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A STUDY ON SET-CORDIAL GRAPHS

Sudev NADUVATH
Department of Mathematics, CHRIST (Deemed to be University), Bengaluru, INDIA

Abstract. For a non-empty ground set X, finite or infinite, the *set-valuation* or *set-labeling* of a given graph G is an injective function $f : V(G) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of the set X. In this paper, we introduce a new type of set-labeling, called set-cordial labeling and study the characteristics of graphs which admit the set-cordial labeling.

1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [11] and for further terminology on graph classes, we refer to [3]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

After the introduction of the notion of β-valuations of graphs in [8], studies on graph labeling problems have emerged as a major research area. It is estimated that more than two thousand research articles have been published since then. Interested readers may refer to [6] for a detailed literature and for further investigation on graph labeling problems.

As an extension of the number valuation of graphs, the notion of *set-indexers* of graphs has been introduced in [1] as an injective set-valued function $f : V(G) \to P(X)$ such that the induced function $f^* : E(G) \to P(X)-\{\emptyset\}$, defined by $f^*(uv) = f(u)*f(v)$ is also injective, where X is a non-empty set, $P(X)$ is the power set of X and $*$ is a binary operation between the elements of $P(X)$. Note that in the literature, $*$ is the symmetric difference of two sets. In [1], it is proved that every graph admits a set-indexer.

In this paper, a set-labeling of a graph G is an injective function $f : V(G) \to P(X)$. Motivated by the studies on the number valuations and set-valuations of

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sudevnk@gmail.com
0000-0001-9692-4053.

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graphs, mentioned above, in this paper, we introduce a particular type of set-labeling called set-cordial labeling and study the characteristics of graphs which admit this type of labeling.

2. Set-Cordial Graphs

We define the notion of the set-cordial labeling of a graph as follows:

Definition 1. Let X be a non-empty set and $f : V(G) \to P(X)$ be a set-labeling defined on a graph G. Then, f is said to be a *strict set-cordial labeling* or simply, a *set-cordial labeling* of G if $|f(v_i)| - |f(v_j)| = \pm 1$ for all $v_iv_j \in E(G)$. A graph which admits a set-cordial labeling is called a *set-cordial graph.*

Definition 2. The minimum cardinality of a ground set X with respect to which a given graph G admits a set-cordial labeling is called *the set-cordiality index* of G, denoted by $\varsigma(G)$.

An illustration of set-cordial graphs is provided in Figure 1.

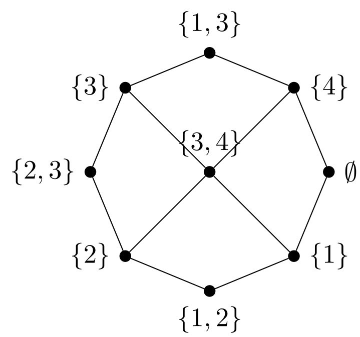

Figure 1. An illustration to a set-cordial graph.

In Figure 1, it can be noticed that the set-cordial index of the graph G is 4 as the minimal ground set is $X = \{1, 2, 3, 4\}$.

Next, we discuss the admissibility of set-cordial labeling by certain fundamental graph classes. In order to consider set-cordial labelings on paths on n vertices, we first show that the hypercube graph Q_n contains a Hamiltonian path.

Lemma 1. *Every hypercube graph Q_n contains a Hamiltonian path. Furthermore, if $n \geq 2$, then Q_n has a Hamiltonian cycle.*

Proof. We first observe that $Q_1 = K_2$ and hence Q_2 itself is a Hamiltonian path. For any positive integer $n \geq 2$, let $v_1 - v_2 - \ldots - v_{2^{n-1}}$ be the list of vertices in a Hamiltonian path in Q_{n-1}. Then, the list of vertices

$$(v_1, 0), (v_2, 0), \ldots, (v_{2^{n-1}}, 0), (v_{2^{n-1}}, 1), (v_{2^{n-1}-1}, 1), \ldots, (v_2, 1), (v_1, 1)$$

is a Hamiltonian path in Q_n, and the list of vertices

$$(v_1, 0), (v_2, 0), \ldots, (v_{2^{n-1}}, 0), (v_{2^{n-1}}, 1), (v_{2^{n-1}-1}, 1), \ldots, (v_2, 1), (v_1, 1), (v_1, 0)$$

is a Hamiltonian cycle in Q_n as required. □

Recall that a connected bipartite graph G with bipartition (X, Y), is called *Hamilton-laceable* (see [9]), if it has a $u-v$ Hamiltonian path for all pairs of vertices $u \in X$ and $v \in Y$. The hypercube Q_n is a bipartite Cayley graph on the Abelian group $\mathbb{Z}_2^n = \prod_n \mathbb{Z}_2$ with the natural generating set $S = \{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 0, 1)\}$. It is proved in [4] that a connected bipartite Cayley graph on an Abelian group is Hamiltonian laceable.

In view of the above-mentioned concepts, the following theorem discusses the admissibility of set-cordial labeling by a path and the corresponding set-cordiality index.

Theorem 1. *Every path P_n is set-cordial. Furthermore, $\varsigma(P_n) = \lceil \log_2 n \rceil$.*

Proof. Let P_n denotes a path of order n, whose vertices are consecutively named by $v_1, v_2, \ldots, v_n$. Let $X = \{x_1, x_2, \ldots, x_{n-1}\}$ be the ground set for labeling. Start labeling the vertex v_1 by the empty set $\emptyset$. For $2 \leq i \leq n$, label vertices v_i by the set $\{x_1, x_2, \ldots, x_{i-1}\}$. Clearly, $f(v_{i+1}) - f(v_i) = \{x_{i-1}\}$ for $0 \leq i \leq n - 1$. Therefore, f is a set-cordial labeling of P_n.

Let $k = \lceil \log_2 n \rceil$. Then $n \leq 2^k < 2n$. By Lemma 1, let $v_1, v_2, \ldots, v_{2^k}$ be the list of vertices in a Hamiltonian path in the hypercube Q_k. Let $X = \{1, 2, 3, \ldots, k\}$. We can identify each vertex of Q_k with a unique element in $P(X)$ and hence we identify the path P_n with the subpath $v_1 - v_2 - \ldots - v_n$ in Q_k. Thus, the set-labeling on P_n given by $f(v_i) = v_i$ is a set-cordial labeling on P_n. Since $n > 2^{k-1}$, there is no set-labeling on P_n that uses a ground set with fewer than k elements. Hence, $\varsigma(Pn) = k = \lceil \log_2 n \rceil$. This completes the proof. □

Theorem 2. *A graph G admits a set-cordial labeling if and only if G is bipartite.*

Proof. Let G be a bipartite graph with bipartition (X, Y). Choose the set $\mathbb{N}$ of natural numbers as the ground set for labeling. For any positive integer k, assign distinct k-element subsets of $\mathbb{N}$ to distinct vertices in X and distinct $(k+1)$-element subsets of $\mathbb{N}$ to distinct vertices in X. Clearly, this labeling is a set-cordial labeling of G.

Conversely, assume that G is a set-cordial graph and let $f : V(G) \to P(A)$ be a set-cordial labeling on G. Let X and Y be the partite sets of G defined by

$$\begin{aligned} X &= \{v \in V(G) : |f(v)| \text{ is even; } \textit{and}\} \\ Y &= \{v \in V(G) : |f(v)| \text{ is odd}\}. \end{aligned}$$

Let $u, v \in X$. Since $f(u)$ and $f(v)$ have an even number of elements, $|f(u)| - |f(v)|$ is even. Thus, $|f(u)| - |f(v)| \neq \pm 1$. Hence, X is an independent set. A similar argument shows that Y is also an independent set. Since $V(G) = X \cup Y$, G is bipartite, completing the proof. □

The following theorem characterises the cycles which admit set-cordial labeling.

Theorem 3. *A cycle C_n admits a set-cordial labeling if and only if n is even. Furthermore, $\varsigma(P_n) = \lceil \log_2 n \rceil$.*

Proof. First part of the theorem is an immediate consequence of Theorem 2. Hence, we shall now determine the set-cordiality index of cycles. By Lemma 1, let $v_1, v_2, \ldots, v_{2^k}$ be the list of vertices in a Hamiltonian path in the hypercube Q_k. Let $X = \{1, 2, 3, \ldots, k\}$.

Let $n = 2m, m \in \mathbb{N}_0$ and $k = \lceil \log_2 n \rceil$. Then, by Lemma 1, we have a list of vertices

$$(v_1, 0), (v_2, 0), \ldots, (v_{2^{k-1}}, 0), (v_{2^{k-1}}, 1), (v_{2^{k-1}-1}, 1), \ldots, (v_2, 1), (v_1, 1), (v_1, 0),$$

which are in a Hamiltonian path in Q_k, where $v_1, v_2, \ldots, v_{2^{k-1}}$ are the vertices in the Hamiltonian path in the hypercube Q_{k-1}. Also, we can identify a cycle of length $n = 2m$ in Q_k, whose vertices are

$$(v_1, 0), (v_2, 0), \ldots, (v_{m-1}, 0), (v_m, 0), (v_m, 1), (v_{m-1}, 1), \ldots, (v_2, 1), (v_1, 1), (v_1, 0).$$

Now, let $X = \{1, 2, 3, \ldots, k\}$. As explained in the proof of Theorem 1, we can identify each vertex of Q_k with a unique element in $P(X)$ and hence we identify the cycle C_n with the sub-cycle in Q_k. Thus, the set-labeling on C_n given by $f(v_i, j) = (v_i, j)$ is a set-cordial labeling on C_n. Since $n > 2^{k-1}$, in this case also, we have no set-labeling on C_n that uses a ground set with fewer than k elements. Hence, $\varsigma(Cn) = k = \lceil \log_2 n \rceil$, completing the proof. □

In view of Theorem 2, we notice that graphs consisting of odd cycles will not admit set-cordial labelings. Therefore, the fundamental graph classes like wheel graphs, friendship graphs and helm graphs do not admit a set-cordial labeling. Also, we note that a complete graph K_n admits a set-cordial labeling if and only if $n \leq 2$.

Suppose that a and b are positive integers such that $a \leq b$. Let $\alpha = \alpha(a, b)$ be the smallest positive integer such that

$$a \leq \binom{2\alpha}{\alpha} \text{ and } b \leq \binom{2\alpha}{\alpha - 1} + \binom{2\alpha}{\alpha + 1}.$$

Similarly, define $\beta = \beta(a, b)$ as the smallest positive integer such that

$$a \leq \binom{2\beta + 1}{\beta + 1} \text{ and } b \leq \binom{2\beta + 1}{\beta} + \binom{2\beta + 1}{\beta + 2}.$$

Using the above notations,the set-cordiality index of a complete bipartite graph is determined in the following theorem.

Theorem 4. *A complete bipartite graph $K_{a,b}$, where $a \leq b$, admits a set-cordial labeling. Furthermore,* $\varsigma(K_{a,b}) = \min\{2\alpha, 2\beta + 1\}$.

Proof. Let (A, B) be the bipartition of $K_{a,b}$ such that $|A| = a \leq |B| = b$. Assume that f is a set-cordial labeling of $K_{a,b}$ with respect to the minimal ground set X. Then, $f(v_i)$ can be an empty set, a single set or a 2-element set. We try to label the vertices of A by singleton subsets of the ground set X and label one vertex of B with empty set and other vertices by 2-element subset of X. This labeling is possible only when $b - 1$ is less than or equal to the number of 2-element subsets of the set $\bigcup_{v \in A} f(v)$. If this condition holds, then f is a set-cordial labeling which yields the minimum ground set $\bigcup_{v \in A} f(v)$. If this condition does not hold, we cannot label the vertices in A by singleton subsets of X and as a result, the vertices of B must be labeled by singleton subsets of X. In this case, $a - 1$ will be less than the number 2-element combinations of the set $\bigcup_{v \in B} f(v)$ and f will be a set-cordial labeling of $K_{a,b}$.

Now, we shall determine the set-cordiality number of $K_{a,b}$. Here, the following two cases are to be addressed.

Case-1: Let n be even, say $n = 2m, m \in \mathbb{N}_0$. Let X be a set containing $n = 2m$ elements, and let $f : V(K_{a,b}) \to P(X)$ be a set-cordial labeling on $K_{a,b}$. Suppose that there exists a vertex u_0 in one partite set of $K_{a,b}$ such that $|f(u_0)| = k$, and there exist vertices v_0 and w_0 in the other partite set of $K_{a,b}$ such that $|f(v_0)| = k-1$ and $|f(w_0)| = k + 1$. Since $|f(u)| - |f(v_0)| = \pm 1$ and $|f(u)| - |f(w_0)| = \pm 1$ for all u in the first partite set, we have $|f(u)| = k$, for all u in the first partite set. Since $|f(v)| - |f(u_0)| = \pm 1$, for all v in the second partite set, we have either $|f(v)| = k-1$ or $|f(v)| = k + 1$. Since $a \leq b$ and

$$\binom{2m}{k} \leq \binom{2m}{k-1} + \binom{2m}{k+1},$$

we have

$$a \leq \binom{2m}{k} \text{ and } b \leq \binom{2m}{k-1} + \binom{2m}{k+1}.$$

Since for all $1 \leq k \leq 2m - 1$,

$$\binom{2m}{k} \leq \binom{2m}{m},$$

we have $k = m$. Thus, $\varsigma(K_{a,b}) = 2m = 2\alpha$.

Case-2: Let n be odd, say $n = 2m + 1, m \in \mathbb{N}_0$. Let X be a set containing $n = 2m+1$ elements and let $f : V(K_{a,b}) \to P(X)$ be a set-cordial labeling on $K_{a,b}$. An argument similar to that in the above paragraph shows that we have $|f(u)| = k$, for all u in one partite set, and either $|f(v)| = k - 1$ or $|f(v)| = k + 1$ for all v in the other partite set. Since $a \leq b$ and

$$\binom{2m+1}{k} \leq \binom{2m+1}{k-1} + \binom{2m+1}{k+1},$$

we have

$$a \leq \binom{2m+1}{k} \text{ and } b \leq \binom{2m+1}{k-1} + \binom{2m+1}{k+1}.$$

Since for all $1 \leq k \leq 2m$,

$$\binom{2m+1}{k} \leq \binom{2m+1}{m+1},$$

we have $k = m+1$. Thus, $\varsigma(K_{a,b}) = 2m+1 = 2\beta+1$. From, the above two cases, we have $\varsigma(K_{a,b}) = \min\{2\alpha, 2\beta+1\}$, completing the proof. □

Figure 2 illustrates a set-cordial labeling of a complete bipartite graph $K_{6,7}$, with respect to the ground set $X = \{1, 2, 3, 4\}$.

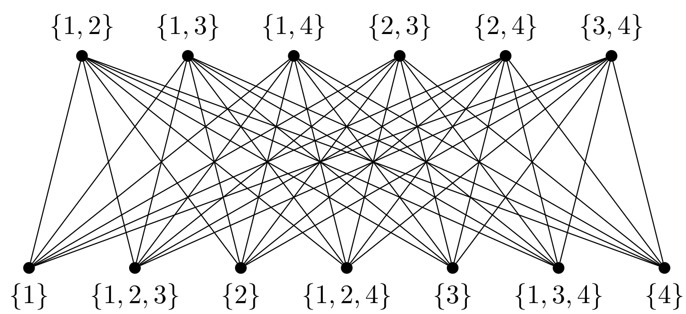

FIGURE 2. An illustration to a set-cordial labeling of $K_{6,7}$.

3. Glutting Number of a Graph

As a consequence of Theorem 2, non-bipartite graphs do not admit a set-cordial labeling. But, by the removal of certain edges from the graph will make the graph set-cordial. Hence, we have the following notion:

Definition 3. The *glutting number* of a graph G, denoted by $\xi(G)$, is the minimum number of edges of G to be removed so that the reduced graph admits a set-cordial labeling.

In view of Theorem 2, we note that the glutting number of a bipartite graph is 0. Therefore, $\xi(P_n) = 0$.
The following discusses the glutting number of a cycle C_n.

Proposition 1.

$$\xi(C_n) = \begin{cases} 0; & \textit{if } n \textit{ is even} \\ 1; & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. The proof is straight forward from Theorem 3. □

We shall now discuss the glutting number of certain fundamental graph classes. Recall that a wheel graph is defined by $W_{1,n} = K_1 + C_n$. The following result discusses the glutting number of a wheel graph.

Proposition 2. $\xi(W_{1,n}) = \begin{cases} \frac{n}{2}; & \textit{if } n \textit{ is even;} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. Note that every edge incident on the central vertex of $W_{1,n}$ is contained in exactly two triangles of $W_{1,n}$. So, removal of any such edge will result in the removal of two triangles in W_n. Also, there are n triangles in $W_{1,n}$. Here, we have to address the following two cases:

Case-1: Let n be even. Then, we need to remove $\frac{n}{2}$ edges incident on the central vertex to make the graph triangle free. Then, the reduced graph has girth 4 and has no odd cycles. Hence, in this case, $\xi(W_{1,n}) = \frac{n}{2}$.

Case-2: Let n be odd. Then, the outer cycle C_n is an odd cycle and hence one edge, say e, must be removed from C_n. Now, there exist $n-1$ triangles in the graph $W_n - e$. Since, $n-1$ is even, we need to remove $\frac{n-1}{2}$ edges from $W_n - e$ to make it triangle free. After the removal of this much edges, the reduced graph has girth 4 and has no odd cycles (see Figure 3, for example). Therefore, in this case, $\xi(W_n) = 1 + \frac{n-1}{2} = \frac{n+1}{2}$. □

A *helm graph* $H_{1,n}$ is the graph obtained from a wheel graph $W_{1,n}$ by attaching one pendant edge to each vertex of the outer cycle C_n of W_n. Then, we have

Proposition 3. $\xi(H_{1,n}) = \begin{cases} \frac{n}{2}; & \textit{if } n \textit{ is even;} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. The proof is exactly as in the proof of Theorem 2. □

A *closed helm* $CH_{1,n}$ is the graph obtained from a helm graph H_n by joining the pendant vertices of H_n so as to form an outer cycle of length n. Then, we have

Proposition 4. $\xi(H_n) = \begin{cases} \frac{n}{2}; & \textit{if } n \textit{ is even;} \\ \frac{n+3}{2} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. In CH_n, the central vertex is contained in all triangles. Hence, the only thing to be noted here is that if n is odd, we need to remove one edge each from inner and outer cycles. Then, the proof is exactly as in the proof of Theorem 2. □

The following theorem determines the glutting number of a complete graph K_n.

Theorem 5. $\xi(K_n) = \begin{cases} \frac{1}{4}n(n-2); & \textit{if } n \textit{ is even;} \\ \frac{1}{4}(n-1)^2; & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. Consider the complete graph K_n. Let G be a spanning subgraph of K_n such that $\xi(K_m) = |E(K_n)| - |E(G)|$. By Theorem 2, G is bipartite. Since $|E(K_n)| - |E(G)|$ is a minimum among all bipartite spanning subgraphs G of K_n, G is a complete bipartite spanning subgraph of K_n. Let A and B be partite vertex sets of G such that $|A| = k$ and $|B| = m - k$. Since A and B are independent sets in G, we have

$$\begin{aligned} \xi(K_n) &= \frac{1}{2}k(k-1) + \frac{1}{2}(n-k)(n-k-1) \\ &= \frac{n^2}{4} - \frac{n}{2} + \left(k - \frac{n}{2}\right)^2 \end{aligned}$$

Here, we have to address the following cases:

Case-1: Let n even. Thus, there exists a positive integer m such that $n = 2m$. Then, $\xi(K_n) = n^2 - n + (k-n)^2$. This value is a minimum when $k = m$. Thus, $\xi(K_n) = m^2 - m = \frac{n^2-n}{4}$.

Case-2: Suppose n is odd. Let m be the positive integer such that $n = 2m+1$. Then, $\xi(K_n) = n^2 - \frac{1}{4} + (k - n - \frac{1}{2})^2$. This value is a minimum when either $k = m$ or $k = m+1$. Thus $\xi(K_n) = m^2 = \frac{(n-1)^2}{4}$. This completes the proof. □

4. Some Variations of Set-Cordial Labeling

Definition 4. Let X be a non-empty set and $f : V(G) \to X$ be a set-labeling defined on a graph G. Then, f is said to be a *weakly set-cordial labeling* of G if $||f(v_i)| - |f(v_j)|| \leq 1$ for all $v_iv_j \in E(G)$. A graph which admits a set-cordial labeling is called a *weakly set-cordial graph.*

Theorem 6. *Every graph G admits a weakly set-cordial labeling.*

Proof. If G is bipartite, the theorem follows by Theorem 2. So, let G be a non-bipartite graph. Let I and be a maximal independent of G. Then, it is possible to choose the ground set X, sufficiently large, in such a way that

(i) all vertices in $G - I$ can be labeled by distinct singleton subsets of X,
(ii) one vertex of I is labeled by the empty set and other vertices can be labeled by distinct 2-element subsets of X.

Clearly, this labeling will be a set-cordial labeling of G, completing the proof. □

Observation 7. It can be noted that the glutting number of G is equal to the number of edges uv in G having $||f(u)| - |f(v)|| = 0$, with respect to a weakly set-cordial labeling f.

Figure 3 depicts a weakly set-cordial labeling of a wheel graph. The dashed lines represent the edges uv with $||f(u)| - |f(v)|| = 0$.

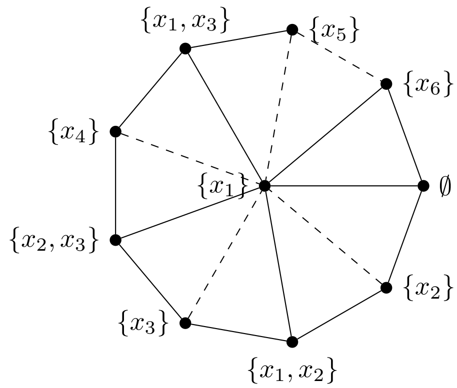

FIGURE 3. A weakly set-cordial labeling of a wheel graph.

5. Conclusion

In this article, we have introduced a particular type of set-labeling, called set-cordial labeling, of graphs and discussed certain properties of graphs which admits this type type of labeling. A couple of new graph parameters, related to the set-cordial labeling have also been introduced. These graph parameters seem to be promising for further studies. The set-cordial labeling of the operations, products and certain derived graphs of given set-cordial graphs can also be studied in detail. The newly introduced parameters can also be studied. All these facts highlight the wide scope for further research in this area.

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References

[1] Acharya, B. D., Set-Valuations and Their Applications, MRI Lecture Notes in Applied Mathematics, The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1983.
[2] Bondy, J. A., Murty, U. S. R., Graph Theory, Springer, New York, 2008.

[3] Brandstädt, A., Le, V. B., Spinrad, J. P., Graph Classes: A Survey, SIAM, Philadelphia, 1999.
[4] Chen, C. C., Quimpo, N. F., On Strongly Hamiltonian Abelian Group Graphs, In: McAvaney K.L., Combinatorial Mathematics VIII, Lecture Notes in Mathematics, vol 884, Springer, Berlin, Heidelberg, 1981.
[5] Deo, N., Graph Theory with Application to Engineering and Computer Science, Prentice Hall of India, New Delhi, 1974.
[6] Gallian, J. A., A dynamic survey of graph labeling, *Electron. J. Combin.*, 2020, # DS-6.
[7] Harary, F., Graph Theory, Narosa Publications, New Delhi, 2001.
[8] Rosa, A., On Certain Valuation of the Vertices of a Graph, Theory of Graphs, Gordon and Breach., Philadelphia, 1967.
[9] Simmons, G. J., Almost All n-Dimensional Rectangular Lattices are Hamilton-Laceable, Proc. 9th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Utilitas Mathematica Publishing, 1978, 649-661.
[10] Weisstein, E. W., CRC Concise Encyclopedia of Mathematics, CRC Press, Boca Raton, 2011.
[11] West, D. B., Introduction to Graph Theory, Pearson Education, Delhi, 2001.
[12] Information System on Graph Classes and Their Inclusions, `http://www.graphclasses.org/smallgraphs`.