Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (2) (2019), 501 – 509
DOI : 10.15672/HJMS.2017.537

Research Article

Semiparallel Submanifolds of a Normal Paracontact Metric Manifold

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Abstract

The object of the present paper is to study invariant semiparallel and 2-semiparallel submanifolds of a normal paracontact metric manifold. We see that parallel submanifolds of a normal paracontact metric manifold are totally geodesic.

Mathematics Subject Classification (2010). 53C15, 53C40, 53C50

Keywords. Riemannian curvature tensor, concircular curvature tensor, invariant submanifold, semiparallel and 2-semiparallel submanifolds

1. Introduction

In the modern geometry, the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. On the other hand, the notion of geodesics plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Therefore, totally geodesic submanifolds are also very much important in physical sciences. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic [4,5]. Later on the invariant submanifolds have been studied by many geometers to different extent [13]. Invariant submanifolds inherit almost all properties of the ambient manifolds.

Arslan K. and et al. [1,11] defined and studied 2-semiparallel surfaces in space forms. Ishıhara I. [7], Yano K. and Kon M. [16] studied anti-invariant submanifolds of a Sasakian space form. In [3–5, 8, 9, 14], authors studied semi-invariant and totally umbilical submanifolds in Sasakian and cosymplectic manifolds. In [2], we discussed the properties of semi-invariant submanifolds of a normal paracontact metric manifold.

Motivated by the above studies, the present paper deals with the study of invariant submanifolds of a normal paracontact metric manifold.

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Received: 20.12.2016; Accepted: 27.11.2017

Let M be a $(2n+1)$-dimensional manifold and ϕ, ξ and η be a tensor field of type $(1,1)$, a vector field and a 1-form on M, respectively. If ϕ, ξ and η satisfy the conditions

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{1.1}$$

for any vector field X on M, then M is said to be an almost contact manifold. In addition, it is called almost contact metric manifold if M has a Riemannian metric tensor such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi), \tag{1.2}$$

for any $X, Y \in \chi(M)$, where $\chi(M)$ denotes set of the differentiable vector fields on M [15].

Furthermore, M is called a normal paracontact metric manifold if we have

$$(\overline{\nabla}_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{1.3}$$

and

$$\overline{\nabla}_X \xi = -\phi X, \tag{1.4}$$

for any $X, Y \in \chi(M)$, where $\overline{\nabla}$ denotes the Levi-Civita connection determined by g.

The concircular curvature tensor, conformal curvature tensor and quasi-conformal curvature tensor of a normal paracontact metric manifold M^{2n+1} are, respectively, defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{1.5}$$

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & - g(X,Z)QY\} + \frac{\tau}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{1.6}$$

$$\begin{aligned} \widetilde{C}(X,Y)Z =& \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ & - \frac{\tau}{2n+1}\{\frac{\lambda}{2n} + 2\mu\}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{1.7}$$

for any $X, Y, Z \in \chi(M)$, where R denotes the Riemannian curvature tensor of M and Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$.

Also, on a normal paracontact metric manifold M^{2n+1}, the following relations are satisfied

$$R(\xi,Y)Z = g(Y,Z)\xi - \eta(Z)Y, \tag{1.8}$$

and

$$R(\xi,Y)\xi = \eta(Y)\xi - Y \tag{1.9}$$

for any $X, Y, Z \in \chi(M)$.

Now let $\overline{M}$ be a submanifold of a normal paracontact metric manifold M with induced metric tensor g. We also denote the induced connections on the tangent bundle $\chi(\overline{M})$ and the normal bundle $\chi^{\perp}(\overline{M})$ by ∇ and $\nabla^{\perp}$, respectively. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.10}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{1.11}$$

for any $X, Y \in \chi(\overline{M})$ and $V \in \chi^{\perp}(\overline{M})$, where h and A_V are second fundamental form and shape operator, respectively, for the immersion of $\overline{M}$ into M [12]. $\overline{M}$ is called totally geodesic submanifold if $h = 0$. h and A_V are related by

$$g(A_V X, Y) = g(h(X, Y), V), \tag{1.12}$$

The covariant derivation of h is defined by

$$(\nabla_X h)(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{1.13}$$

for any $X, Y, Z \in \chi(\overline{M})$. h is said to be parallel if $(\nabla_X h)(Y, Z) = 0$.

For a submanifold $\overline{M}$ of a normal paracontact metric manifold M, if for any $X \in \chi(\overline{M})$, then we can write

$$\phi X = fX + \omega X, \tag{1.14}$$

where fX and ωX are the tangent and normal components of ϕX, respectively. $\overline{M}$ is said to be an invariant submanifold if $\omega = 0$ [6]. Throughout this paper, we assume that $\overline{M}$ is an invariant submanifold of a normal paracontact metric manifold M. In this case, we have $\phi(\chi(\overline{M})) \subseteq \chi(\overline{M})$ and $\phi(\chi^{\perp}(\overline{M})) \subseteq \chi^{\perp}(\overline{M})$ [10].

2. Preliminaries

Let (M, g) be a Riemannian manifold and $\overline{M}$ be a submanifold of M. We denote the Levi-Civita connection of g and the second fundamental form of $\overline{M}$ by ∇ and h, respectively. The submanifold $\overline{M}$ said to be semiparallel if

$$R(X, Y) \cdot h = 0, \tag{2.1}$$

for any $X, Y \in \chi(\overline{M})$, where R denotes the Riemannian curvature tensor of M and $R(X, Y) \cdot h = 0$ is defined by

$$(R(X, Y) \cdot h)(Z, U) = R^{\perp}(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U),$$

for any $X, Y, Z, U \in \chi(\overline{M})$.

In [1] Arslan et al. defined and studied 2-semiparallel submanifolds. Such submanifolds are defined as, a Riemannian submanifold $\overline{M}$ is said to be 2-semiparallel if the following relation holds

$$R(X, Y) \cdot \nabla h = 0, \tag{2.2}$$

for any $X, Y \in \chi(\overline{M})$, where

$$\begin{aligned}(R(X, Y) \cdot \nabla h)(Z, U, W) = &R^{\perp}(X, Y)(\nabla_Z h)(U, W) - (\nabla_{R(X,Y)Z} h)(U, W) \\ &- (\nabla_Z h)(R(X, Y)U, W) - (\nabla_Z h)(U, R(X, Y)W),\end{aligned} \tag{2.3}$$

for any $X, Y, Z, U, W \in \chi(\overline{M})$.

Now, let us assume that normal paracontact metric manifold M^{2n+1} is conformal flat. Then from (1.6) we have

$$\begin{aligned}R(X, Y)\xi = &\frac{1}{2n-1}\{S(Y, \xi)X - S(X, \xi)Y + \eta(Y)QX - \eta(X)QY\} \\ &- \frac{\tau}{2n(2n-1)}\{\eta(Y)X - \eta(X)Y\},\end{aligned} \tag{2.4}$$

which implies that

$$\eta(Y)X - \eta(X)Y = \frac{2n}{2n-1}\{\eta(Y)X - \eta(X)Y\} + \frac{1}{2n-1}\{\eta(Y)QX - \eta(X)QY\} - \frac{\tau}{2n(2n-1)}\{\eta(Y)X - \eta(X)Y\}. \tag{2.5}$$

This is equivalent to

$$\eta(Y)QX - \eta(X)QY = \{\eta(Y)X - \eta(X)Y\}\{\frac{\tau}{2n} - 1\}. \tag{2.6}$$

For $Y = \xi$, we obtain

$$QX = \left(\frac{\tau}{2n} - 1\right)X + \left(2n + 1 - \frac{\tau}{2n}\right)\eta(X)\xi, \tag{2.7}$$

that is, conformally flat normal paracontact metric manifold is an Einstein manifold and the Ricci tensor is given by

$$S(X,Y) = \left(\frac{\tau}{2n} - 1\right)g(X,Y) + \left(2n + 1 - \frac{\tau}{2n}\right)\eta(X)\eta(Y). \tag{2.8}$$

The scalar curvature τ of M^{2n+1} is obtained by

$$\tau = \left(\frac{\tau}{2n} - 1\right)(2n+1) + \left(2n + 1 - \frac{\tau}{2n}\right). \tag{2.9}$$

Thus we have the following theorem for later use.

Theorem 2.1. *Conformally flat a normal paracontact metric manifold is always an η-Einstein manifold.*

Now, let us suppose that normal paracontact metric manifold be Quasi-Conformally flat. Then from (1.7), we have

$$R(X,Y)Z = -\frac{\mu}{\lambda}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{\tau}{n\lambda}(\frac{\lambda}{n-1} + 2\mu)\{g(Y,Z)X - g(X,Z)Y\},$$

for any $X, Y, Z \in \chi(M)$. By the direct calculations, we obtain

$$\tau = \frac{2n(n-1)\{\lambda + \mu(2n-1)\}}{2\lambda + 3\mu(n-1)} \tag{2.10}$$

provided that $2\lambda + 3\mu(n-1) \neq 0$.

3. Invariant submanifolds of a normal paracontact metric manifold

In this section, we study of invariant submanifolds of a normal paracontact metric manifold satisfying the $\widetilde{Z}(X,Y) \cdot h = 0$ and $\widetilde{Z}(X,Y) \cdot \nabla h = 0$. Finally we see that these conditions are satisfied if and only if invariant submanifold is totally geodesic.

Proposition 3.1. *Let $\overline{M}$ be an invariant submanifold of a normal paracontact metric manifold M. Then the following relations holds:*
1) $\nabla_X\xi = -fX, \quad h(X,\xi) = 0$
2) $\phi h(X,Y) = h(X, fY)$
3) $(\nabla_X f)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(Y)\eta(X)\xi,$
for any $X, Y \in \chi(\overline{M})$.

Proof. By using (1.4) and taking into account of $\overline{M}$ being invariant submanifold, 1) statement is obvious. On the other hand, making use of (1.3) and (1.10), we have

$$\begin{aligned}(\overline{\nabla}_X\phi)Y &= \overline{\nabla}_X\phi Y - \phi\overline{\nabla}_X Y\\ &= h(X,fY) + \overline{\nabla}_X fY - \phi h(X,Y) - f\nabla_X Y\\ &= -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,\end{aligned}$$

for any $X, Y \in \chi(\overline{M})$, which proves 2) and 3) statements. □

Thus we have the following conclusion.

Corollary 3.2. *Every invariant submanifold of a normal paracontact metric manifold has a normal paracontact metric structure.*

Theorem 3.3. *Let $\overline{M}$ be an invariant submanifold of a normal paracontact metric manifold M. Then the second fundamental form of $\overline{M}$ is parallel if and only if $\overline{M}$ is a totally geodesic submanifold.*

Proof. If the second fundamental form h of $\overline{M}$ is parallel, then we have

$$\nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0, \tag{3.1}$$

for any $X, Y, Z \in \chi(\overline{M})$. Setting $Z = \xi$ in (3.1) and taking into account that Proposition 3.1, we get $h(Y, \nabla_X\xi) = -h(X, fY) = 0$, which implies that $\overline{M}$ is a totally geodesic submanifold. The converse statement is obvious. □

Theorem 3.4. *Let $\overline{M}$ be an invariant submanifold of a normal paracontact metric manifold M. Then $\overline{M}$ is semiparallel if and only if $\overline{M}$ is a totally geodesic submanifold.*

Proof. If $\overline{M}$ is semiparallel, then $R \cdot h = 0$. This implies that

$$\begin{aligned}(R(X,Y)\cdot h)(Z,U) =& R^{\perp}(X,Y)h(Z,U) - h(R(X,Y)Z,U)\\ &- h(Z, R(X,Y)U),\end{aligned} \tag{3.2}$$

for any $X, Y, Z, U \in \chi(\overline{M})$. Putting $X = U = \xi$ in (3.2), we obtain

$$R^{\perp}(\xi,Y)h(Z,\xi) - h(R(\xi,Y)Z,\xi) - h(Z, R(\xi,Y)\xi) = 0. \tag{3.3}$$

Here taking into account of (1.8) and (1.9), we reach at

$$\eta(Z)h(Y,\xi) - g(Y,Z)h(\xi,\xi) + h(Y,Z) - \eta(Y)h(Z,\xi) = 0. \tag{3.4}$$

Here, from (3.4) we conclude $h(Y,Z) = 0$, that is, the submanifold is a totally geodesic. Conversely, if $h = 0$, then M is semiparallel. □

Theorem 3.5. *Let $\overline{M}$ be an invariant submanifold of a normal paracontact metric manifold M. Then $\overline{M}$ is 2-semiparallel if and only if $\overline{M}$ is a totally geodesic submanifold.*

Proof. Let us suppose $\overline{M}$ be 2-semiparallel. This implies that

$$\begin{aligned}(R(X,Y)\cdot\nabla h)(Z,U,W) =& R^{\perp}(X,Y)(\nabla_Z h)(U,W) - (\nabla_{R(X,Y)Z}h)(U,W)\\ &- (\nabla_Z h)(R(X,Y)U,W) - (\nabla_Z h)(U, R(X,Y)W),\end{aligned} \tag{3.5}$$

for all $X, Y, Z, U, W \in \chi(\overline{M})$. Here taking $X = U = \xi$ and we calculate each expression as follows

$$\begin{aligned}R^{\perp}(\xi,Y)(\nabla_Z h)(\xi,W) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_Z h(\xi,W) - h(\nabla_Z\xi,W) - h(\xi,\nabla_Z W)\}\\ &= R^{\perp}(\xi,Y)h(fZ,W),\end{aligned} \tag{3.6}$$

$$\begin{aligned}(\nabla_{R(\xi,Y)Z}h)(\xi,W) &= \nabla^{\perp}_{R(\xi,Y)Z}h(\xi,W) - h(\nabla_{R(\xi,Y)Z}\xi,W) - h(\nabla_{R(\xi,Y)Z}W,\xi)\\ &= -h(\nabla_{-\eta(Z)Y+g(Y,Z)\xi}\xi,W)\\ &= \eta(Z)h(\nabla_Y\xi,W) - g(Y,Z)h(\nabla_\xi\xi,W)\\ &= -\eta(Z)h(fY,W), \end{aligned} \tag{3.7}$$

$$\begin{aligned}(\nabla_Z h)(R(\xi,Y)\xi,W) =& \nabla^{\perp}_Z h(R(\xi,Y)\xi,W) - h(\nabla_Z R(\xi,Y)\xi,W)\\ &- h(R(\xi,Y)\xi,\nabla_Z W)\\ =& -\nabla^{\perp}_Z h(Y,W) + \nabla^{\perp}_Z h(\eta(Y)\xi,W) + h(\nabla_Z Y,W)\\ &- h(\eta(Y)\xi,W) + h(Y,\nabla_Z W) - h(\nabla_Z W,\eta(Y)\xi)\\ =& -(\nabla_Z h)(Y,W), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}(\nabla_Z h)(\xi,R(\xi,Y)W) =& \nabla^{\perp}_Z h(\xi,R(\xi,Y)W) - h(\nabla_Z\xi,R(\xi,Y)W)\\ &- h(\nabla_Z R(\xi,Y)W,\xi)\\ =& -h(\nabla_Z\xi,-\eta(W)Y + g(Y,W)\xi)\\ =& -\eta(W)h(fZ,Y). \end{aligned} \tag{3.9}$$

Thus, by combining (3.6),(3.7),(3.8) and (3.9), we derive

$$\begin{aligned}(R(\xi,Y)\cdot\nabla h)(Z,\xi,W) =& R^{\perp}(\xi,Y)h(fZ,W) + \eta(Z)h(fY,W)\\ &+ (\nabla_Z h)(Y,W) + \eta(W)h(fZ,Y). \end{aligned} \tag{3.10}$$

Since $\overline{M}$ is 2-semiparallel and for $W=\xi$, we obtain $h(fY,W)=0$. This proves our assertion. The converse is obvious. □

Theorem 3.6. *Let $\overline{M}$ be an invariant submanifold of a paracontact metric manifold M with $\tau \neq 2n(2n+1)$. Then $\widetilde{Z}(X,Y)\cdot h = 0$ if and only if $\overline{M}$ is totally geodesic submanifold.*

Proof. $\widetilde{Z}(X,Y)\cdot h = 0$ implies that

$$\begin{aligned}(\widetilde{Z}(X,Y)\cdot h)(Z,U) =& R^{\perp}(X,Y)h(Z,U) - h(\widetilde{Z}(X,Y)Z,U)\\ &- h(Z,\widetilde{Z}(X,Y)U), \end{aligned} \tag{3.11}$$

for any $X,Y,Z,U \in \chi(\overline{M})$. By using (1), we have

$$\widetilde{Z}(\xi,Y)Z = \left(1 - \frac{\tau}{2n(2n+1)}\right)(g(Y,Z)\xi - \eta(Z)Y). \tag{3.12}$$

Thus

$$\begin{aligned}0 &= R^{\perp}(\xi,Y)h(Z,\xi) - h(\widetilde{Z}(\xi,Y)Z,\xi) - h(Z,\widetilde{Z}(\xi,Y)\xi)\\ &= \left(1 - \frac{\tau}{2n(2n+1)}\right)(h(-\eta(Z)Y + g(Y,Z)\xi,\xi) - h(Z,-Y+\eta(Y)\xi))\\ &= \left(1 - \frac{\tau}{2n(2n+1)}\right)h(Y,Z). \end{aligned} \tag{3.13}$$

This proves our assertion. □

Theorem 3.7. *Let $\overline{M}$ be an invariant submanifold of a paracontact metric manifold M with $\tau \neq 2n(2n+1)$. Then $\widetilde{Z}(X,Y)\cdot\nabla h = 0$ if and only if $\overline{M}$ is totally geodesic submanifold.*

Proof. $\widetilde{Z}(X,Y)\cdot\nabla h=0$ means that

$$R^{\perp}(X,Y)(\nabla_Z h)(U,W)-(\nabla_{\widetilde{Z}(X,Y)Z}h)(U,W)-(\nabla_Z h)(\widetilde{Z}(X,Y)U,W)\\ -(\nabla_Z h)(U,\widetilde{Z}(X,Y)W)=0, \tag{3.14}$$

for any $X,Y,Z,U,W\in\chi(\overline{M})$. Here,

$$R^{\perp}(\xi,Y)(\nabla_Z h)(\xi,W)=R^{\perp}(\xi,Y)\{\nabla_Z^{\perp}h(\xi,W)-h(\nabla_Z\xi,W)-h(\nabla_Z W,\xi)\}\\ =R^{\perp}(\xi,Y)h(fZ,W), \tag{3.15}$$

$$(\nabla_{\widetilde{Z}(\xi,Y)Z}h)(\xi,W)=\nabla^{\perp}_{\widetilde{Z}(\xi,Y)Z}h(\xi,W)-h(\nabla_{\widetilde{Z}(\xi,Y)Z}\xi,W)-h(\nabla_{\widetilde{Z}(\xi,Y)Z}W,\xi)\\ =-\left(1-\frac{\tau}{2n(2n+1)}\right)h(\nabla_{-\eta(Z)Y+g(Y,Z)\xi}\xi,W)\\ =-\left(1-\frac{\tau}{2n(2n+1)}\right)\eta(Z)h(fY,W), \tag{3.16}$$

$$(\nabla_Z h)(\widetilde{Z}(\xi,Y)\xi,W)=\nabla_Z^{\perp}h(\widetilde{Z}(\xi,Y)\xi,W)-h(\nabla_Z\widetilde{Z}(\xi,Y)\xi,W)\\ -h(\nabla_Z W,\widetilde{Z}(\xi,Y)\xi)\\ =\left(1-\frac{\tau}{2n(2n+1)}\right)\{\nabla_Z^{\perp}h(-Y+\eta(Y)\xi,W)\\ -h(\nabla_Z-Y+\eta(Y)\xi,W)-h(\nabla_Z W,-Y+\eta(Y)\xi)\}\\ =\left(1-\frac{\tau}{2n(2n+1)}\right)\{-\nabla_Z^{\perp}h(Y,W)+\nabla_Z^{\perp}(\eta(Y)h(\xi,W))\\ +h(\nabla_Z Y,W)-\eta(Y)h(\xi,W)+h(\nabla_Z W,Y)\\ -\eta(Y)h(\nabla_Z W,\xi)\}\\ =-\left(1-\frac{\tau}{2n(2n+1)}\right)(\nabla_Z h)(Y,W) \tag{3.17}$$

and

$$(\nabla_Z h)(\xi,\widetilde{Z}(\xi,Y)W)=\nabla_Z^{\perp}h(\xi,\widetilde{Z}(\xi,Y)W)-h(\nabla_Z\xi,\widetilde{Z}(\xi,Y)W)\\ -h(\nabla_Z\widetilde{Z}(\xi,Y)W,\xi)\\ =\left(1-\frac{\tau}{2n(2n+1)}\right)\{\eta(W)h(\nabla_Z\xi,Y)\\ -g(Y,W)h(\nabla_Z\xi,\xi)\}\\ =-\left(1-\frac{\tau}{2n(2n+1)}\right)\eta(W)h(fZ,Y). \tag{3.18}$$

Thus we obtain

$$R^{\perp}(\xi,Y)h(fZ,W)+\left(1-\frac{\tau}{2n(2n+1)}\right)\{\eta(Z)h(fY,W)\\ +(\nabla_Z h)(Y,W)+\eta(W)h(fZ,Y)\}=0. \tag{3.19}$$

Here choosing $W=\xi$, we conclude

$$\left(1+\frac{\tau}{2n(2n+1)}\right)\{h(fZ,Y)-(\nabla_Z h)(Y,\xi)\}=\left(1+\frac{\tau}{2n(2n+1)}\right)h(fZ,W).$$

The converse is obvious. This proves our assertion. □

Example 3.8. Let $\overline{M}$ be a submanifold of $\mathbb{R}^7$ is given by the equation

$$\begin{aligned}\phi(x_1, y_1, s) =& \big(\cos x_1 \sinh y_1, \sin y_1 \sinh x_1, \cos x_1 \sinh y_1,\\ & \sin x_1 \cosh y_1, \cos y_1 \cosh x_1, \sin x_1 \cosh y_1, s\big).\end{aligned}$$

Then tangent space of $\overline{M}$ is spanned by the vectors

$$\begin{aligned}e_1 =& -\sin x_1 \sinh y_1 \frac{\partial}{\partial x_1} + \sin y_1 \cosh x_1 \frac{\partial}{\partial x_2} - \sin x_1 \sinh y_1 \frac{\partial}{\partial x_3}\\ &+ \cos x_1 \cosh y_1 \frac{\partial}{\partial y_1} + \cos y_1 \sinh x_1 \frac{\partial}{\partial y_2} + \cos x_1 \cosh y_1 \frac{\partial}{\partial y_3},\end{aligned}$$

$$\begin{aligned}e_2 =& \cos x_1 \cosh y_1 \frac{\partial}{\partial x_1} + \cos y_1 \sinh x_1 \frac{\partial}{\partial x_2} + \cos x_1 \cosh y_1 \frac{\partial}{\partial x_3}\\ &+ \sin x_1 \sinh y_1 \frac{\partial}{\partial y_1} - \sin y_1 \cosh x_1 \frac{\partial}{\partial y_2} + \sin x_1 \sinh y_1 \frac{\partial}{\partial y_3},\end{aligned}$$

$$e_3 = \xi = \frac{\partial}{\partial s}.$$

We define the almost paracontact structure of $\mathbb{R}^7$ by

$$\phi(x_1, x_2, x_3, y_1, y_2, y_3, s) = (-y_1, -y_2, -y_3, x_1, x_2, x_3, 0),$$

then we have $\phi^2 X = -X + \eta(X)\xi$ for any $X \in \chi(\mathbb{R}^7)$. By direct calculations,

$$\begin{aligned}\phi e_1 =& \big(-\cos x_1 \cosh y_1, -\cos y_1 \sinh x_1, -\cos x_1 \cosh y_1,\\ & -\sin x_1 \sinh y_1, \sin y_1 \cosh x_1, -\sin x_1 \sinh y_1, 0\big)\\ =& -e_2,\end{aligned}$$

$$\begin{aligned}\phi e_2 =& \big(-\sin x_1 \sinh y_1, \sin y_1 \cosh x_1, -\sin x_1 \sinh y_1,\\ & \cos x_1 \cosh y_1, \cos y_1 \sinh x_1, \cos x_1 \cosh y_1, 0\big)\\ =& e_1.\end{aligned}$$

Thus $\overline{M}$ is 3−dimensional an invariant submanifold of $\mathbb{R}^7$. On the other hand, Lie-bracket the vector fields of e_1 and e_2 is

$$\begin{aligned}[e_1, e_2] =& \sinh(2y_1) \frac{\partial}{\partial x_1} + \sin(2x_1) \frac{\partial}{\partial y_1}\\ &- \big(2 \sin x_1 \cos y_1 \sinh y_1 \cosh x_1 + 2 \cos x_1 \sin y_1 \sinh x_1 \cosh y_1\big) \frac{\partial}{\partial x_2}\\ &+ \big(2 \sin x_1 \sin y_1 \sinh x_1 \sinh y_1 - 2 \cos x_1 \cos y_1 \cosh x_1 \cosh y_1\big) \frac{\partial}{\partial y_2}\\ &+ \sinh(2y_1) \frac{\partial}{\partial x_3} + + \sin(2x_1) \frac{\partial}{\partial y_3}.\end{aligned}$$

By using Kozsul-formulae, we obtain

$$\begin{aligned}\nabla_{e_1} e_2 =& \big[-\cos x_1 \sinh(2x_1) \cosh y_1 - \sin(2y_1) \sin x_1 \sinh y_1\\ & -\sin x_1 \sinh y_1 \sinh(2y_1) + \cos x_1 \cosh y_1 \sin 2x_1\\ & + \frac{1}{2} \cos x_1 \cosh y_1 \sinh(2x_1) + \frac{1}{2} \sin x_1 \sinh y_1 \sin(2y_1)\big] e_1\\ & + \big[\sin x_1 \sinh y_1 \sin(2x_1) - \frac{1}{2} \sin x_1 \sinh y_1 \sinh(2x_1)\\ & + \cos x_1 \cosh y_1 \sinh(2y_1) + \frac{1}{2} \cos x_1 \cosh y_1 \sin(2y_1)\big] e_2\end{aligned}$$

Since $\overline{M}$ is a totally-geodesic submanifold, $\overline{M}$ is a semiparallel and 2-semiparallel submanifold of $\mathbb{R}^7$. This verifies the statements of Theorem 3.4 and Theorem 3.5.

Acknowledgment. The authors would like to thank the referee for many helpful suggestions and editors for their patience.

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