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# ON $\beta-$SEMIGROUP GENERATED BY FOURIER- BESSEL TRANSFORM AND RIESZ POTENTIAL ASSOCIATED WITH $\beta-$SEMIGROUP

# FOURIER- BESSEL DÖNÜŞÜMÜ TARAFINDAN ÜRETİLEN $\beta-$SEMİGRUBU VE $\beta$- SEMİGUP TARAFINDAN DOĞRULAN RİESZ POTANSİYELİ ÜZERİNE

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## ABSTRACT

$\beta$ - semigroup associated with singular Laplace- Bessel differential operator

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left( \frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n} \right)$$

is introduced. Representations of the Riesz potentials $\left(-\Delta_B\right)^{\frac{-\alpha}{2}}\varphi$, $\left(\mathrm{Re}\,\alpha > 0\right)$ via the $\beta$-semigroup are obtained.

**Mathematics Subject Classification:** 26A33, 44A35

**Key Words and Phrases:** Riesz Potentials, Generalized Shift, Fourier- Bessel Transform, Laplace- Bessel Differential Operator.

## ÖZET

Bu çalışmada öncelikle

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left( \frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n} \right)$$

Laplace- Bessel Diferansiyel operatorü tarafından üretilen $\beta$ - semigrup tanımlanmıştır. Tanımlanmış olan $\beta$ - semigrup vasıtasıyla $\left(-\Delta_B\right)^{\frac{-\alpha}{2}}\varphi$, $\left(\mathrm{Re}\,\alpha > 0\right)$ Riesz potansiyelinin yeni bir gösterimi elde edilmiştir.

**Anahtar Kelimeler:** Genelleşmiş kayma operatörü, Riesz Potansiyeli, Fourier- Bessel dönüşümü, Laplace- Bessel diferansiyel operatörü.

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## 1. INTRODUCTION

In the Fourier- Harmonic Analysis some of the important tools as well as singular integral operators are the Riesz, Bessel, Parabolic Potentials, etc.

The Riesz Potenatials, representing the negative ( fractional) powers of the Laplace- Bessel Differenatial operator $\left(-\Delta_B\right)$, are defined in terms of Fourier- Bessel multiplier by $\varphi \in S(R_+^n)$

$$\left(-\Delta_B\right)\varphi = F_B^{-1}\left(|x|^2 F_B\varphi\right)$$

are applied to the theory of functions, partial differential equations and other areas of mathematics as well as harmonic analysis. ( Samko and Kilbas, 1987; Stein, 1970; Rubin 1996).

The singular integral operators and potentials associated with the Laplace- Bessel differential operator ( Levitan, 1951; Muckenhoupt and Stein, 1965; Kipriyanov, 1967).

$$\Delta_B = \sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k^2} + \left(\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}\right), \nu > 0$$

are known to be important differential operators in analysis and its applications, have been research areas for many mathematicians such as I. Kipriyanov and M. Klyuchantsev (1970), L.Lyakhov (1984), A. D. Gadjiev and I. A. Aliev (1987, 1988, 1994), V. S. Guliyev (1998) and others.

I. A. Aliev and S. Sezer (2010) have studied a new characterization of the Riesz Potenatial generated by Fouier transform. They have introduced the so- called $\beta$ – semigroup

$$\left(W_t^{(\beta)} f\right)_{t>0}(x) = \int_{R^n} \omega^{(\beta)}\left(|y|, t\right) f(x-y)dy$$

generated by the radial kernel

$$\omega^{(\beta)}\left(|y|, t\right) = \left(2\Pi\right)^{-n}\int_{R^n} e^{-t|x|^{\beta}} e^{iy.x}dx$$

$y.x = y_1x_1 + ... + y_nx_n$ and using this $\beta$ - semigroup $\left(W_t^{(\beta)}\right)_{t>0}$ they have obtained a new integral representation of the Riesz potentials. Note that; for $\beta=1$ and $\beta=2$ . $\left(W_t^{(\beta)}\right)_{t>0}$ coincides with the well known Poisson and Gauss- Weierstrass integrals, respectively.

The purpose of this article is to define $\beta$ - semigroup $\left(B_t^{(\beta)}\right)_{t>0}$ generated by Fourier- Bessel transform and obtain a new representation of the Riesz potentials with aid of this $\beta$ - semigroup.

## 2. AUXILIARY DEFINITIONS, NOTATIONS AND RESULTS

1) Let $R_+^n=\{x=(x_1,...,x_n)\in R^n : x_n\geq 0\}$ and denote by $S(R_+^n)$ the space of functions

which are the restrictions to $R_+^n$ of the test functions of Schwartz that are even in the last variable $x_n$. The space $C(R_+^n)$ of continious functions is defined similiary. The closure of the space $S(R_+^n)$ in the seminorm

$$\|f\|_{p,\nu}=\left(\int_{R_+^n}|f(x)|^p x_n^{2\nu}dx\right)^{\frac{1}{p}} \tag{1}$$

is denoted by $L_{p,\nu}\left(R_+^n\right)\equiv L_{p,\nu}$, where $\nu>0$ is a fixed parameter, $1\leq p<\infty$ and $dx=dx_1...dx_n$.

2) Let $\Delta_B\equiv\Delta_B(x)$ denote the singular differential operator of Laplace- Bessel as

fallows:

$$\Delta_B=\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k^2}+\left(\frac{\partial^2}{\partial x_n^2}+\frac{2\nu}{x_n}\frac{\partial}{\partial x_n}\right),\nu>0$$

3) Denote by $T^y$ the generalized shift operator (GSO), acting according to the law

$$T^y\varphi(x)=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}\int_0^{\Pi}\varphi\left(x'-y';\sqrt{x_n^2-2x_ny_n\cos\alpha+y_n^2}\right)(\sin\alpha)^{2\nu-1}d\alpha$$

where $x = (x', x_n)$, $y = (y', y_n)$ and $x', y' \in R^{n-1}$. ( Levitan, 1951; Kipriyanov and Klyuchantsev, 1970; Klyuchantsev, 1970).

We remark that $T^y$ is closely connected with Bessel differential operator ( Levitan, 1951)

$$B_t = \frac{d^2}{dt^2} + \frac{2\nu}{t}\frac{d}{dt}$$

The convolution ( B- convolution) generated by GSO is defined on the space $S(R_+^n)$ by

$$(\varphi \otimes \psi)(x) = \int_{R_+^n} \varphi(\varsigma) T^{\varsigma} \psi(x) \varsigma_n^{2\nu} d\varsigma \qquad (2)$$

where $d\varsigma = d\varsigma_1 ... d\varsigma_n$.

It is easy to prove the following Young's inequality

$$\|\varphi \otimes \psi\|_{r,\nu} \le \|\varphi\|_{p,\nu} \|\psi\|_{q,\nu}$$

where $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ by using lemma of Muchenhoupt and E. Stein (1965) about convolution or the Marcinkiewicz interpolation theorem.

4) The Fourier- Bessel transformation ( Kipriyanov, 1967) is defined on the space of test functions $S(R_+^n)$ by

$$(F_B\varphi)(z) = \int_{R_+^n} \varphi(x) e^{-ix'.z'} j_{\nu - \frac{1}{2}}(x_n z_n) x_n^{2\nu} dx \qquad (3)$$

and its inverse is defined by

$$(F_B^{-1}\varphi)(z) = c_\nu(n)(F_B\varphi)(-z) \qquad (4)$$

where $x'.z' = x_1 z_1 + ... + x_{n-1} z_{n-1}$, $c_\nu(n) = \left[ (2\Pi)^{n-1} 2^{2\nu-1} \Gamma^2\left(\nu + \frac{1}{2}\right) \right]^{-1}$.

The function $j_p(t)$ $\left(t>0, p>\frac{-1}{2}\right)$ in (3) is connected with the Bessel function (of the first kind) $J_p(t)$ as follows ( Levitan, 1951)

$$j_p(t)=2^p\Gamma(p+1)\frac{J_p(t)}{t^p}$$

The acting of the Fourier- Bessel transformation to B- convolution is as follows:

$$F_B(\varphi\otimes\psi)=F_B(\varphi)F_B(\psi),\ \varphi,\psi\in S(R_+^n) \qquad (5)$$

Let $P(t_1,t_2,...,t_{n-1},t_n^2)$ be a polynomial in the n variables with constant coefficients and even according in the last variable. Then,

$$P\left(\frac{\partial}{\partial y_1},...\frac{\partial}{\partial y_{n-1}},B_{y_n}\right)(F_B\varphi)(y)=F_B\left[P\left(-ix_1,...,-ix_{n-1},-x_n^2\right)\varphi(x)\right](y)$$

$$F_B\left[P\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_{n-1}},B_{x_n}\right)\varphi(x)\right](y)=P\left(iy_1,...,iy_{n-1},-y_n^2\right)(F_B\varphi)(y)$$

In particular, for the Laplace- Bessel differential operator $\Delta_B$

$$(-\Delta_B)\varphi=F_B^{-1}\left(|\varsigma|^2F_B\varphi\right),\ \varphi\in S(R_+^n) \qquad (6)$$

is proved ( Bayrakci and Aliev, 1998).

## 3. ON THE $\beta$ - SEMIGROUPS GENERATED BY FOURIER-BESSEL TRANSFORM

The $\beta$ - semigroup associated with $\Delta_B$ is an integral operator of convolution type generated by the generalized shift. The kernel $\omega^{(\beta)}(|.|,t)$ of this operator is defined as the inverse Fourier- Bessel transformation of the function $e^{-t|x|^\beta}$, $(x\in R_+^n, t>0)$

$$\omega^{(\beta)}(|y|,t)=F_B^{-1}\left(e^{-t|x|^{\beta}}\right)(y) \tag{7}$$

where $y\in R_+^n, t>0, \beta>0$.

**Lemma 1** Let $y\in R_+^n, t>0, \beta>0, \nu>0$. Then

$$\omega^{(\beta)}(|y|,t)=t^{\frac{-(2\nu+n)}{\beta}}\omega^{(\beta)}\left(t^{\frac{-1}{\beta}}|y|,1\right) \tag{8}$$

**Proof** From (3), (4) and (7) we get

$$\omega^{(\beta)}(|y|,t)=c_\nu(n)F_B\left(e^{-t|x|^{\beta}}\right)(-y)$$

$$=c_\nu(n)\int_{R_+^n} e^{-t|z|^{\beta}}e^{-iy'.z'}j_{\nu-\frac{1}{2}}(y_n z_n)z_n^{2\nu}dz$$

Changing the variable as $z_i=t^{\frac{-1}{\beta}}m_i$ , we get $\omega^{(\beta)}(|y|,t)=t^{\frac{-(2\nu+n)}{\beta}}\omega^{(\beta)}\left(t^{\frac{-1}{\beta}}|y|,1\right)$.

The $\beta$ - semigroup $\{B_t^{(\beta)}\}_{t>0}$ generated by the kernel $\omega^{(\beta)}(|y|,t)$ is defined by

$$\left(B_t^{\beta}f\right)(x)=\int_{R_+^n}\omega^{(\beta)}(|y|,t)T^y f(x)y_n^{2\nu}dy \tag{9}$$

where $dy=dy_1...dy_n$.

## 4. ON THE RIESZ POTENTIALS GENERATED BY THE LAPLACE- BESSEL DIFFERENTIAL OPERATOR AND THEIR REPRESENTATION VIA THE $\beta$ - SEMIGROUP $\{B_t^{(\beta)}\}_{t>0}$

According to the formula (6) we have

$$(-\Delta_B)\varphi = F_B^{-1}\left(|x|^2 F_B\varphi\right) \text{ for all } \varphi \in S(R_+^n)$$

Thus we define the negative ( fractional) powers of the Laplace-Bessel operator $(-\Delta_B)$ by

$$(-\Delta_B)^{-\frac{\alpha}{2}}\varphi = F_B^{-1}\left(|x|^{-\alpha} F_B\varphi\right),\ (\alpha > 0)$$

We denote the last operator by $I_B^\alpha$ as in ( Gadjiev and Aliev, 1988);

$$\left(I_B^\alpha\varphi\right) = F_B^{-1}\left(|x|^{-\alpha} F_B\varphi\right),\ \ (\varphi \in S(R_+^n), \alpha > 0)$$

Many known results for the classic Riesz potentials such as the Hardy- Littlewood- Sobolev theorem, etc are also valid for the Riesz potential $I_B^\alpha$. See for example ( Gadjiev and Aliev, 1988; Guliyev, 1998).

**Theorem 2** ( Gadjiev and Aliev, 1988) For all $\varphi \in S(R_+^n)$ and $0 < \alpha < n + 2\nu$

$$\left(I_B^\alpha\varphi\right)(x) = \frac{1}{\gamma_n(\alpha,\nu)}\left(\frac{1}{|x|^{n+2\nu-\alpha}} \otimes \varphi\right)$$

$$= \frac{1}{\gamma_n(\alpha,\nu)}\int_{R_+^n}\varphi(y)T^y\left(|x|^{-n-2\nu+\alpha}\right)y_n^{2\nu}dy \qquad (10)$$

where

$$\gamma_n(\alpha,\nu) = \left[2^{1-\alpha}\Pi^{\frac{1-n}{2}}\Gamma\left(\frac{n+2\nu-\alpha}{2}\right)\right]^{-1}\left[\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\right]$$

A normalizing coefficient $\frac{1}{\gamma_n(\alpha,\nu)}$ is chosen in such a way that

$$F_B\left(I_B^\alpha \varphi\right)(y) = |y|^{-\alpha} F_B(\varphi)(y) \tag{11}$$

As mentioned above, our main purpose is to determine a new formula representing the Riesz Potentials associated with $\left\{B_t^{(\beta)}\right\}_{t>0}$ $\beta$ - semigroups.

## 4.1 MAIN THEOREM

**Theorem 3** $0<\alpha<n+2\nu$ , $f \in L_{p,\nu}\left(R_+^n\right)$ ; $1 \le p < \frac{n+2\nu}{\alpha}$ , $\left(t>0, x \in R_+^n\right)$ and $I_B^\alpha f$ be the Riesz potenatial associated with the Laplace- Bessel differential operator. Then

$$\left(I_B^\alpha f\right)(x) = \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty t^{\frac{\alpha}{\beta}-1} \left(B_t^{(\beta)} f\right)(x) dt$$

**Note:** A similar formula representing a relation between the calssical $\beta$- semigroup and the Riesz Potential associated with the Laplace differential operator $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ was proved by S. Sezer and I. A. Aliev (2010).

**Proof.** By changing the order of integration, we have

$$\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty t^{\frac{\alpha}{\beta}-1} \left(B_t^{(\beta)} f\right)(x) dt = \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty t^{\frac{\alpha}{\beta}-1} \left( \int_{R_+^n} \omega^{(\beta)}(|y|,t) T^y f(x) y_n^{2\nu} dy \right) dt$$

$$= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty \int_{R_+^n} t^{\frac{\alpha}{\beta}-1} \omega^{(\beta)}(|y|,t) T^y f(x) y_n^{2\nu} dy dt$$

$$=\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_{R_+^n}\left(\int_0^\infty t^{\frac{\alpha}{\beta}-1}\omega^{(\beta)}(|y|,t)dt\right)T^y f(x) y_n^{2\nu}dy$$

$$=\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_{R_+^n}T^y f(x)\left(\int_0^\infty t^{\frac{-(2\nu+n)+\alpha}{\beta}-1}\omega^{(\beta)}\left(t^{\frac{-1}{\beta}}|y|,1\right)dt\right)y_n^{2\nu}dy$$

$$\left(t^{\frac{-1}{\beta}}|y|=\tau, dt=(-\beta)|y|^{\beta}\tau^{-\beta-1}d\tau\right)$$

$$=\frac{-\beta}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_{R_+^n}T^y f(x)\left(\int_{-\infty}^0 |y|^{\alpha-2\nu-n-\beta}\tau^{n+2\nu-\alpha+\beta}\omega^{\beta}(\tau,1)|y|^{\beta}\tau^{-\beta-1}d\tau\right)y_n^{2\nu}dy$$

$$=\frac{\beta}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_{R_+^n}T^y f(x)|y|^{\alpha-2\nu-n}\left(\int_0^\infty \tau^{n+2\nu-\alpha-1}\omega^{\beta}(\tau,1)d\tau\right)y_n^{2\nu}dy$$

$$=\frac{\beta}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty \tau^{n+2\nu-\alpha-1}\omega^{\beta}(\tau,1)d\tau\left(\int_{R_+^n}T^y f(x)|y|^{\alpha-2\nu-n}y_n^{2\nu}dy\right)$$

Therefore, we have

$$\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty t^{\frac{\alpha}{\beta}-1}\left(B_t^{(\beta)}f\right)(x)dt=d_n(\alpha,\nu)\left(\int_{R_+^n}T^y f(x)|y|^{\alpha-2\nu-n}y_n^{2\nu}dy\right) \tag{12}$$

where

$$d_n(\alpha,\nu)=\frac{\beta}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty \tau^{n+2\nu-\alpha-1}\omega^{(\beta)}(\tau,1)d\tau$$

In accordance with (10) we must show that

$$d_n(\alpha,\nu)=\frac{1}{\gamma_n(\alpha,\nu)}$$

We will show the last equality not by straight calculation but indirectly using the Fourier- Bessel transform. Since (12) holds for all $f\in L_{p,\nu}\left(R_+^n\right)$, $1\le p<\dfrac{n+2\nu}{\alpha}$, it holds in particular for Schwartz test functions. Thus, assuming $f\in S\left(R_+^n\right)$ we get

$$F_B\left(\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty t^{\frac{\alpha}{\beta}-1}\left(B_t^{(\beta)}f\right)(y)dt\right)=\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty t^{\frac{\alpha}{\beta}-1}F_B\left(B_t^{(\beta)}f\right)(y)dt$$

$$=\frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty t^{\frac{\alpha}{\beta}-1}F_B\left(\omega^{(\beta)}(|.|,t)\right)(y)(F_Bf)(y)dt$$

$$=\frac{(F_Bf)(y)}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty t^{\frac{\alpha}{\beta}-1}e^{-t|y|^\beta}dt$$

$$\left(t\to t|y|^{-\beta}\right)$$

$$=\frac{(F_Bf)(y)}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty t^{\frac{\alpha}{\beta}-1}e^{-t}|y|^{-\alpha}dt$$

$$=\frac{(F_Bf)(y)}{\Gamma\left(\frac{\alpha}{\beta}\right)}|y|^{-\alpha}\int_0^\infty t^{\frac{\alpha}{\beta}-1}e^{-t}dt$$

$$=(F_Bf)(y)|y|^{-\alpha}=F_B\left(I_B^\alpha f\right)(z)$$

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