

Determination of the laplace transform for the first falling moment to zero level of a semi-Markov random process

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Abstract: One of the important problems of stochastic process theory is to define the Laplace transformations for the distribution of this process. With this purpose, we will investigate a semi-Markov random processes with positive tendency and negative jump in this article. The first falling moment into the zero-level of this process is constructed as mathematically and the Laplace transformation of this random variable is obtained.

Keywords: Semi-Markov process, Laplace transform, Erlang distribution.

1 Introduction

In recent years, random walks with one or two barriers are used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, mathematical biology etc. Many good monographs in this field exist in literature (see references [1]-[4] and etc.).

In particular, a number of very interesting problems of stock control, queues and reliability theories can be expressed by means of random walks with two barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite volumes or queueing systems with finite waiting time or sojourn time by means of random walks with two delaying barriers. Furthermore, the functioning of stochastics systems with spare equipment can be given by random walks with two barriers, one of them is delaying and the other one is any type barrier.

Numerous studies have been done about step processes of semi-Markovian random walk with two barriers of their practical and theoretical importance. But in the most of these studies the distribution of the process has free distribution. Therefore, the obtained results in this case are cumbersome and they will not be useful for applications ([1], [2], [3], [4] and etc.).

The investigations of the distributions for the processes of semi-Markov random process have an important value in the random process theory. There are number of works devoted to definition of the Laplace transforms for the distribution of the first passage of the zero level. Some authors used the asymptotic, factorization and etc. methods (see references [1], [2] and [5]). But other authors narrowing the class of distributions of walking found the evident form for Laplace transforms for distributions and its main characteristics (see [3], [4]).

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The purpose of the present article is to find the Laplace transforms for Erlang distribution of the semi-Markov random processes with positive tendency and negative jump. The first passage of the zero level of the semi-Markov process with positive tendency and negative jumps will be included as a random variable. The Laplace transform for the distribution of this random variable is defined.

2 Construction of the process

Suppose that $\{(\xi_i, \eta_i)\}, i = 1, 2, 3, \dots$ is a sequence of identically and independently distributed pairs of random variables, defined on any probability space such that ξ_i 's are positive valued, i.e., $P\{\xi_i > 0\} = 1, i = 1, 2, 3, \dots$. In addition, the random variables ξ_i and η_i are mutually independent as well. Also let us denote the distribution function of ξ_1 and η_1

$$\Phi(t) = P\{\xi_1 < t\}, F(x) = P\{\eta_1 < x\}, t \in R^+, x \in R, \tag{1}$$

respectively. By the means of these random variables, we can construct the following process:

$$X(t) = z + t - \sum_{i=1}^{k-1} \eta_i, \text{ if } \sum_{i=1}^{k-1} \xi_i \leq t < \sum_{i=1}^k \xi_i, k = (1, \infty), \tag{2}$$

where the number $z > 0$ is given. $X(t)$ process is a (asymptotic) semi-Markov random processes with positive tendency and negative jumps. One of the realizations of the process $X(t)$ will be in the following form

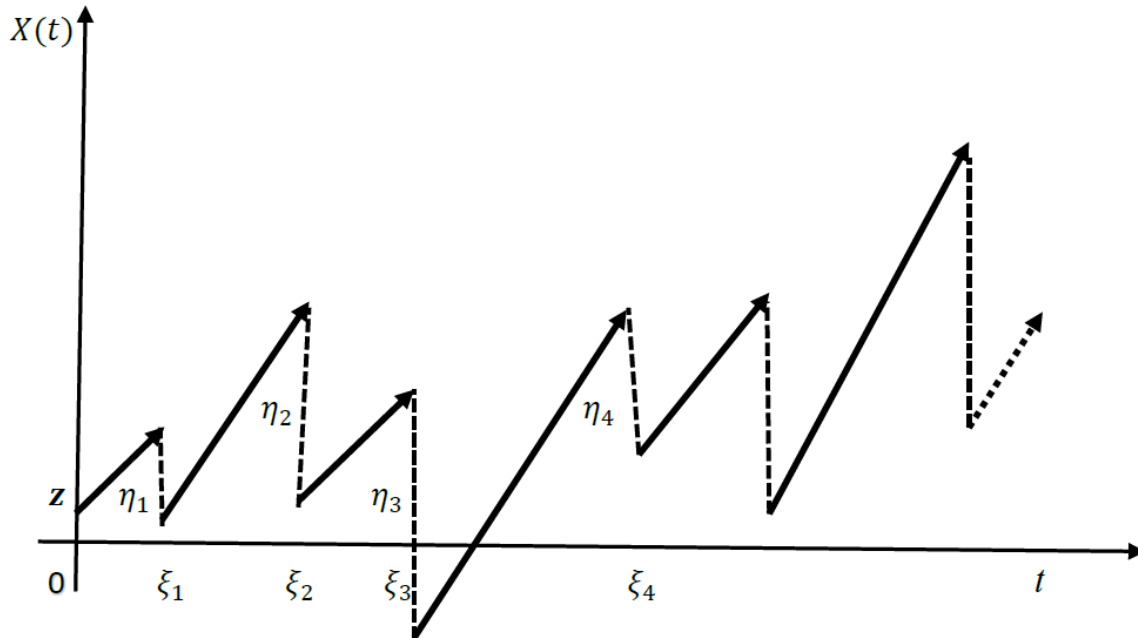


Fig. 1: A View of the Semi-Markov Random Process with positive tendency and negative jumps.

Now, we introduce the random variable as below.

$$\tau_0 = \tau_0(w) = \inf\{t : X(t) \leq 0\}, \tag{3}$$

where τ_0 is the first falling time to the zero-level of the process $X(t)$. Furthermore, we get $\tau_0 = +\infty$ when $X(t) > 0$ for every t . It is obvious that the random variable τ_0 is the time of the first crossing of the the process $X(t)$ into the zero level. In this case τ_0 is rewritten as $\tau_0 = \min\{k \geq 1 : X(t) = 0\}$. Thus, for example, we can write

$$\tau_0 = \begin{cases} \xi_1, z + \xi_1 - \eta_1 < 0 \\ \xi_1 + T, z + \xi_1 - \eta_1 > 0 \end{cases}$$

where ξ_1 and T are identically and independently distributed random variables.

Note that τ_0 is important from a scientific and practical point of view and it is an important boundary functional of the process $X(t)$. This random variable play an important role in solving of most probability problems arising in control of random levels of stocks in a warehouse which is functioning according to the process $X(t)$. For this reason, the consideration with detailed of random variable τ_0 seems very interesting from scientific and practical point of view.

The aim of the present work is to determine the Laplace transform of the distribution function of the random variable τ_0 . Let us denote the Laplace transform of the distribution function of the random variable τ_0 by

$$L(\theta) = E[e^{-\theta\tau_0}], \theta > 0, \tag{4}$$

and let us denote the Laplace transform of the conditional distribution of the random variable τ_0 by

$$L(\theta|z) = E[e^{-\theta\tau_0} | X(0) = z], \theta > 0, z \geq 0. \tag{5}$$

According to the total probability formula, we can write

$$L(\theta) = \int_{z=0}^{\infty} L(\theta|z) dP(X(0) < z).$$

Finally, let us denote the Laplace transformation of the random variable ξ_1 by

$$\varphi(\theta) = E[e^{-\theta\xi_1}].$$

3 Derivation of the Integral Equation for $L(\theta|z)$

In this section we give an integral equation for the Laplace transform of the conditional distribution of the random variable τ_0 . According to the formula of total probability, we have

$$\begin{aligned} L(\theta|z) &= E \left[e^{-\theta\tau_0} \mid X(0) = z \right] = \int_{\Omega} e^{-\theta t} P \{ dw \mid X(0) = z \} \\ &= \int_{\{w: z + \xi_1 - \eta_1 < 0\}} e^{-\theta\xi_1} P \{ dw \} + \int_{\{w: z + \xi_1 - \eta_1 > 0\}} e^{-\theta(\xi_1 + T)} P \{ dw \}. \end{aligned}$$

Applying some substitutions, we obtain that

$$\begin{aligned} E \left[e^{-\theta\tau_0} \mid X(0) = z \right] &= \int_{s=0}^{\infty} \int_{y=z+s}^{\infty} e^{-\theta s} P\{\xi_1 \in ds; \eta_1 \in dy\} + \int_{s=0}^{\infty} \int_{y=0}^{z+s} \int_{\beta=0}^{\infty} e^{-\theta(s+\beta)} P\{\xi_1 \in ds; \eta_1 \in dy; T \in d\beta\} \\ &= \int_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} \int_{y=z+s}^{\infty} P\{\eta_1 \in dy\} + \int_{s=0}^{\infty} e^{-\theta s} \int_{y=0}^{z+s} dP\{\eta_1 < y\} dP\{\xi_1 < s\} L(\theta|z+s-y) \\ &= \int_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} P\{\eta_1 > z+s\} + \int_{s=0}^{\infty} e^{-\theta s} \int_{\beta=z+s}^0 L(\theta|\beta) dP\{\eta_1 < z+s-\beta\} dP\{\xi_1 < s\}, \end{aligned}$$

or, equivalently,

$$L(\theta|z) = \int_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} P\{\eta_1 > z+s\} + \int_{s=0}^{\infty} e^{-\theta s} \int_{y=0}^{z+s} L(\theta|z+s-y) P\{\eta_1 \in dy\} P\{\xi_1 \in ds\}.$$

By substituting $z+s-y = \alpha$, we get the integral equation for $L(\theta|z)$.

$$L(\theta|z) = \int_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} P\{\eta_1 > z+s\} + \int_{s=0}^{\infty} e^{-\theta s} \int_{y=0}^{z+s} L(\theta|\alpha) P\{\eta_1 < z+s-y\} dP\{\xi_1 < s\}. \quad (6)$$

4 The solution of the Integral Equation for $L(\theta|z)$

The integral equation given in (6) for $L(\theta|z)$ can be solved by method of successive approximations for arbitrarily distributed random variables ξ_i and $\eta_i, i \geq 1$, but it is unsuitable for applications. On the other hand, this equation has a solution in explicit form in the classes of Erlang distributions. Therefore, this integral equation will be solved in the cases that the random variables ξ_1 and η_1 have an Erlang distribution of second order with the parameter λ and first order with the parameter μ , respectively. In this case, we have

$$\begin{aligned} P\{\xi_1(\omega) < t\} &= \left[1 - (1 + \lambda t)e^{-\lambda t} \right] \varepsilon(t), \lambda > 0, \\ P\{\zeta_1(\omega) < t\} &= [1 - e^{-\mu t}] \varepsilon(t), \mu > 0 \end{aligned}$$

where

$$\varepsilon(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

Thus, according to the total probability formula, we have the following integral equation for the Laplace transform of the conditional distribution of random variable τ_0

$$L(\theta|z) = \frac{\lambda^2 e^{-\mu z}}{(\lambda + \mu + \theta)^2} - \lambda^2 \mu e^{-\mu z} \int_{s=0}^{\infty} s e^{-(\lambda + \mu + \theta)s} \int_{\alpha=0}^{z+s} e^{\mu \alpha} L(\theta|\alpha) d\alpha ds \quad (7)$$

Now, we will receive a differential equation. For this aim, we will multiply both sides of equation (7) by $e^{\lambda z}$ and derive on z .

$$\mu L(\theta|z) + L'(\theta|z) = \lambda^2 \mu \int_{s=0}^{\infty} s e^{-(\lambda + \theta)s} L(\theta|z+s) ds$$

Then we will multiply both sides of last equation by $e^{-(\lambda+\mu+\theta)z}$ and derive on z . In this case we will receive the differential equation with 3-th construction

$$L'''(\theta|z) - [2(\lambda + \theta) - \mu]L''(\theta|z) + (\lambda + \theta)(\lambda + \theta - 2\mu)L'(\theta|z) + (2\lambda + \theta)\mu\theta L(\theta|z) = 0 \tag{8}$$

The general solution of integral equation (7) has the form

$$L(\theta|z) = C_1(\theta)e^{k_1(\theta)z} + C_2(\theta)e^{k_2(\theta)z} + C_3(\theta)e^{k_3(\theta)z} \tag{9}$$

where $k_i(\theta), i = 1, 2, 3$ are the roots of characteristic equation of

$$k^3(u) - [2(\lambda + \theta) - \mu]k^2(u) + (\lambda + \theta)(\lambda + \theta - 2\mu)k(u) + (2\lambda + \theta)\mu\theta L(\theta|z) = 0$$

By finding $C_i(\theta), i = 1, 2, 3$ from equation (6) we will get the following system of algebraic equations.

$$\begin{cases} L(\theta|0) = \frac{\lambda^2}{(\lambda+\mu+\theta)^2} + \lambda^2\mu \int_{s=0}^{\infty} se^{-(\lambda+\mu+\theta)s} \int_{\alpha=0}^0 e^{\mu\alpha} d\alpha ds \\ L'(\theta|0) = -\mu L(\theta|0) + \lambda^2\mu \int_{s=0}^{\infty} s^2 e^{-(\lambda+\theta)s} L(\theta|s) ds \\ L''(\theta|0) = \mu(\lambda + \theta)L(\theta|0) + (\lambda + \theta - \mu)L'(\theta|0) - \lambda^2\mu \int_{s=0}^{\infty} e^{-(\lambda+\theta)s} L(\theta|s) ds \end{cases} \tag{10}$$

From these conditions, taking into account expression (6), we obtain the following system of equations with respect to $C_i(\theta)$,

$$\begin{cases} \sum_{i=1}^3 \left[1 - \frac{\lambda^2\mu[2(\lambda+\theta)+\mu-k_i(\theta)]}{[\lambda+\theta-k_i(\theta)]^2[\lambda+\mu+\theta]^2} \right] C_i(\theta) = \frac{\lambda^2}{(\lambda+\mu+\theta)^2} \\ \sum_{i=1}^3 \left[\mu + k_i(\theta) - \frac{\lambda\mu}{\lambda+\theta-k_i(\theta)} \right] C_i(\theta) = 0 \\ \sum_{i=1}^3 \left[k_i^2(\theta) - (\lambda + \theta - \mu)k_i(\theta) - \mu(\lambda + \theta) + \frac{\lambda^2\mu}{\lambda+\theta-k_i(\theta)} \right] C_i(\theta) = 0 \end{cases} \tag{11}$$

Now we will proof linear dependence of this algebraic system. In this case, if we consider the following equalities

$$\begin{aligned} k_1(\theta) + k_2(\theta) + k_3(\theta) &= (\lambda + \theta) - \mu \\ k_1(\theta)k_2(\theta) + k_1(\theta)k_3(\theta) + k_2(\theta)k_3(\theta) &= 2(\lambda + \theta)(\lambda + \theta - \mu) \end{aligned}$$

and

$$k_1(\theta)k_2(\theta)k_3(\theta) = (\lambda + \theta)(\lambda + \theta - 2\mu)$$

then we can write

$$[\mu + k_1(\theta)][\mu + k_2(\theta)][\mu + k_3(\theta)] = \lambda^2\mu.$$

Therefore, after some transformations, the boundary conditions given the equation (11) take the following form

$$\begin{cases} \sum_{i=1}^3 (\lambda - k_i(\theta) + \theta)^2 C_i(\theta) = \lambda^2 \\ \sum_{i=1}^3 0 \times C_i(\theta) = 0 \\ \sum_{i=1}^3 0 \times C_i(\theta) = 0 \end{cases} \quad (12)$$

Thus, the system (11) is a linear dependent equations system, as

$$\begin{aligned} C_2(\theta) = C_3(\theta) = 0, \theta > 0, \\ C_1(\theta) = \frac{\lambda^2}{[\lambda + \theta - k_1(\theta)]^2}. \end{aligned}$$

So, the general solution of integral equation (6) will be as follows. This is the Laplace transform for relative distribution of random variable τ_0 .

Since random variable has a first order Erlang distribution with the parameter μ , the Laplace transform for non-relative distribution of random variable τ_0 will be

$$\begin{aligned} L(\theta) &= \int_{z=0}^{\infty} L(\theta|z) \lambda^2 z e^{-\lambda z} dz = \int_{z=0}^{\infty} C_1(\theta) e^{k_1(\theta)z} \lambda^2 z e^{-\lambda z} dz \\ &= C_1(\theta) \lambda^2 \int_{z=0}^{\infty} z e^{[k_1(\theta) - \lambda]z} dz = \frac{\lambda^2}{[\lambda - k_1(\theta)]^2} C_1(\theta) \end{aligned}$$

We now determine $E[\tau_0]$ and $D[\tau_0]$, expectation and standart diversion of the random variable τ_0 . Since is a Laplace transformation, it is written

$$E[\tau_0] = -L'(0) \quad \text{and} \quad D[\tau_0] = -L''(0) - [L'(0)]^2.$$

Therefore, we can write expectation and standard diversion of random variable τ_0 for $\lambda > 2\mu$, as follows.

$$\begin{aligned} E[\tau_0] &= -L'(0) = \frac{2(\lambda + \mu)}{\lambda(\lambda - 2\mu)} \\ E[\tau_0|z] &= \frac{2(1 + z\mu)}{\lambda - 2\mu} \end{aligned}$$

and

$$\begin{aligned} L''(0) &= \frac{8\mu^2}{\lambda^2(\lambda - 2\mu)^2} + \frac{2\mu}{\lambda(\lambda - \mu)(\lambda - 2\mu)} + \frac{2\lambda}{(\lambda - 2\mu)^3} \\ D[\tau_0] &= \frac{8\mu^2}{\lambda^2(\lambda - 2\mu)^2} + \frac{16\mu}{\lambda(\lambda - \mu)(\lambda - 2\mu)} + \frac{3\lambda}{(\lambda - 2\mu)^3} - \frac{4(\lambda + 2\mu)\mu}{\lambda^2(\lambda - 2\mu)^2} \\ D[\tau_0|z] &= \frac{2}{(\lambda - 2\mu)^2} + \frac{(2 + z\lambda)\mu}{(\lambda - 2\mu)^3}. \end{aligned}$$

5 Conclusion

In this article we have defined Laplace transforms for Erlang distribution of the first falling time to zero level of semi-Markov random process with positive tendency and negative jump. The obtained results can be applied in the theory of queuing, in insurance theory, in the theory of finance, and also in the theory of inventory management.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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