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On characterization of boundedness of superposition operators on the Maddox space $C_{r0}(p)$ of double sequences

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Abstract: In this paper, we discuss a scale of necessary and sufficient conditions for the local boundedness and boundedness of superposition operator $P_g : C_{r0}(p) \to \mathscr{L}(q)$, where $p = (p_{ks})$ and $q = (q_{ks})$ are bounded double sequences of positive numbers.

Keywords: Superposition operators, local boundedness, boundedness, double sequence spaces.

1 Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and real numbers, respectively. A real double sequence is a function acting from $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and briefly denoted by (x_{ks}). Let Ω denotes the space of all real double sequences with coordinatewise addition and scalar multiplication. Let $x = (x_{ks}) \in \Omega$ be any sequence. If, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{ks} - l| < \varepsilon$ for all $k, s \geq n_\varepsilon$, then real double sequence $x = (x_{ks})$ is said to be converging to $l \in \mathbb{R}$ in *Pringsheim's sense* and denoted by $p - \lim x_{ks} = l$. Let the double sequence $x = (x_{ks})$ converges in Pringsheim's sense and the iterated limits $\lim_k x_{ks}$ and $\lim_s x_{ks}$ exist. Then the double sequence $x = (x_{ks})$ is called *regularly convergent* and denoted by $r - \lim x_{ks}$. By C_r, we denote the space of all regularly convergent double sequences. The Maddox space $C_{r0}(p)$ is defined by

$$C_{r0}(p) = \{x = (x_{ks}) \in \Omega : r - \lim |x_{ks}|^{p_{ks}} = 0\}$$

where $p = (p_{ks})$ is a bounded sequence of positive numbers. Also, $\|.\|_{C_{r0}(p)} : C_{r0}(p) \to \mathbb{R}$ is defined as

$$\|x\|_{C_{r0}(p)} = \sup_{k,s\in\mathbb{N}} |x_{ks}|^{\frac{p_{ks}}{M_1}},$$

where $M_1 = \max\left\{1, \sup_{k,s\in\mathbb{N}} p_{ks}\right\}$. The convergence of the partial sums sequence (s_{nm}) ,where $s_{nm} = \sum_{k=1}^{n}\sum_{s=1}^{m} x_{ks}$ $(n, m \in \mathbb{N})$ implies that the double series $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} x_{ks}$ is convergent. By v, we denote convergence notions, i.e., in Pringsheim's sense or regularly convergent. If the partial sums sequence (s_{nm}) is convergent to a real number s in v-sense, i.e.

$$v - \lim_{n,m} \sum_{k=1}^{n}\sum_{s=1}^{m} x_{ks} = s,$$

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then the series $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} x_{ks}$ is called $v-$convergent and it's denoted by

$$\sum_{k,s=1}^{\infty} x_{ks} = s.$$

If the series $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} x_{ks}$ is $v-$convergent, then the $v-$limit of (x_{ks}) equals to zero. The remaining term of the series $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} x_{ks}$ is defined by

$$R_{nm} = \sum_{k=1}^{n-1}\sum_{s=m}^{\infty} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=1}^{m-1} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=m}^{\infty} x_{ks}. \tag{1}$$

and briefly denoted by

$$\sum_{\max\{k,s\}\geq N} x_{ks}$$

for $n = m = N$. It is known that if the series $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} x_{ks}$ is $v-$convergent, then the $v-$limit of the remaining term $\sum_{\max\{k,s\}\geq N} x_{ks}$ is zero.

The double sequence space $\mathscr{L}_p$ is defined as follows

$$\mathscr{L}_p := \left\{ x = (x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^p < \infty \right\}$$

and this space is a Banach space with the norm

$$\|x\|_p = \left(\sum_{k,s=1}^{\infty} |x_{ks}|^p \right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$. The Maddox space $\mathscr{L}(q)$ of double sequences is defined as

$$\mathscr{L}(q) = \left\{ x = (x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^{q_{ks}} < \infty \right\},$$

where $q = (q_{ks})$ is a bounded sequence of positive numbers. Also, $\|.\|_{\mathscr{L}(q)} : \mathscr{L}(q) \to \mathbb{R}$ is defined by

$$\|x\|_{\mathscr{L}(q)} = \sum_{k,s=1}^{\infty} |x_{ks}|^{\frac{q_{ks}}{M_2}},$$

where $M_2 = \max\left\{1, \sup_{k,s\in\mathbb{N}} q_{ks}\right\}$. For more details see [1],[2],[3],[7],[9],[12],[18].

Let X, Y be two double sequence spaces. A superposition operator P_g on X is a mapping from X into Ω defined by $P_g(x) = (g(k,s,x_{ks}))_{k,s=1}^{\infty}$, where $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies condition (1) in below. (1) $g(k,s,0) = 0$ for all $k,s \in \mathbb{N}$.

If $P_g(x) \in Y$ for all $x \in X$, we say that P_g acts from X into Y and write $P_g : X \to Y$ [13]. Also, we shall use some of the following conditions:

(2) $g(k,s,.)$ is continuous for all $k,s \in \mathbb{N}$;

(2′) $g(k,s,.)$ is bounded on every bounded subset of $\mathbb{R}$ for all positive integers k,s.

One can easily see that if the function $g(k,s,.)$ satisfies the propety (2), then g satisfies $(2')$. Also, if the function $g(k,s,.)$ is locally bounded on $\mathbb{R}$, then g satisfies $(2')$.

Boundedness of the superposition operators on some sequence spaces was studied by Samae [16], Sağır and Güngör [14] and Chew [4], [5], [6], [8], [10], [11], [18]. Sağır and Güngör [15] characterized the superposition operators P_g on $C_{r0}(p)$ as follows

Theorem 1. *Let $g:\mathbb{N}^2\times\mathbb{R}\to\mathbb{R}$ satisfies $(2')$. Then $P_g: C_{r0}(p)\to\mathscr{L}_1$ if and only if there exist $\alpha>0$ and $(c_{ks})_{k,s=1}^{\infty}\in\mathscr{L}_1$ such that*

$$|g(k,s,t)|\leq c_{ks} \text{ whenever } |t|\leq\alpha$$

for all $k,s\in\mathbb{N}$.

Theorem 2. *Let $g:\mathbb{N}^2\times\mathbb{R}\to\mathbb{R}$. Then $P_g: C_{r0}(p)\to\mathscr{L}(q)$ if and only if there exist $N\in\mathbb{N}$ and $\alpha>0$ such that*

$$\sum_{\max\{k,s\}\geq N}\ \sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\infty.$$

2 Conclusion

2.1 Superposition Operators of $C_{r0}(p)$ into $\mathscr{L}_1$

Theorem 3. *Let $P_g: C_{r0}(p)\to\mathscr{L}_1$. Then P_g is locally bounded on $C_{r0}(p)$ if and only if g satisfies $(2')$.*

Proof. Suppose that g satisfies $(2')$ and let $z=(z_{ks})\in C_{r0}(p)$. By Theorem 1, there exist $(c_{ks})\in\mathscr{L}_1$ and $\alpha>0$ such that

$$|g(k,s,t)|\leq c_{ks} \tag{2}$$

whenever $|t|\leq\alpha$ for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. Let $x=(x_{ks})\in C_{r0}(p)$ satisfies the following relation;

$$\|z-x\|_{C_{r0}(p)}\leq\frac{\alpha^{\frac{p_{ks}}{M_1}}}{2}.$$

Thus, we have

$$\sup_{k,s\in\mathbb{N}}|z_{ks}-x_{ks}|^{\frac{p_{ks}}{M_1}}\leq\frac{\alpha^{\frac{p_{ks}}{M_1}}}{2}. \tag{3}$$

Since $r-\lim z_{ks}=0$, there exists $N\in\mathbb{N}$ such that $|z_{ks}|^{p_{ks}}\leq\frac{\alpha^{p_{ks}}}{2^{M_1}}$ for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. Hence,

$$\sup_{\max\{k,s\}\geq N}|z_{ks}|^{\frac{p_{ks}}{M_1}}\leq\frac{\alpha^{\frac{p_{ks}}{M_1}}}{2}. \tag{4}$$

Using the relations (3) and (4), we get

$$|x_{ks}|^{\frac{p_{ks}}{M_1}}\leq\sup_{\max\{k,s\}\geq N}|x_{ks}|^{\frac{p_{ks}}{M_1}}\leq\sup_{k,s\in\mathbb{N}}|z_{ks}-x_{ks}|^{\frac{p_{ks}}{M_1}}+\sup_{\max\{k,s\}\geq N}|z_{ks}|^{\frac{p_{ks}}{M_1}}<\alpha^{\frac{p_{ks}}{M_1}}$$

for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. From (1), we have that

$$|g(k,s,x_{ks})|\leq c_{ks}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. Therefore,

$$\sum_{\max\{k,s\} \geq N} |g(k,s,x_{ks})| \leq \sum_{\max\{k,s\} \geq N} c_{ks} \leq \sum_{k,s=1}^{\infty} c_{ks} = \|c_{ks}\|_1 . \tag{5}$$

Let $m_{ks} = \sup_{|t-z_{ks}| \leq \frac{\alpha}{2^{\frac{M_1}{p_{ks}}}}} |g(k,s,t)|$. Since g satisfies $(2')$, we have that $m_{ks} < \infty$ for all $k,s \in \mathbb{N}$ and so

$$|g(k,s,x_{ks})| \leq m_{ks} \tag{6}$$

for each $k,s \in \mathbb{N}$. By (5) and (6), we obtain

$$\begin{aligned} \left\|P_g(x)\right\|_1 &= \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})| = \sum_{k,s=1}^{N-1} |g(k,s,x_{ks})| + \sum_{\max\{k,s\} \geq N} |g(k,s,x_{ks})| \\ &\leq \sum_{k,s=1}^{N-1} m_{ks} + \sum_{k,s=1}^{\infty} c_{ks} = \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1 . \end{aligned}$$

Therefore,

$$\begin{aligned} \left\|P_g(x) - P_g(z)\right\|_1 &\leq \left\|P_g(x)\right\|_1 + \left\|P_g(z)\right\|_1 \\ &\leq \left\|P_g(z)\right\|_1 + \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1 . \end{aligned}$$

Let $\gamma = \left\|P_g(z)\right\|_1 + \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1$, then $\left\|P_g(x) - P_g(z)\right\|_1 \leq \gamma$. It means that P_g is locally bounded on $C_{r0}(p)$.

Conversely, let P_g be locally bounded on $C_{r0}(p)$. It is enough to show that g is locally bounded on $\mathbb{R}$. Let $y = (y_{ks})$ be as

$$y_{ks} = \begin{cases} a, & k = n \text{ and } s = m \\ \frac{1}{k} + \frac{1}{s}, & \text{others} \end{cases}$$

for all $k,s \in \mathbb{N}$ and $a \in \mathbb{R}$. Thus $y = (y_{ks}) \in C_{r0}(p)$. By the hypothesis, there exists $\alpha, \beta > 0$ such that

$$\left\|P_g(x) - P_g(y)\right\|_1 \leq \beta \tag{7}$$

whenever $\|x - y\|_{C_{r0}(p)} \leq \alpha$. If we take $x = (x_{ks})$ such that

$$x_{ks} = \begin{cases} b, & k = n \text{ and } s = m \\ \frac{1}{k} + \frac{1}{s}, & \text{others} \end{cases}$$

for all $k,s \in \mathbb{N}$ and $b \in \mathbb{R}$ with $|b - a| \leq \alpha^{\frac{M_1}{p_{ks}}}$, we have $x = (x_{ks}) \in C_{r0}(p)$. Thus, we get

$$\|x - y\|_{C_{r0}(p)} = \sup_{k,s \in \mathbb{N}} |x_{ks} - y_{ks}|^{\frac{p_{ks}}{M_1}} = |b - a|^{\frac{p_{ks}}{M_1}} \leq \alpha,$$

which means that $\left\|P_g(x) - P_g(y)\right\| \leq \beta$ by (7). Then, we obtain

$$|g(k,s,b) - g(k,s,a)| \leq \sum_{k,s=1}^{\infty} |g(k,s,x_{ks}) - g(k,s,y_{ks})| = \left\|P_g(x) - P_g(y)\right\| \leq \beta.$$

Since $b \in \mathbb{R}$ is arbitrary, $g(k,s,.)$ is locally bounded on $\mathbb{R}$.

Theorem 4. *Let $P_g : C_{r0}(p) \to \mathscr{L}_1$. Then P_g is bounded on $C_{r0}(p)$ if and only if for every $\beta > 0$ there exists a sequence $c(\beta) = c_{ks}(\beta) \in \mathscr{L}_1$ such that*

$$|g(k,s,t)| \leq c_{ks}(\beta)$$

whenever $|t|^{\frac{p_{ks}}{M_1}} \leq \beta$ for all $k,s \in \mathbb{N}$.

Proof. Suppose that the condition holds. Let $\beta > 0$ and $x = (x_{ks}) \in C_{r0}(p)$ such that $\|x\|_{C_{r0}(p)} \leq \beta$. Then, $|x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \beta$ for each $k,s \in \mathbb{N}$. By hypothesis, there exists a sequence $c(\beta) = c_{ks}(\beta) \in \mathscr{L}_1$ such that $|g(k,s,x_{ks})| \leq c_{ks}(\beta)$ for all $k,s \in \mathbb{N}$. Therefore, we get

$$\left\|P_g(x)\right\|_1 = \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})| \leq \sum_{k,s=1}^{\infty} c_{ks}(\beta) = \|c(\beta)\|_1.$$

Hence, P_g is bounded on $C_{r0}(p)$.

Conversely, assume that P_g is bounded on $C_{r0}(p)$. Let $\beta > 0$ and let define $A(\beta)$ and $c_{ks}(\beta)$ as follows

$$A(\beta) = \left\{ t \in \mathbb{R} : |t|^{\frac{p_{ks}}{M_1}} \leq \beta \right\},$$

and

$$c_{ks}(\beta) = \sup\{|g(k,s,t)| : t \in A(\beta)\}$$

for all $k,s \in \mathbb{N}$. Therefore, we have $|g(k,s,t)| \leq c_{ks}(\beta)$ whenever $|t|^{\frac{p_{ks}}{M_1}} \leq \beta$. Since g satisfies $(2')$, we get $0 \leq c_{ks}(\beta) < \infty$ for all $k,s \in \mathbb{N}$. Hence, for each $\varepsilon > 0$, there exists a sequence $x = (x_{ks}) \in C_{r0}(p)$ with $|x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \beta$ such that

$$c_{ks}(\beta) < |g(k,s,x_{ks})| + \frac{\varepsilon}{2^{k+s}} \tag{8}$$

for all $k,s \in \mathbb{N}$. By assumption, there exists $\alpha(\beta) > 0$ such that $\sum_{k,s=1}^{\infty} |g(k,s,x_{ks})| \leq \alpha(\beta)$. Then, by (2.7) we find

$$\sum_{k,s=1}^{\infty} c_{ks}(\beta) < \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})| + \sum_{k,s=1}^{\infty} \frac{\varepsilon}{2^{k+s}} \leq \alpha(\beta) + \varepsilon.$$

Hence, we obtain $\mathrm{c}(\beta) = c_{ks}(\beta) \in \mathscr{L}_1$. The proof is completed.

Example 1. Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$g(k,s,t) = \frac{|t|^{\frac{p_{ks}}{M_1}}}{4^{k+s}}$$

for all $k,s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Since g satisfies $(2')$, P_g is locally bounded on $C_{r0}(p)$ by Theorem 3. Let take $|t|^{\frac{p_{ks}}{M_1}} \leq \beta$ and $c_{ks}(\beta) = \frac{\beta}{4^{k+s}}$ for all $k,s \in \mathbb{N}$. Then, the condition in Theorem 4 holds and so the superposition operator P_g is bounded on $C_{r0}(p)$.

2.2 Superposition Operators of $C_{r0}(p)$ into $\mathscr{L}(q)$

Theorem 5. *Let $P_g : C_{r0}(p) \to \mathscr{L}(q)$. Then P_g is locally bounded on $C_{r0}(p)$ if and only if g satisfies $(2')$.*

Proof. Let g satisfies $(2')$ and let $z = (z_{ks}) \in C_{r0}(p)$. By Theorem 2, there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that

$$\sum_{\max\{k,s\}\geq N} \sup_{|t|\leq \alpha^{\frac{1}{p_{ks}}}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} < \infty. \tag{9}$$

Let $x = (x_{ks}) \in C_{r0}(p)$ such that $\|z-x\|_{C_{r0(p)}} \leq \frac{\alpha^{\frac{1}{M_1}}}{2^{\frac{p_{ks}}{M_1}}}$. Then, we have

$$\sup_{k,s\in\mathbb{N}} |z_{ks}-x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \frac{\alpha^{\frac{1}{M_1}}}{2^{\frac{p_{ks}}{M_1}}}. \tag{10}$$

Since $r-\lim z_{ks} = 0$, there exists $N \in \mathbb{N}$ such that $|z_{ks}|^{p_{ks}} \leq \frac{\alpha}{2^{p_{ks}}}$ for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. Hence,

$$\sup_{\max\{k,s\}\geq N} |z_{ks}| \leq \frac{\alpha^{\frac{1}{p_{ks}}}}{2}. \tag{11}$$

Using the relations (9) and (10), we get

$$|x_{ks}| \leq \sup_{\max\{k,s\}\geq N} |x_{ks}| \leq \sup_{k,s\in\mathbb{N}} |z_{ks}-x_{ks}| + \sup_{\max\{k,s\}\geq N} |z_{ks}| < \alpha^{\frac{1}{p_{ks}}}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. By (8), we have

$$\sum_{\max\{k,s\}\geq N} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \leq \sum_{\max\{k,s\}\geq N} \sup_{|t|\leq \alpha^{\frac{1}{p_{ks}}}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} < \infty \tag{12}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. Let $m_{ks} = \sup_{|t-z_{ks}|\leq \frac{\alpha^{\frac{1}{p_{ks}}}}{2}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}}$. Since g satisfies $(2')$, we can easily see that $m_{ks} < \infty$ for all $k,s \in \mathbb{N}$. Hence, we have

$$|g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \leq m_{ks} \tag{13}$$

for each $k,s \in \mathbb{N}$. By (12) and (13), we obtain

$$\begin{aligned} \|P_g(x)\|_{\mathscr{L}(q)} &= \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} = \sum_{k,s=1}^{N-1} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} + \sum_{\max\{k,s\}\geq N} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \\ &\leq \sum_{k,s=1}^{N-1} m_{ks} + \sum_{\max\{k,s\}\geq N} \sup_{|t|\leq \alpha^{\frac{1}{p_{ks}}}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} < \infty. \end{aligned}$$

Let $A = \sum_{k,s=1}^{N-1} m_{ks} + \sum_{\max\{k,s\}\geq N} \sup_{|t|\leq \alpha^{\frac{1}{p_{ks}}}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} < \infty$. Then, we get

$$\begin{aligned}\left\|P_g(x) - P_g(z)\right\|_{\mathscr{L}(q)} &\leq \left\|P_g(x)\right\|_{\mathscr{L}(q)} + \left\|P_g(z)\right\|_{\mathscr{L}(q)} \\ &\leq \left\|P_g(z)\right\|_{\mathscr{L}(q)} + A.\end{aligned}$$

Let $\gamma = \left\|P_g(z)\right\|_{\mathscr{L}(q)} + A$, then we have $\left\|P_g(x) - P_g(z)\right\|_{\mathscr{L}(q)} \leq \gamma$. Hence, P_g is locally bounded on $C_{r0}(p)$.

Conversely, assume that P_g is locally bounded on $C_{r0}(p)$. To complete the proof, it is sufficient that g is locally bounded on $\mathbb{R}$. Let $y = (y_{ks})$ be as follows

$$y_{ks} = \begin{cases} a, & k = n \text{ and } s = m \\ \frac{1}{k} + \frac{1}{s}, & \text{others} \end{cases}$$

for all $k,s \in \mathbb{N}$ and $a \in \mathbb{R}$. Then, it is clear that $y = (y_{ks}) \in C_{r0}(p)$. By the hypothesis, there exists $\alpha, \beta > 0$ such that

$$\left\|P_g(x) - P_g(y)\right\|_{\mathscr{L}(q)} \leq \beta, \tag{14}$$

whenever $\|x-y\|_{C_{r0}(p)} \leq \alpha$. Let $x = (x_{ks})$ be as follows

$$y_{ks} = \begin{cases} b, & k = n \text{ and } s = m \\ \frac{1}{k} + \frac{1}{s}, & \text{others} \end{cases}$$

for all $k,s \in \mathbb{N}$ and $b \in \mathbb{R}$ with $|b-a| \leq \alpha^{\frac{M_1}{p_{ks}}}$. Thus $x = (x_{ks}) \in C_{r0}(p)$. Hence, we get

$$\|x-y\|_{C_{r0}(p)} = \sup_{k,s\in\mathbb{N}} |x_{ks} - y_{ks}|^{\frac{p_{ks}}{M_1}} = |b-a|^{\frac{p_{ks}}{M_1}} \leq \alpha.$$

Therefore, by (3.6) we get $\left\|P_g(x) - P_g(y)\right\|_{\mathscr{L}(q)} \leq \beta$. Then, we obtain

$$\begin{aligned}|g(k,s,b) - g(k,s,a)|^{\frac{q_{ks}}{M_2}} &\leq \sum_{k,s=1}^{\infty} |g(k,s,x_{ks}) - g(k,s,y_{ks})|^{\frac{q_{ks}}{M_2}} \\ &= \left\|P_g(x) - P_g(y)\right\|_{\mathscr{L}(q)} \leq \beta.\end{aligned}$$

Since $b \in \mathbb{R}$ is arbitrary, $g(k,s,.)$ is locally bounded on $\mathbb{R}$.

Theorem 6. *Let $P_g : C_{r0}(p) \to \mathscr{L}(q)$. Then P_g is bounded on $C_{r0}(p)$ if and only if for every $\beta > 0$ there exists a sequence $c(\beta) = c_{ks}(\beta) \in \mathscr{L}$*

$$|g(k,s,t)|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta),$$

whenever $|t|^{\frac{p_{ks}}{M_1}} \leq \beta$ for all $k,s \in \mathbb{N}$.

Proof. Assume that the condition holds. Let $\beta > 0$ and let $x = (x_{ks}) \in C_{r0}(p)$ such that $\|x\|_{C_{r0}(p)} \leq \beta$. Then, $|x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \beta$ for each $k,s \in \mathbb{N}$. By hypothesis, there exists a sequence $c(\beta) = c_{ks}(\beta) \in \mathscr{L}_1$ such that $|g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta)$ for all $k,s \in \mathbb{N}$. Therefore, we have

$$\left\|P_g(x)\right\|_{\mathscr{L}(q)} = \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \leq \sum_{k,s=1}^{\infty} c_{ks}(\beta) = \|c(\beta)\|_1,$$

which implies that P_g is bounded on $C_{r0}(p)$.

Conversely, assume that P_g is bounded on $C_{r0}(p)$. Let $\beta > 0$. Let define $A(\beta)$ and $c_{ks}(\beta)$ as follows;

$$A(\beta) = \left\{ t \in \mathbb{R} : |t|^{\frac{p_{ks}}{M_1}} \leq \beta \right\}$$

and

$$c_{ks}(\beta) = \sup \left\{ |g(k,s,t)|^{\frac{q_{ks}}{M_2}} : t \in A(\beta) \right\}$$

for all $k,s \in \mathbb{N}$. Therefore, we get $|g(k,s,t)|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta)$ whenever $|t|^{\frac{p_{ks}}{M_1}} \leq \beta$. Since g satisfies $(2')$, it is easy seen that $0 \leq c_{ks}(\beta) < \infty$ for all $k,s \in \mathbb{N}$. Hence, for each $\varepsilon > 0$, there exists a sequence $x = (x_{ks}) \in C_{r0}(p)$ with $|x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \beta$ such that

$$c_{ks}(\beta) < |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} + \frac{\varepsilon}{2^{k+s}} \tag{15}$$

for all $k,s \in \mathbb{N}$. By assumption, there exists $\alpha(\beta) > 0$ such that $\sum_{k,s=1}^{\infty} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \leq \alpha(\beta)$. Then, we have

$$\sum_{k,s=1}^{\infty} c_{ks}(\beta) < \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} + \sum_{k,s=1}^{\infty} \frac{\varepsilon}{2^{k+s}} \leq \alpha(\beta) + \varepsilon.$$

Hence, we obtain $\mathrm{c}(\beta) = c_{ks}(\beta) \in \mathscr{L}_1$. This completes the proof.

Example 2. Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ be as follows

$$g(k,s,t) = \left(\frac{|t|^{p_{ks}}}{2^{k+s}} \right)^{\frac{M_2}{q_{ks}}}$$

for all $k,s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Since g satisfies $(2')$, P_g is locally bounded on $C_{r0}(p)$ by Theorem 5. Let take $|t|^{\frac{p_{ks}}{M_1}} \leq \beta$ and $c_{ks}(\beta) = \frac{\beta^{M_1}}{4^{k+s}}$ for all $k,s \in \mathbb{N}$. Then, the condition in Theorem 6 holds. Hence, the superposition operator P_g is bounded on $C_{r0}(p)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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