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Research Article

Limit theorem for a semi - Markovian stochastic model of type (s,S)

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Abstract

In this study, a semi-Markovian inventory model of type (s, S) is considered and the model is expressed by means of renewal-reward process $(X(t))$ with an asymmetric triangular distributed interference of chance and delay. The ergodicity of the process $X(t)$ is proved and the exact expression for the ergodic distribution is obtained. Then, two-term asymptotic expansion for the ergodic distribution is found for standardized process $W(t) \equiv (2X(t))/(S - s)$. Finally, using this asymptotic expansion, the weak convergence theorem for the ergodic distribution of the process $W(t)$ is proved and the explicit form of the limit distribution is found.

Keywords: Inventory model of type (s, S), Renewal-reward process, Weak convergence, Asymmetric triangular distribution, Asymptotic expansion.

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1. Introduction

A number of very interesting problems arising in the theories of inventory, stock control, queuing theory, reliability, mathematical insurance, stochastic finance etc., can be expressed by means of renewal processes, renewal-reward processes, random walk processes and their modifications. There are many important theoretical results about these subjects in literature (Borovkov (1984), Brown and Solomon (1975), Feller (1971), Gihman and Skorohod (1975), Janssen and Leuwaarden (2007), Khaniyev et al. (2008,

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2013), Lotov (1996), Rogozin (1964), etc.). The results of these studies have complex mathematical structures and they are not useful for applied problems.

To avoid this difficulty, in recent years, the asymptotic methods have started to be applied to these problems. There are several valuable studies about using asymptotic methods, as well (e.g., Janssen and Leuwaarden (2007), Lotov (1996), Khaniyev (2005), Khaniyev and Atalay (2010), Khaniyev and Mammadova (2006), Khaniyev et al. (2008, 2013)).

Lately, the inventory models of type (s,S) have been extensively considered and some of their characteristics have been investigated in the literature (see, e.g., Khaniyev and Atalay (2010), Khaniyev and Mammadova (2006), Khaniyev et al. (2013)). Especially, in the studies Khaniyev and Atalay (2010) and Khaniyev et al. (2013), an inventory model of type (s,S) with triangular distributed interference of chance is tackled. In Khaniyev and Atalay (2010), the weak convergence theorem for the considered process is proved and in Khaniyev et al. (2013), three-term asymptotic expansions for the moments of ergodic distribution are obtained. These results are not only remarkable from the theoretical point of view but are also very useful in the application. Unfortunately, in these both studies, discrete interference has a symmetric triangular distribution and it is assumed that the lead time is zero. Note that, the processes having this restricted properties cannot adequately express real world problems arising in applied sciences. It can be observed in the following example which is given by Khaniyev et al. (2013).

1.1. The Real-World Model. A company operating in the energy sector produces, stores, fills, and distributes liquefied petroleum gas (LPG). Domestic LPG distribution is carried out via pipelines and transported from the LPG production center (a city in Turkey) to the 30 dealers by tankers with the capacities of $22m^3$ (approximately 10-11 tons) and $35m^3$ (approximately 17-18 tons). The tankers are kept under surveillance with the GPS (Global Positioning Systems) 24 hours a day and 7 days a week. After delivering the needed amount of gas to the dealer, if more than 10% of the capacity of the tanker is left over, the tanker waits in its position until the next order of any dealer. Each dealer has a storage capacity of $S = 30\ m^3$. Random amounts of LPG (η_n) are sold from these storage tanks at random times $T_n = \sum_{i=1}^{n} \xi_i$. When at random moments $\tau_n,\ n \geq 1$ the level of LPG in the tank of the dealer falls below the control level $s = S/5$, a demand signal is automatically sent online to the production center. As a response to this demand, the nearest tanker to the dealer is directed to the demanding dealer. If there is no tanker near to the dealer, a full tanker is sent from the production center. For safety concerns (in order not to allow the gas pressure to reach its maximum value), the dealers, most of the time, fill about 85% of the capacity (S) of their tanks. However, with a low probability, by taking a risk the dealers fill their tanks to the full capacity when the need arises. On the other hand, even if the amount of gas in the tanker does not meet 85% of the dealer's tank, the existing amount of gas in the tanker is loaded into the dealer's tank.

The concept of filling the depot approximately 85% indicates the necessity of using an asymmetric triangular distributed interference of chance for modeling this problem. Therefore, in our opinion, the process that expresses the working principle of the depot can best be modelled as a stochastic process with an asymmetric triangular distributed interference of chance.

Moreover, the existing studies in literature, assumes that the lead time is equal to zero. However, in real world problems, it is not possible to refill a depot immediately everytime. This delay time may be due to transportation, trying to provide demands from suppliers, etc. Therefore, to solve certain real world problems, the following assumptions should be satisfied.

1. The random variable which represents the discrete interference of chance has an asymmetric triangular distribution.

2. Lead time takes positive values.

The studies existing in literature, unfortunately do not satisfy these assumptions. To fill this gap, in this study, a semi-Markovian inventory model of type (s, S) is considered under these two assumptions. In Section 2, the stochastic process $X(t)$ which expresses this model is constructed mathematically. Next, under some weak conditions, the ergodicity of the process is proved and the exact form of the ergodic distribution is found in Section 3. Finally, in Section 4, the standardized process $W(t)$ is defined and two-term asymptotic expansion for the ergodic distribution of $W(t)$ is obtained. Then, weak convergence theorem which is the main aim of the study is proved. Additionally, the explicit form of the limit distribution is found. Before stating these results, let us first define the process mathematically.

2. Mathematical Construction of the Process $X(t)$

Let $\{\xi_n\}$, $\{\eta_n\}$, $\{\zeta_n\}$, $\{\theta_n\}$, $n = 1, 2, ...$ be four sequences of random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ such that variables in each sequences are independent and identically distributed. Additionally, ξ_n, η_n, ζ_n and θ_n are also mutually independent and can take only non-negative values. Denote the distribution functions of ξ_1, η_1, ζ_1, θ_1 by

$$\Phi(t) = P\{\xi_1 \le t\}; \quad F(x) = P\{\eta_1 \le x\}; \quad \pi(z) = P\{\zeta_1 \le z\}; \quad H(t) = P\{\theta_1 \le t\},$$

respectively. Define the renewal sequences $\{T_n\}$ and $\{Y_n\}$ using $\{\xi_n\}$ and $\{\eta_n\}$ as follows:

$$T_0 = Y_0 = 0; \; T_n = \sum_{i=1}^{n} \xi_i; \; Y_n = \sum_{i=1}^{n} \eta_i; \; n = 1, 2, ...$$

and a sequence of integer-valued random variables $\{N_n\}$; $n = 0, 1, 2, ...$ as:

$$\begin{aligned} N_0 &= 0; \; N_1 = N_1(z) = \inf\{k \ge 1 : z - Y_k < s\}; z \in [s, S]; \\ N_{n+1} &= N_{n+1}(\zeta_n) = \inf\{k \ge N_n + 1 : \zeta_n - (Y_k - Y_{N_n}) < s\}; \; n = 1, 2.. \end{aligned}$$

Define

$$\tau_0 = 0; \; \zeta_0 = z \in [s, S]; \; \tau_n = \tau_n(\zeta_{n-1}) = \sum_{i=0}^{N_n} \xi_i;$$

$$\gamma_n = \tau_n + \theta_n, \; n = 1, 2, ...$$

$$\nu(t) = \max\{n \ge 0 : T_n \le t\}, \; t > 0.$$

Now let us construct the stochastic process $X(t)$:

$$X(t) = \sum_{n=0}^{\infty} \max\left\{s, \zeta_n - \left(Y_{\nu(t)} - Y_{N_n}\right)\right\} I_{[\gamma_n; \gamma_{n+1}]}(t).$$

Here, indicator function $I_A(t)$ of the set A is defined as

$$I_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}$$

Considered process $X(t)$ is known as "Renewal-reward process with a discrete interference of chance" in literature. In this study, it is assumed that the random variable ζ_1 has asymmetric triangular distribution with parameters (s, m, S). For this reason, this process can be called "Renewal-reward process with an asymmetric triangular distributed interference of chance". A sample trajectory of this process is shown in Figure 1.

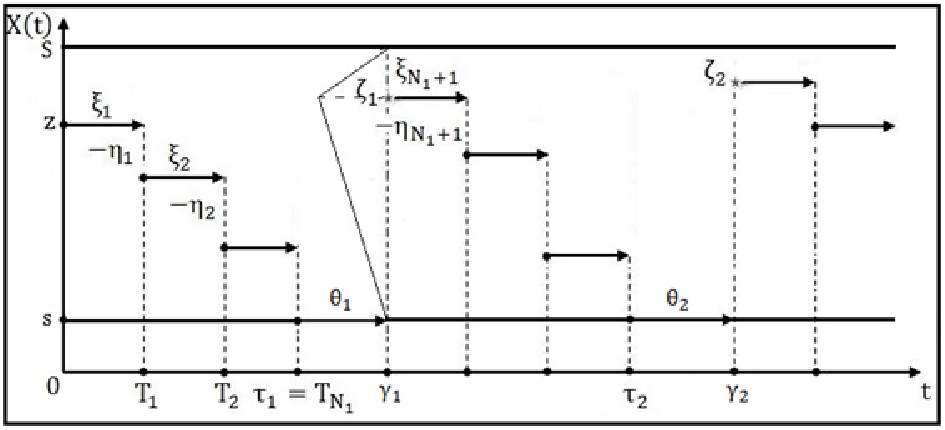

Figure 1. A trajectory of the process $X(t)$

3. The Ergodicity of the Process $X(t)$

In order to study the stationary characteristics of the process, it is required to show that the process is ergodic under some weak conditions. For this aim, first of all, we are going to prove the ergodicity of the process.

3.1. Proposition. *Let the initial sequences of the random variables* $\{\xi_n\}$, $\{\eta_n\}$, $\{\zeta_n\}$, $\{\theta_n\}$, $n = 1, 2, ...$ *satisfy the following supplementary conditions:*

(i) $0 < E(\xi_1) < +\infty$;
(ii) $0 \leq E(\theta_1) < +\infty$;
(iii) $E(\eta_1) > 0$;
(iv) η_1 *is a non-arithmetic random variable;*
(v) *random variable* ζ_1 *has asymmetric triangular distribution with parameters* $(s, m, S), 0 \leq s < m < S < +\infty$.

Then, the process $X(t)$ *is ergodic and the following relation holds with probability* 1, *for each bounded and measurable function* $f(x)$ $(f : [0, +\infty) \rightarrow R)$*:*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(u))\, du = \frac{1}{E(\gamma_1)} \int_{z=s}^{S} \int_{v=s}^{S} \int_{t=0}^{+\infty} f(v) P_z \{\gamma_1 > t; X(t) \in dv\}\, dt d\pi(z). \tag{3.1}$$

Proof. The process $X(t)$ belongs to a wide class of processes is called "stochastic processes with a discrete interference of chance". This notion is first introduced to literature by A. N. Kolmogorov. For this class, the general ergodic theorem of type Smith's "key renewal theorem" exists in the literature (Gihman and Skorohod (1975), p.243). According to this theorem, to prove the ergodicity of the processes with a discrete interference of chance, it is sufficient to show that the following two assumptions hold.

Assumption 1. Choosing a sequence of ascending random times is required such that the values of the process $X(t)$ at these times form an embedded Markov chain which is ergodic. For this purpose, it is sufficient to choose the sequence of random times $\{\gamma_n,\ n \geq 0\}$, defined in Section 2. The values of the process $X(t)$ at these times are equal to $\zeta_n = X(\gamma_n)$, $n \geq 1$ which form an embedded Markov chain. In our case, the embedded Markov chain $\{\zeta_n\}$, $n \geq 1$ is ergodic with a stationary distribution $\pi(z) = \lim_{n\to\infty} P\{\zeta_n \leq z\} = P\{\zeta_1 \leq z\}$, because the random variables $\{\zeta_n\}$, $n = 1, 2, ...$ are independent and identically distributed random variables in the interval $[s, S]$. Therefore, the first assumption of the general ergodic theorem (Gihman and Skorohod (1975)) is satisfied.

Assumption 2. The expected value of the times between successive stopping times $\{\gamma_n\}$, $n = 1, 2, ...$ should be finite, that is $E(\gamma_n - \gamma_{n-1}) < \infty$, $n = 1, 2, ...$ For this aim, it is sufficient to show that

$$\begin{aligned} E\left(\gamma_1(z)\right) &= E\left((\tau_1(z) + \theta_1)\right) = E\left(\tau_1(z)\right) + E(\theta_1) < \infty; \\ E\left(\gamma_n - \gamma_{n-1}\right) &= E\left(\gamma_1(\zeta_1)\right) = E\left(\tau_1(\zeta_1)\right) + E\left(\theta_1\right) < \infty; \;\; n = 2, 3, ... \end{aligned}$$

By the conditions of Proposition 3.1, $E(\xi_1) < \infty$ and $E(\theta_1) < \infty$. From Wald identity (Feller (1971)), it is hold that $E\left(\tau_1(z)\right) = E\left(\sum_{i=1}^{N_1(z)} \xi_i\right) = E(\xi_1) E\left(N_1(z)\right)$. Note that, $E\left(N_1(z)\right) \equiv U_\eta(z - s) < \infty$ for each finite z (Feller (1971), p.185). Here, $U_\eta(x)$ is a renewal function generated by the sequence of the random variables $\{\eta_n\}$, $n \geq 1$. Therefore, $E\left(\gamma_1(z)\right) < \infty$. At the same time, the renewal function $U_\eta(x)$ is a non-decreasing function. Therefore, for each $z \in [s, S]$, $U_\eta(z - s) < \infty$ is provided. Hence, we have the following relation:

$$E\left(\tau_1\left(\zeta_1\right)\right) = E(\xi_1) \int_s^S U_\eta(z - s) d\pi(z) \leq E(\xi_1) U_\eta(S - s) < \infty$$

Thus, for each $n = 2, 3, ...$; $E\left(\gamma_n - \gamma_{n-1}\right) < \infty$. This shows that Assumption 2 is also satisfied. Thereby, the process $X(t)$ is ergodic and the relation in Eq. (3.1) holds. This concludes the proof of Proposition 3.1. □

From this theorem, many valuable results can be obtained. Some of them can be given as follows.

3.2. Corollary. *Under the conditions Proposition 3.1, for each $x \in [s, S]$, the exact expression of the ergodic distribution function of $X(t)$ is given as follows:*

$$Q_X(x) = 1 - \frac{E\left(U_\eta\left(\zeta_1 - x\right)\right)}{K + E\left(U_\eta\left(\zeta_1 - s\right)\right)} \tag{3.2}$$

Here, $Q_X(x) \equiv \lim_{t\to\infty} P\{X(t) \leq x\}$ is the ergodic distribution of process $X(t)$. Moreover, $K = E(\theta_1)/E(\xi_1)$; $\pi(z) = P\{\zeta_1 \leq z\}$ and

$$\pi'(z) \equiv p(z) = \begin{cases} \frac{2(x-s)}{(m-s)(S-s)}, & s < x \leq m \\ \frac{2(S-x)}{(S-m)(S-s)}, & m < x \leq S \end{cases}$$

Let us denote $\tilde{X}(t) \equiv X(t) - s$ and $\tilde{\zeta_1} = \zeta_1 - s$. Then, state the following corollary.

3.3. Corollary. *Under the conditions of Proposition 3.1, the process $\tilde{X}(t)$ is ergodic and the exact expression of its ergodic distribution $(Q_{\tilde{X}}(x))$ can be shown as follows:*

$$Q_{\tilde{X}}(x) \equiv \lim_{t\to\infty} P\left\{\tilde{X}(t) \leq x\right\} = 1 - \frac{E\left(U_\eta\left(\tilde{\zeta}_1 - x\right)\right)}{K + E\left(U_\eta\left(\tilde{\zeta}_1 - s\right)\right)}; \quad x \in [0, S - s]$$

4. Weak Convergence Theorem for the Ergodic Distribution of the Process $X(t)$

The aim of this section is to prove the weak convergence theorem for the ergodic distribution of standardized process $W(t) \equiv \tilde{X}(t)/\beta$ where $\tilde{X}(t) \equiv X(t) - s$ and $\beta = (S-s)/2$, when $\beta \to \infty$. Denote the ergodic distribution of $W(t)$ with $Q_W(x)$. Then, the exact expression of $Q_W(x)$ can be written as follows ($x \in [0,2]$):

$$(4.1) \qquad Q_W(x) \equiv \lim_{t\to\infty} P\{W(t) \le x\} = Q_{\tilde{X}}(\beta x) = 1 - \frac{E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right)}{K + E\left(U_\eta\left(\tilde{\zeta}_1 - s\right)\right)}.$$

Here, $K = E(\theta)/E(\xi_1)$ is delay coefficient and random variable $\tilde{\zeta}_1$ has the asymmetric triangular distribution with parameters $(0, m-s, S-s)$ and its probability density function ($\tilde{p}(z)$) is written as follows:

$$(4.2) \qquad \tilde{p}(z) \equiv \tilde{\pi}'(z) = \pi'(s+z) = \begin{cases} \frac{z}{2\alpha\beta^2}, & 0 < z \le 2\alpha\beta \\ \frac{2\beta - z}{2(\alpha-1)\beta^2}, & 2\alpha\beta < x \le 2\beta \end{cases}$$

where $\alpha = (m-s)/(S-s)$ and $\beta = (S-s)/2$.

Before proving the weak convergence theorem, define the following functions $A(z) = \int_0^z U_\eta(y)dy$ and $V(z) = \int_0^z yU_\eta(y)dy$, then give the following propositions.

4.1. Proposition. *The Laplace transforms $\tilde{A}(\lambda)$ and $\tilde{V}(\lambda)$ of the function $A(z)$ and $V(z)$ can be represented as follows:*

$$(4.3) \qquad \tilde{A}(\lambda) = \frac{1}{\lambda^2(1-\varphi(\lambda))}; \; \tilde{V}(\lambda) = \frac{1-\varphi(\lambda) - \lambda\varphi'(\lambda)}{\lambda^3(1-\varphi(\lambda))^2}.$$

Here, $\varphi(\lambda) = E(\exp(-\lambda\eta_1))$; $\lambda > 0$.

Proof. Proof of the Proposition 4.1 can be derived from both definitions of $A(z)$, $V(z)$ and properties of Laplace transform. □

4.2. Proposition. *Assume that $m_3 \equiv E\left(\eta_1^3\right) < \infty$. Then, the following asymptotic expansions can be written, when $z \to \infty$:*

$$(4.4) \qquad \begin{aligned} A(z) &= \frac{1}{2m_1}z^2 + \frac{m_2}{2m_1^2}z + \frac{A_1}{m_1} + o(1); \\ V(z) &= \frac{1}{3m_1}z^3 + \frac{m_2}{4m_1^2}z^2 + o(z) \end{aligned}$$

Here, $A_1 = m_{21}^2 - m_{31}/2$; $m_{k1} = m_k/(km_1)$; $m_k = E(\eta_1^k)$, $k = 1, 2, ...$

Proof. Since $m_3 < +\infty$ is satisfied, then the following asymptotic expansions for $\varphi(\lambda)$ and $\varphi'(\lambda)$ can be written, when $\lambda \to 0$ (Feller (1971)):

$$(4.5) \qquad \varphi(\lambda) = E\left(e^{-\lambda\eta_1}\right) = 1 - \lambda m_1 + \frac{\lambda^2}{2}m_2 - \frac{\lambda^3}{6}m_3 + o\left(\lambda^3\right);$$

$$(4.6) \qquad \varphi'(\lambda) = E(-\eta_1 e^{-\lambda\eta_1}) = -m_1 + \lambda m_2 - \frac{\lambda^2}{2}m_3 + o\left(\lambda^2\right).$$

Substituting Eq. (4.5) and Eq. (4.6) in Eq. (4.3) and applying Tauber-Abel Theorem to Eq. (4.3), the asymptotic expansions in Eq.(4.4) are obtained. This concludes the proof of Proposition 4.2. □

4.3. Lemma. *In addition to the conditions of Proposition 3.1, let $m_3 < \infty$ be also satisfied. Then, the asymptotic expansion of $E\left(U_\eta\left(\tilde{\zeta}_1\right)\right)$ can be written as follows, when $\beta \to \infty$:*

$$(4.7) \qquad E\left(U_\eta\left(\tilde{\zeta}_1\right)\right) = \frac{2+2\alpha}{3m_1}\beta + \frac{m_2}{2m_1^2} + o\left(\frac{1}{\beta}\right)$$

Here, $m_k = E(\eta_1^k)$; $k = 1, 2, 3$.

Proof. Present $E\left(U_\eta\left(\tilde{\zeta}_1\right)\right)$ as follows:

$$(4.8) \qquad E\left(U_\eta\left(\tilde{\zeta}_1\right)\right) = \int_0^{2\beta} U_\eta(z)\tilde{p}(z)dz = \int_0^{2\alpha\beta} U_\eta(z)\tilde{p}(z)dz + \int_{2\alpha\beta}^{2\beta} U_\eta(z)\tilde{p}(z)dz$$

Using Proposition 4.2, calculate the first integral in Eq.(4.8) as follows:

$$(4.9) \qquad I_1 \equiv \int_0^{2\alpha\beta} U_\eta(z)\tilde{p}(z)dz = \frac{1}{2\alpha\beta^2}V\left(2\alpha\beta\right) = \frac{(2\alpha)^2}{3m_1}\beta + \frac{\alpha m_2}{2m_1^2} + o(1)$$

Now, with the help of Proposition 4.2, calculate the second integral in Eq.(4.8) as follows:

$$(4.10) \qquad I_2 \equiv \int_{2\alpha\beta}^{2\beta} U_\eta(z)\tilde{p}(z)dz = \frac{1}{2(1-\alpha)\beta^2}\left[D(2\beta) - D(2\alpha\beta)\right]$$

Here, $D(x) \equiv \int_0^x U_\eta(z)(2\beta - z)dz$. From Proposition 4.2, $D(2\beta)$ can be written as follows:

$$(4.11) \qquad D(2\beta) = 2\beta A(2\beta) - V(2\beta) = \frac{(2\beta)^3}{6m_1} + \frac{m_2\,(2\beta)^2}{4m_1^2} + \frac{2\beta A_1}{m_1} + o(\beta)$$

Here, $A_1 = m_{21}^2 - m_{31}/2$; $m_{k1} = m_k/(km_1)$; $m_k = E(\eta_1^k)$, $k = 1, 2, ...$

In a similar way, $D(2\alpha\beta)$ can be presented as follows:

$$(4.12) \qquad D(2\alpha\beta) = 2\beta A(2\alpha\beta) - V(2\alpha\beta) = \frac{8\alpha^2(3-2\alpha)}{6m_1} + \frac{4\alpha m_2(2-\alpha)}{4m_1^2}\beta^2 + \frac{2A_1}{m_1}\beta + o(\beta)$$

Considering Eq.(4.11) and Eq.(4.12) into Eq.(4.10), the following asymptotic expansion can be obtained, when $\beta \to \infty$:

$$(4.13) \qquad I_2 = \frac{1}{2(1-\alpha)\beta^2}\left[D(2\beta) - D(2\alpha\beta)\right] = \frac{2(1+\alpha-2\alpha^2)}{3m_1}\beta + \frac{m_2(1-\alpha)}{2m_1^2} + o(1)$$

Substituting Eq.(4.9) and Eq.(4.13) in Eq. (4.8), the following asymptotic expansion is obtained, when $\beta \to \infty$:

$$(4.14) \qquad E\left(U_\eta\left(\tilde{\zeta}_1\right)\right) = I_1 + I_2 = \frac{2+2\alpha}{3m_1}\beta + \frac{m_2}{2m_1^2} + o(1)$$

Therefore, Lemma 4.3 is proved. □

4.4. Lemma. *Suppose that $m_3 < +\infty$ is also satisfied in addition to the conditions of Proposition 3.1 Then, two-term asymptotic expansion for $E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right)$ can be written as follows, when $\beta \to \infty$:*

$$(4.15) \qquad E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right) = \begin{cases} \frac{8\alpha^2+8\alpha-12\alpha x+x^3}{12m_1\alpha}\beta + \frac{m_2(4\alpha-x^2)}{8m_1\alpha} + o(1), & x \in (0, 2\alpha) \\ \frac{(2-x)^3}{12(1-\alpha)}\beta + \frac{m_2(2-x)^2}{8(1-\alpha)m_1^2} + o(1), & x \in (2\alpha, 2) \end{cases}$$

Here $\alpha = (m-s)/(S-s)$.

Proof. For each $x \in (0, 2\alpha)$, write $E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right)$ as follows:

$$E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right) = J_{11}(x) + J_{12}(x) \tag{4.16}$$

Here,

$$J_{11}(x) \equiv \int_{\beta x}^{2\alpha\beta} U_\eta(z - \beta x)\tilde{p}(z)dz; \quad J_{12}(x) \equiv \int_{2\alpha\beta}^{2\beta} U_\eta(z - \beta x)\tilde{p}(z)dz;$$

First of all, using Proposition 4.2, calculate $J_{11}(x)$:

$$\begin{aligned} J_{11}(x) &= \frac{1}{2\alpha\beta^2}\int_{\beta x}^{2\alpha\beta} U_\eta(z - \beta x)zdz \\ &= \frac{1}{2\alpha\beta^2}\left\{\beta x\left[A\left((2\alpha - x)\beta\right) + V\left((2\alpha - x)\beta\right)\right]\right\} \\ &= \frac{16\alpha^3 - 12\alpha^2 x + x^3}{12\alpha m_1}\beta + \frac{m_2(4\alpha^2 - x^2)}{8\alpha^2} + o(1) \end{aligned} \tag{4.17}$$

Here, $A(z) = \int_0^z U_\eta(y)dy$ and $V(z) = \int_0^z yU_\eta(y)dy$.

Now, with the similar method, calculate $J_{12}(x)$:

$$\begin{aligned} J_{12}(x) &= \int_{2\alpha\beta}^{2\beta} U_\eta\left(z - \beta x\right)\tilde{p}(z)dz = \frac{1}{2(1-\alpha)\beta^2}\int_{2\alpha\beta}^{2\beta} U_\eta\left(z - \beta x\right)(2\beta - z)dz \\ &= \frac{1}{2(1-\alpha)\beta^2}\int_{(2\alpha - x)\beta}^{(2-x)\beta} U_\eta\left(y\right)(2\beta - \beta x - y)dy \\ &= \frac{1}{2(1-\alpha)\beta^2}\{\int_0^{(2-x)\beta} U_\eta(y)(2\beta - \beta x - y)dy \\ &- \int_0^{(2\alpha - x)\beta} U_\eta(y)(2\beta - \beta x - y)dy\} \\ &= \frac{1}{2(1-\alpha)\beta^2}\left[B\left((2-x)\beta\right) - B\left((2\alpha - x)\beta\right)\right] \end{aligned} \tag{4.18}$$

Here, $B(t) \equiv \int_0^t U_\eta(y)(2\beta - \beta x - y)dy$ for simplicity.

With the help of Proposition 4.2, compute $B((2-x)\beta)$ and $B((2\alpha - x)\beta)$, as follows:

$$\begin{aligned} B((2-x)\beta) &= (2\beta - \beta x)A((2-x)\beta) - V((2-x)\beta) \\ &= \frac{(2-x)^3}{6m_1}\beta^3 + \frac{m_2(2-x)^2}{4m_1^2}\beta^2 + \frac{A_1}{m_1}\beta + o(\beta) \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} B((2\alpha - x)\beta) &= (2-x)\beta A((2\alpha - x)\beta) - V((2\alpha - x)\beta) \\ &= \frac{(2\alpha - x)^2(6 - x - 4\alpha)}{6m_1}\beta^3 + \frac{m_2(2-x)^2(4 - x - 2\alpha)}{4m_1^2}\beta^2 \\ &+ \frac{A_1(2-x)}{m_1}\beta + o(\beta). \end{aligned} \tag{4.20}$$

Here, $A_1 = m_{21}^2 - m_{31}/2$; $m_{k1} = m_k/(km_1)$; $m_k = E(\eta_1^k)$, $k = 1, 2, ...$

By considering Eq.(4.19) and Eq.(4.20) in Eq.(4.18), $J_{12}(x)$ can be written as follows:

$$\begin{aligned} J_{12}(x) &= \frac{1}{2(1-\alpha)\beta^2}\left[B((2-x)\beta) - B((2\alpha - x)\beta)\right] \\ &= \frac{16\alpha^3 - 24\alpha^2 - 12\alpha^2 x + 24\alpha x - 12x + 8}{12(1-\alpha)m_1}\beta + \frac{m_2(4\alpha - x^2)}{8\alpha m_1^2} + o(1) \end{aligned} \tag{4.21}$$

Hence, the first part of Eq.(4.15) holds. Now, we can obtain the second part of the Eq.(4.15) in a similar way.
For each $x \in (2\alpha; 2)$, $\beta x \in (2\alpha\beta; 2\beta)$ holds. Then, using Proposition 4.2, $E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right)$ is calculated, as follows:

$$\begin{aligned} E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right) &= \int_{\beta x}^{2\beta} U_\eta(z-\beta x)\tilde{p}(z)dz \\ &= \frac{1}{2(1-\alpha)\beta^2}\int_{\beta x}^{2\beta} U_\eta(z-\beta x)(2\beta - z)dz \\ &= \frac{1}{2(1-\alpha)\beta^2}\left[(2-x)\beta A((2-x)\beta) - V((2-x)\beta)\right] \\ &= \frac{(2-x)^3}{12(1-\alpha)m_1}\beta + \frac{m_2(2-x)^2}{8(1-\alpha)m_1^2} + o(1). \end{aligned}$$

Here, $m_k = E(\eta_1^k)$, $k = 1, 2$. Thus, the second part of the Eq.(4.15) is obtained. □

With the help of the given propositions above, weak convergence theorem can be stated as follows.

4.5. Theorem. *In addition to the conditions of Proposition 3.1, let $m_3 < \infty$ be also satisfied. Then, the following two-term asymptotic expansion for the ergodic distribution of $W(t)$ can be written, when $\beta \to \infty$, i.e.,*

$$(4.22)\qquad Q_W(x) = \begin{cases} R_1(x) + \frac{D_1(x)}{\beta} + o\left(\frac{1}{\beta}\right), & x \in (0; 2\alpha) \\ R_2(x) + \frac{D_2(x)}{\beta} + o\left(\frac{1}{\beta}\right), & x \in (2\alpha; 2) \end{cases}$$

Here,

$$\begin{aligned} R_1(x) &= \frac{12\alpha x - x^3}{8\alpha(1+\alpha)}; \quad R_2(x) = 1 - \frac{(2-x)^3}{8(1-\alpha)^2}; \\ D_1(x) &= \frac{3m_{21}(4\alpha - x^2)}{8\alpha(1+\alpha)} - \frac{3(8\alpha^2 - 8\alpha - 12\alpha x + x^3)(Km_1 + m_{21})}{16\alpha(1+\alpha)^2}; \\ D_2(x) &= \frac{3m_{21}(2-x)^2 - 3(2-x)^3(Km_1 + m_{21})}{8(1-\alpha^2)}; \ K = E(\theta_1)/E(\xi_1). \end{aligned}$$

Proof. As in shown in Eq. (4.3), the exact expression for the ergodic distribution of the process $W(t)$ is as follows:

$$(4.23)\qquad Q_W(x) = 1 - \frac{E\left(U_\eta\left(\tilde{\zeta}_1 - \beta x\right)\right)}{K + E\left(U_\eta\left(\tilde{\zeta}_1\right)\right)}$$

Using the asymptotic expansion of $E\left(U_\eta\left(\tilde{\zeta}_1\right)\right)$ in Eq.(4.14), the following asymptotic expansion can be written:

$$(4.24)\qquad \frac{1}{K + E\left(U_\eta\left(\tilde{\zeta}_1\right)\right)} = \frac{3m_1}{2(1+\alpha)\beta}\left[1 - \frac{3(2Km_1^2 + m_2)}{4(1+\alpha)m_1}\frac{1}{\beta} + \left(\frac{1}{\beta}\right)\right]$$

With the help of the Eq.(4.15) and Eq.(4.24), asymptotic expansion for $Q_W(x)$ in Eq.(4.22) is found. Thus, the Theorem 4.5 is proved. □

Now, let us give the following proposition.

4.6. Proposition. *Suppose that $m_2 < \infty$ and $K < \infty$ holds. Then, for each $x \in (0;2\alpha)$, the inequality $|D_1(x)| \leq 5m_{21} + 4Km_1 < \infty$ and for each $x \in (2\alpha;2)$, the inequality $|D_2(x)| \leq 3m_{21} + Km_1 < \infty$ are satisfied. Here, $K = E(\theta_1)/E(\xi_1)$ is delay coefficient.*

Proof. Using the inequality $|a-b| \leq |a|+|b|$, since $m_2 < \infty$; $E(\theta_1) < \infty$ and $0 < E(\xi_1) < +\infty$ are satisfied, then the following inequalities can be written:

$$\begin{aligned} |D_1(x)| &\leq \frac{12\alpha m_{21}}{8\alpha(1+\alpha)} + \frac{3(8\alpha^2+8\alpha+24\alpha^2+8\alpha^3)(Km_1+m_{21})}{16\alpha(1+\alpha)} \\ &\leq 5m_{21} + 4Km_1 < \infty; \end{aligned}$$

$$|D_2(x)| \leq \frac{3m_{21}(2-x)^2}{8(1-\alpha^2)} + \frac{3(2-x)^3(Km_1+m_{21})}{8(1-\alpha^2)} \leq 3m_{21} + Km_1 < \infty.$$

□

Finally, let us give the following theorem which is the main goal of this study.

4.7. Theorem. *(Weak Convergence Theorem) Assume that the conditions of Theorem 4.5 are satisfied. Then, the ergodic distribution $(Q_W(x))$ of $W(t)$ weakly converges to limit distribution $R(x)$, for each $x \in (0;2)$, when $\beta \to \infty$, i.e.,*

$$\lim_{\beta\to\infty} Q_W(x) = R(x) = \begin{cases} R_1(x) & x \in (0,2\alpha) \\ R_2(x) & x \in (2\alpha,2) \end{cases} \tag{4.25}$$

Here,

$$R_1(x) = \frac{12\alpha x - x^3}{8\alpha(1+\alpha)}; \;\; R_2(x) = 1 - \frac{(2-x)^3}{8(1-\alpha^2)}; \;\; \alpha = \frac{m-s}{S-s}; \;\; \beta = \frac{S-s}{2}.$$

Proof. According to Lemma 4.3, $|D_1(x)| < \infty$ and $|D_2(x)| < \infty$ are satisfied. Then, from Theorem 4.5, the following inequalities can be obtained, when $\beta \to \infty$:

a) For each $x \in (0;2\alpha)$

$$|Q_W(x) - R_1(x)| \leq \frac{|D_1(x)|}{\beta} + \left|o\left(\frac{1}{\beta}\right)\right| \leq 2\left(\frac{5m_{21}+4Km_1}{\beta}\right). \tag{4.26}$$

b) For each $x \in (2\alpha;2)$

$$|Q_W(x) - R_2(x)| \leq 2\frac{|D_2(x)|}{\beta} \leq 2\left(\frac{3m_{21}+Km_1}{\beta}\right). \tag{4.27}$$

According to the conditions of Theorem 4.7, $K < \infty$ and $m_3 < \infty$. Therefore, the right hand sides of the inequalities in Eq.(4.26) and Eq.(4.27) are finite. Then, as $\beta \to \infty$, the right hand sides of the inequalities in Eq.(4.26) and Eq.(4.27) converge to zero. Hence, the following relation holds:

$$\lim_{\beta\to\infty} Q_W(x) = R(x) \equiv \begin{cases} R_1(x) & x \in (0,2\alpha) \\ R_2(x) & x \in (2\alpha,2) \end{cases}$$

That is, as $\beta \to \infty$, the ergodic distribution of the process $W(t)$ weakly converges to the limit distribution $R(x)$, for each $x \in (0;2)$. Thus, Theorem 4.7 is proved. □

5. Conclusion

In this study, a semi - Markovian inventory model of type (s, S) is considered. This model is expressed by means of the renewal - reward process $(X(t))$ with an asymmetric triangular distributed interference of chance. Ergodicity of this model is proved and the exact expression for the ergodic distribution function is found. Using the exact expression for the ergodic distribution of the process, two-term asymptotic expansion for the ergodic distribution is obtained and the weak convergence theorem is proved. As a result, the explicit form for the limit distribution function is found in Eq.(4.25). In the case when the interference has a symmetric triangular distribution, the parameter $\alpha = 0.5$. Then, from Eq.(4.25), limit distribution $R(x)$ can be extracted as follows:

$$R(x) = \begin{cases} x - \frac{x^3}{6}; & x \in (0, 1] \\ 1 - \frac{(2-x)^3}{6}; & x \in (1, 2) \end{cases} \tag{5.1}$$

The result in Eq.(5.1) coincides with the limit distribution given in Khaniyev and Atalay (2010). It means that our result includes the results of Khaniyev and Atalay (2010) as a special case. On the other hand, in the real-world problem introduced in Section 1, the parameter α which characterizes degree of asymmetry of the triangular distribution, is equal to 0.85. Hence, the limit distribution for real-world model in Section 1 can be written as follows:

$$R(x) = \begin{cases} \frac{10.2x - x^3}{12.58}; & x \in (0; 1.7] \\ 1 - \frac{(2-x)^3}{2.22}; & x \in (1.7; 2) \end{cases} \tag{5.2}$$

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