$ or to module $(T(Y), Y)$, as moreover both X and Y are indecomposable modules. We next apply Corollary 3.23.

Point (2) can be proved analogously. □

It is worthwhile noticing that in [11] are also introduced K-modules of the type $(X, H(X))$ and $(H(Y), Y)$, where $H(X) = \mathrm{Hom}_R(M, X)$, $H(Y) = \mathrm{Hom}_S(N, Y)$.

Corollary 3.26. *Suppose that M is a projective R-module. Then the K-module $(X, H(X))$ is ω-Hopfian if, and only if, X is an ω-Hopfian R-module. A similar assertion is valid for the K-module $(H(Y), Y)$, provided projectivity of the S-module N.*

Proof. It imitates the same idea as that in Corollary 3.23. □

Corollary 3.27. *Let M and N be a projective R-module and S-module, respectively. If all indecomposable injective R-modules and S-modules are ω-Hopfian, then any indecomposable injective K-module is also ω-Hopfian.*

Proof. For an arbitrary injective K-module (A, B) there are an injective R-module X and an injective S-module Y having the property $(A, B) \cong (X, H(X)) \oplus (H(Y), Y)$ (cf. [11, Corollary 5.7]). If now (A, B) is an indecomposable module, then it is isomorphic to either one of modules $(X, H(X))$ or $(H(Y), Y)$. Likewise, modules X and Y are indecomposable as well. Furthermore, Corollary 3.26 applies to get the claim. □

3.3. n-co-Hopfian groups

Proposition 3.28. *A direct summand of an ω-co-Hopfian group (respectively, of an n-co-Hopfian group) is again ω-co-Hopfian (respectively, n-co-Hopfian).*

Proof. It is identical to that in Proposition 3.6 stated above. □

Proposition 3.29. *A non-zero torsion-free group is not ω-co-Hopfian.*

Proof. Since any torsion-free co-Hopfian group must be divisible of finite rank, for any integer $k \neq 0$ the map $k \cdot 1$ is its monomorphism and $(k \cdot 1)^n = k^n \cdot 1 \neq 1$ for all naturals n, so that the group is not ω-co-Hopfian. □

Proposition 3.30. *A non-zero divisible group is not ω-co-Hopfian. In addition, ω-co-Hopfian groups are reduced.*

Proof. Using the structure theorem for divisible groups (e.g., cf. [5]) accomplished with Propositions 3.28 and 3.29, we need just consider the (p-)torsion case. However, the group $\mathbb{Z}(p^\infty)$ is co-Hopfian but has an automorphism group which is not torsion being isomorphic to the unit group of the ring of p-adic integers. Thus $\mathbb{Z}(p^\infty)$ is not ω-co-Hopfian, which substantiates our initial claim.

The second part is now immediate by taking into account Proposition 3.28. □

We remark that it follows from this statement that $\mathbb{Z}(p^\infty)$ is an example of a co-Hopfian group which is not ω-co-Hopfian.

Proposition 3.31. *If $A = B \oplus C$, where B and C are fully invariant n-co-Hopfian and m-co-Hopfian groups, respectively, then A is an $[m, n]$-co-Hopfian group, where $[m, n]$ is the $LCM(m, n)$.*

Proof. Since we have that $E(A) \cong E(B) \times E(C)$, the result follows without any difficulty. □

Proposition 3.32. *A direct sum of cyclic groups is ω-co-Hopfian if, and only if, it is finite. In particular, such a group is n-co-Hopfian for some $n \in \mathbb{N}$.*

Proof. In virtue of Propositions 3.28 and 3.29, we may restrict our attention on p-groups. But the co-Hopfian direct sum of cyclic p-groups is finite. The finiteness of the number of p-components now follows from Example 2.2. □

Theorem 3.33. *Any sp-group is not ω-co-Hopfian.*

Proof. Every monomorphism ϕ of such a group A can be written as $\phi = (..., \phi_p, ...)$, where ϕ_p is a monomorphism of the p-component A_p. Since the number of these A_p is infinite, for each natural n there exists a prime p with the property that if $\phi_p^{n_p} = 1$ for some $n_p \in \mathbb{N}$, then $n_p > n$. Certainly, $\phi^n \neq 1$ for every $n \in \mathbb{N}$, which substantiates our claim. □

Proposition 3.34. *If G is an n-co-Hopfian (an ω-co-Hopfian) group, then kG is an n-co-Hopfian (an ω-co-Hopfian) group for any $k \in \mathbb{N}$.*

Proof. If $f : kG \to kG$ is a monomorphism of kG, then in view of [4, Proposition 113.3] there exists a monomorphism φ of G whose restriction $\varphi \,|\, kG = f$. Since $\varphi^n = 1_G$, we conclude that $\varphi^n \,|\, kG = f^n \,|\, kG = 1_{kG}$, as required. □

It is worthwhile noticing that the converse implication is not, however, true: Indeed, any infinite k-bounded group is not necessarily ω-co-Hopfian.

Theorem 3.35. *Any torsion ω-co-Hopfian group is finite.*

Proof. Our argumentation is similar to that from Theorem 3.2. □

Proposition 3.36. *If $A = B \oplus C$ is a group, where B is fully invariant in A, B, C are ω-co-Hopfian and the group $Hom(C, B)$ is torsion, then A is ω-co-Hopfian.*

Proof. Since B and C are both co-Hopfian groups, then A is co-Hopfian as well. In fact, every endomorphism of A can be presented as $f = \begin{pmatrix} \varphi & \psi \\ 0 & \eta \end{pmatrix}$, where $\varphi \in \mathrm{E}(B)$, $\psi \in \mathrm{Hom}(C, B)$, $\eta \in \mathrm{E}(C)$. If f is a monomorphism, then φ is a monomorphism, and hence an automorphism. If $\eta(c) = 0$ for some $0 \neq c \in C$, then $\psi(c) \neq 0$ and $\varphi(b) = \psi(c)$ for some $b \in B$. Consequently, $f(b-c) = \varphi(b)-\psi(c) = 0$ for $b-c \neq 0$, a contradiction. Thus η is also a monomorphism, whence, an automorphism. Therefore, any monomorphism of A is its automorphism, which gives our claim about co-Hopficity of A. Now, the periodicity of $\mathrm{Aut}(A)$ follows directly from the formula $\mathrm{Aut}(A) = \mathrm{Hom}(C, B) \rtimes [\mathrm{Aut}(B) \times \mathrm{Aut}(C)]$, as required. □

We emphasize that Hopfian algebraically compact groups are described in ([7], [8]). However, to the authors' knowledge, the complete description of co-Hopfian algebraically compact groups is not known to principally exist in the literature, so we offer a weaker version of it at the next statement.

Proposition 3.37. *An algebraically compact ω-co-Hopfian group is finite, and vice versa.*

Proof. According to Propositions 3.28 and 3.30, such a group is reduced. We hereafter may adapt the idea for proof from Theorem 3.16.

The converse part is trivial. □

Proposition 3.38. *Let $0 \to H \to G \to K \to 0$ be an exact sequence. If H, K are both ω-co-Hopfian groups and if H is invariant under each injection $\psi : G \to G$, then G is ω-co-Hopfian.*

Proof. The proof is essentially dual to that of Proposition 3.20, so we omit it and leave to the interested reader. □

3.4. ω-co-Hopfian modules over the formal matrix ring

There are some analogies for ω-co-Hopfian modules to statements stated above. In fact, for ω-co-Hopfian modules one can deduce analogical assertions to Proposition 3.22 and Corollaries 3.23, 3.24 and 3.25, respectively. Just we need additionally to assume that the modules N_R and M_S are flat.

Besides, there are analogies to Corollaries 3.26 and 3.27, where instead of above, no conditions on M and N are needed.

4. Left-open problems

As a concluding discussion, it is worthwhile noticing that we may locate our work within the context of groups with torsion automorphism group and relate it to the work of A.L.S. Corner on groups with finite automorphism group. This is possible because, in other terms, a group is n-Hopfian (respect., n-co-Hopfian) provided the multiplicative semigroup of epimorphisms (respect., monomorphisms) is n-bounded.

It is well know that (cf. [4], v. II, Chapter 116, Exercise 3) any elementary 2-group G_β of power 2^β, where β is a cardinal strictly less than the first strongly intangible cardinal number, can be realized as the group of the automorphisms of some torsion-free group. That is why, one can pose the following (compare with Corollary 3.17 alluded to above):

Problem 1. For which cardinals β there exists a Hopfian group A_β with the property $\mathrm{Aut}(A_\beta) \cong G_\beta$?

It is clear that such groups A_β have to be 2-Hopfian.

Problem 2. Does there exist a Hopfian (respectively, an ω-Hopfian, an n-Hopfian) group A whose p-components and the factor-group $A/t(A)$ are not Hopfian?

We close the considerations with our final query:

Problem 3. If G is an ω-Hopfian group (or an n-Hopfian group for some $n \in \mathbb{N}$), is it true that its automorphism group (respectively its endomorphism group) is also ω-Hopfian (or m-Hopfian for some $m \in \mathbb{N}$)?

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