

GEÇİKEN DEĞİŞKENLİ BİR SINIR DEĞER PROBLEMİNİN YAKLAŞIK ÇÖZÜMÜ ÜZERİNE

ON APPROXIMATE SOLUTION OF A BOUNDARY VALUE PROBLEM WITH RETARDED ARGUMENT

Seda İĞRET ARAZ^{1*}, Arzu AYKUT²

Atatürk Üniversitesi, Fen Fakültesi, Matematik Bölümü, 25240, Erzurum
Atatürk Üniversitesi, Fen Fakültesi, Matematik Bölümü, 25240, Erzurum

Geliş Tarihi: 07 Kasım 2013 **Kabul Tarihi:** 24 Nisan 2014

ÖZET

Bu çalışmada geciken değişkenli bir sınır değer probleminin çözümü için basit ardışık yaklaşıklar metodunu kullandık:

$$\left. \begin{aligned} x''(t) + a(t)x(t - \tau(t)) &= f(t), \quad (0 \leq t \leq T) \\ x(t) = \varphi(t) \quad (\lambda_0 \leq t \leq 0), \quad x(T) &= x(c) \quad (0 < c < T) \end{aligned} \right\} \quad (1)$$

Burada, $(0 \leq t \leq T)$ ve $a(t) \geq 0, f(t) \geq 0, \tau(t) \geq 0, (0 \leq t \leq T)$ ve $\varphi(t), (\lambda_0 \leq t \leq 0)$ bilinen sürekli fonksiyonlardır.

Anahtar Kelimeler: Fredholm-Volterra integral denklemler, geciken değişkenli diferansiyel denklemler, basit ardışık yaklaşıklar metodu.

ABSTRACT

In this paper, we used ordinary successive approximation method for the solution of a boundary value problem with retarded argument:

$$\left. \begin{aligned} x''(t) + a(t)x(t - \tau(t)) &= f(t), \quad (0 < t < T) \\ x(t) = \varphi(t) \quad (\lambda_0 \leq t \leq 0) \quad x(T) &= x(c) \quad (0 < c < T) \end{aligned} \right\} \quad (1)$$

where $(0 \leq t \leq T)$ and $a(t) \geq 0, f(t) \geq 0, \tau(t) \geq 0, (0 \leq t \leq T)$ and $\varphi(t), (\lambda_0 \leq t \leq 0)$ are known continuous functions.

Key words: Fredholm-Volterra integral equations, differential equation with retarded argument, ordinary successive approximation method.

1. INTRODUCTION

Differential equations with retarded argument appeared as far back as the eighteenth century in connection with the solution of the

*Sorumlu Yazar: seda.igret@atauni.edu.tr

problem of Euler on the investigation of the general for of curves similar to their own evolutes. However, until recently there were no foundations of the general theory of differential equations with retarded arguments; there was not even a precise formulation of initial value problems. This was first done in the dissertation of (Myskis, 1946).

A common method used for the analytical solution of the boundary value problems is the integral equation method (Norkin, 1972; Mamedov, 1994). By this method, we obtain an integral equation that is equivalent to the boundary value problem (1) and the solution of the integral equation is equivalent to the solution of the boundary value problem. The equivalent integral equation is usually a Fredholm equation in the classical theory. In this study we obtain a Fredholm-Volterra integral equation different from classical theory for the problem (1).

The Fredholm operator included in the equivalent integral equation is an operator with a degenerated kernel. We applied the ordinary successive approximation method for problem (1). In this study, this method was applied for the boundary value problems with retarded argument. The problem (1) has been studied for $\tau(t) \equiv 0$ in (Memmedov ve Kaçar, 1993), $\tau(t) = constant$ in (Kaçar ve Memmedov, 1994). Furthermore, problem (1) has been studied with varied boundary conditions (Aykut, 1995). It has been solved ordinary and modified successive approximation method in (Aykut ve Yıldız, 1998). Problem used in dissertation of (Aykut, 1995) solved with ordinary successive approximation method and then it converted Padé series in (Celik vd., 2003). Similarly, it was solved with the modified two sided approximations method and Padé approximants in (Celik vd., 2003) and then it solved with the modified successive approximations method and pade approximants in (Celik vd., 2004). It was studied as comparison of ordinary and modified successive approximation method in (Aykut, 2007). We also investigated the solution of boundary value problem (1) for arbitrary continuous function $\tau(t)$ under determined conditions.

2. MATERIAL AND METHOD

2.1. An equivalent integral equation

In the problem (1), if we take $\lambda(t) = t - \tau(t)$ then $t_0 \in [0, T]$ is a point located at the left side of T such that conditions $\lambda(t_0) = 0$ and $\lambda(t) \leq 0, 0 \leq t \leq t_0$ are satisfied, where, $\lambda_0 = \min_{0 \leq t \leq t_0} \lambda(t)$. We assume that $\lambda(t)$ is a nondecreasing function in the interval $[t_0, T]$ and the equation $\lambda(t) = \sigma$ has a differentiable continuous solution $t = \gamma(\sigma)$ for arbitrary $\sigma \in [0, \lambda(t)]$. It can be seen that if $x^*(t)$ is a solution of the boundary value problem (1) then $x^*(t)$ is also the solution of the equation

$$x(t) = h^*(t) + t \int_0^T \frac{T-s}{T-c} a(s) x(s - \tau(s)) ds - t \int_0^c \frac{c-s}{T-c} a(s) x(s - \tau(s)) ds - \int_0^t (t-s) a(s) x(s - \tau(s)) ds \quad (2)$$

Here,

$$h^*(t) = \varphi(0) - t \int_0^T \frac{T-s}{T-c} a(s) x(s - \tau(s)) ds + t \int_0^c \frac{c-s}{T-c} f(s) ds + \int_0^t (t-s) f(s) ds$$

Let $\sigma = s - \tau(s)$. Therefore Equation (2) can be written as follows:

$$\begin{aligned}
x^*(t) = & H(t) + t \int_0^{\lambda(T)} \frac{(T-\gamma(\sigma))}{T-c} a(\gamma(\sigma)) x^*(\sigma) \gamma'(\sigma) d\sigma \\
& - t \int_0^{\lambda(c)} \frac{(T-\gamma(\sigma))}{T-c} a(\gamma(\sigma)) x^*(\sigma) \gamma'(\sigma) d\sigma \\
& - \int_0^{\lambda(t)} (t-\gamma(\sigma)) a(\gamma(\sigma)) x^*(\sigma) \gamma'(\sigma) d\sigma
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
H(t) = & h^*(t) + \int_{\lambda_0}^0 \frac{(T-\gamma(\sigma))}{T-c} a(\gamma(\sigma)) \varphi(\sigma) \gamma'(\sigma) d\sigma \\
& - \int_{\lambda_0}^0 \frac{(T-\gamma(\sigma))}{T-c} a(\gamma(\sigma)) \varphi(\sigma) \gamma'(\sigma) d\sigma \\
& - \int_{\lambda_0}^0 (t-\gamma(\sigma)) a(\gamma(\sigma)) \varphi(\sigma) \gamma'(\sigma) d\sigma
\end{aligned} \tag{4}$$

Let

$$K_1(\sigma) = \frac{T-\gamma(\sigma)}{T-c} a(\gamma(\sigma)) \gamma'(\sigma),$$

$$K_2(\sigma) = -\frac{c-\gamma(\sigma)}{T-c} a(\gamma(\sigma)) \gamma'(\sigma),$$

$$K(t, \sigma) = -(t-\gamma(\sigma)) a(\gamma(\sigma)) \gamma'(\sigma).$$

Therefore we write

$$\begin{aligned}
x^*(t) = & H(t) + t \int_0^{\lambda(T)} K_1(\sigma) x^*(\sigma) d\sigma + t \int_0^{\lambda(c)} K_2(\sigma) x^*(\sigma) d\sigma \\
& + \int_0^{\lambda(t)} K(t, \sigma) x^*(\sigma) d\sigma
\end{aligned} \tag{5}$$

or

$$x(t) = H(t) + tF_\lambda^T x + tF_\lambda^c x + V_\lambda x \tag{6}$$

where,

$$F_{\lambda}^T x \equiv \int_0^{\lambda(T)} K_1(\sigma)x(\sigma)d\sigma, \quad F_{\lambda}^c x \equiv \int_0^{\lambda(c)} K_2(\sigma)x(\sigma)d\sigma$$

are the Fredholm operator,

$$V_{\lambda} x \equiv \int_0^{\lambda(T)} K(t, \sigma)x(\sigma)d\sigma$$

is the Volterra operator. Equation (6) is a Fredholm-Volterra integral equation and it is equivalent to the problem (1).

2.2. The ordinary successive approximation method

In this section we will show the existence and the uniqueness of solution for the problem (1). However, we will also prove that the solution of the problem (1) converges to the solution of the ordinary successive approximations,

$$x_n(t) = h(t) + \int_0^{\lambda(T)} G_c(t, \gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)x_{n-1}(\sigma)d\sigma \quad n = (1, 2, \dots)$$

for the arbitrary continuous function $x_0(t) \quad 0 \leq t \leq T$

where

$$G_c(t, s) = \begin{cases} \frac{c(t-s) + s(T-t)}{T-c}, & 0 \leq s \leq t \\ \frac{t(T-s)}{T-c}, & t \leq s \leq T \end{cases}$$

is Green function. We will use the Fredholm integral equation in order to show the existence and the uniqueness of solution for the problem (1).

Lemma 1. We assume that E is a Banach space and $A : E \rightarrow E$ a contraction mapping. Then the equation

$$x = A(x) \tag{8}$$

has a unique solution x^* and the ordinary approximations $\{x_n\}$ which are defined by

$$x_n = A(x_{n-1}), \quad (n=1, 2, \dots) \quad (9)$$

and these approximations converge to x^* where the first approximation is $x_0 \in E$.

Proof. First of all we will show that $\{x_n\}$ is well defined, that is $x_n \in E$. By the hypothesis $x_0 \in E$, we assume that $x_{n-1} \in E$. Since $x_{n-1} \in E$ is defined in E than. From $x_{n-1} \in E$ Equation (9) we can write $x_n = A(x_{n-1}) \in E$. By induction it can be seen that $x_n \in E$ for $n=1, 2, \dots$ Now we will show that x_n is a Cauchy sequence. Let us write the inequality:

$$\|x_{n+1} - x_n\| = \|A(x_n) - A(x_{n-1})\| \leq \alpha \|x_n - x_{n-1}\| \quad (10)$$

$$\|x_{n+1} - x_n\| \leq \alpha \|x_n - x_{n-1}\|$$

where

$$\|x_2 - x_1\| \leq \alpha \|x_1 - x_0\|$$

for $n=1$. Similarly for $n=2, 3, 4, \dots$ we can write

$$\|x_n - x_{n-1}\| \leq \alpha^{n-1} \|x_1 - x_0\|$$

or

$$\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|.$$

If we use Equation (10) than we obtain

$$\|x_{n+k} - x_n\| \leq (\alpha^k - 1) \frac{\alpha^n}{\alpha - 1} \|x_1 - x_0\| \quad (11)$$

Since $\alpha < 1$ we write

$$\lim_{n \rightarrow \infty} \|x_{n+k} - x_n\| = 0$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space

$$\lim_{n \rightarrow \infty} x_n = x^*$$

from the Lipschitz condition we may write

$$\|A(x_{n-1}) - A(x^*)\| \leq \alpha \|x_{n-1} - x^*\|$$

and from Equation (9) we have $x^* = A(x^*)$ as n goes to ∞ . Therefore, x^* is the solution of Equation (8).

Now we will prove that this solution is unique. Suppose that Equation (9) has two solutions as x^* and y^* , i.e. $x^* = A(x^*)$ and $y^* = A(y^*)$. Then we can write

$$\|x^* - y^*\| = \|A(x^*) - A(y^*)\| \leq \alpha \|x^* - y^*\|.$$

Since $\alpha < 1$, we get $x^* = y^*$. Thus lemma is proved.

Theorem 1. Suppose that $a = a(t)$ is a continuous function in the interval $0 \leq t \leq T$ and

$$\ell = \frac{T^4}{8(T-c)^2} \|a\| < 1.$$

Therefore problem (1) has a unique solution and the approximation (7) converge to the solution of the problem (1) and the speed of the convergence is determined by

$$\|x_n - x\| \leq \ell^n \|x_0 - x\|.$$

Proof. By using Lemma 1 we are going to show that

$$Ax \equiv \int_0^{\lambda(T)} G_c(t, \gamma(\sigma)) a(\gamma(\sigma)) \gamma'(\sigma) x(\sigma) d\sigma$$

is a contraction mapping in $C[0, T]$. Suppose that $x(t), y(t) \in C[0, T]$. Then

$$\begin{aligned}
|Ax(t) - Ay(t)| &\leq \int_0^{\lambda(T)} G_c(t, \gamma(\sigma)) |a(\gamma(\sigma))| |x(\sigma) - y(\sigma)| d\gamma(\sigma) \\
&\leq a \left[\int_0^{\lambda(T)} G_c(t, \gamma(\sigma)) d\gamma(\sigma) \right] \|x - y\|
\end{aligned}$$

and

$$\|Ax - Ay\| \leq \ell \|x - y\|$$

Therefore we have just proved the existence and the uniqueness of the solution. Now we are going to determine the convergence speed of the solution. If we write

$$\begin{aligned}
|x_n(t) - x(t)| &\leq \int_0^{\lambda(T)} G_c(t, \gamma(\sigma)) |a(\gamma(\sigma))| |x_{n-1}(\sigma) - x(\sigma)| d\gamma(\sigma) \\
&\leq a \left[\int_0^{\lambda(T)} G_c(t, \gamma(\sigma)) d\gamma(\sigma) \right] \|x_{n-1} - x\| \\
&\leq \ell \|x_{n-1} - x\|
\end{aligned}$$

and $n = 1, 2, 3, \dots$

$$\|x_n - x\| \leq \ell \|x_{n-1} - x\|$$

then we get

$$|x_n(t) - x(t)| \leq \ell^n \|x_0 - x\|$$

where $n = 1, 2, \dots$

3. Findings

Example 1. Let us consider the boundary value problem:

$$\left. \begin{aligned} x''(t) + tx \left(t - \frac{1}{2} \sqrt{t} \right) &= -2t^3 + 2t^{5/2} + \frac{5}{2}t^2 - \frac{3}{2}t^{3/2} - 4, \quad (0 \leq t \leq 1) \\ x(t) &= 0 \quad (-1/16 \leq t \leq 0), \quad x(1) = x\left(\frac{1}{2}\right) \end{aligned} \right\} \quad 12)$$

This equation can be written as the Fredholm-Volterra integral equation

$$\begin{aligned} x(t) &= 2.872933230t - 2t^2 - 0.1714285714t^{7/2} + 0.2083333333t^4 \\ &\quad + 0.1269841270t^{9/2} - 0.1 \\ &\quad + \frac{t}{8} \int_0^{1/2} \left[3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}} \right] x(\sigma) d\sigma \\ &\quad - \frac{t}{8} \int_0^{1/2 - \sqrt{2}/4} \left[1 - 4\sigma - 16\sigma^2 + \frac{1 + 4\sigma - 80\sigma^2}{\sqrt{1+16\sigma}} \right] x(\sigma) d\sigma \\ &\quad - \frac{1}{16} \int_0^{t - \sqrt{t}/2} \left[(4t-1) + (16t-12)\sigma - 16\sigma^2 + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}} \right] x(\sigma) d\sigma \end{aligned} \quad (13)$$

Let

$$\begin{aligned} h(t) &= 2.872933230t - 2t^2 - 0.1714285714t^{(7/2)} + 0.2083333333t^4 \\ &\quad + 0.1269841270t^{(9/2)} - 0.1t^5, \end{aligned}$$

$$K_1(\sigma) = \frac{1}{8} \left(3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}} \right),$$

$$K_2(\sigma) = -\frac{1}{8} \left(1 - 4\sigma - 16\sigma^2 + \frac{1 + 4\sigma - 80\sigma^2}{\sqrt{1+16\sigma}} \right),$$

$$K(t, \sigma) = -\frac{1}{16} \left[(4t-1) + (16t-12)\sigma - 16\sigma^2 + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}} \right]$$

and

$$F_{\lambda}^T x \equiv \int_0^{1/2} K_1(\sigma)x(\sigma)d\sigma, \quad F_{\lambda}^c x \equiv \int_0^{1/2-\sqrt{2}/4} K_2(\sigma)x(\sigma)d\sigma$$

$$V_{\lambda}x \equiv \int_0^{t-\sqrt{t}/2} K(t,\sigma)x(\sigma)d\sigma$$

Therefore, the integral equation (13) can be written as

$$x(t) = h(t) + tF_{\lambda}^T x + tF_{\lambda}^c x + V_{\lambda}x$$

and this equation is equivalent to problem (12). Some values of the solution of this equation are obtained by using the ordinary successive approximations of third which are given in Table 1, where the first approximations is $x_0(t) = 2.872933230t$.

Table 1: Values at some point in the interval $[0,1]$.

t_i	$x(t_i)$	$x_3(t_i)$	$\varepsilon(t_i)$
0.00	0.000000	0.000428	0.000428
0.25	0.625000	0.625486	0.000486
0.50	1.000000	1.000451	0.000451
1.00	1.000000	0.957439	0.042561

$x(t_i)$ is the exact solution, $x_3(t_i)$ is the ordinary successive approximations of third. $\varepsilon(t_i)$ is error value of the ordinary successive approximations of third.

4. CONCLUSIONS

The fundamental aim of this paper has been to construct an approximation to the solution of differential equation with retarded argument. To solve boundary value problem (1), we obtained an integral equation that is equivalent to the boundary value problem (1). In this paper, we obtained a Fredholm-Volterra integral equation for the problem (1). After we obtained integral equation, we used

ordinary successive approximation method. With this method, we found successive approximate solutions or problem (1). As is seen from Table 1, values calculated for problem (1) are coincide with exact solution. Our aim has been achieved by carrying out ordinary successive approximation method. The computations associated with the example mentioned above were carried out by using Maple.

5. REFERENCES

- Ayktut A. 1995. İkinci mertebeden geciken değişkenli lineer adi diferansiyel denklemlerin yaklaşık metotlarla çözümü, Atatürk Üniversitesi.
- Ayktut A., Yıldız B. 1998. On a boundary value problem for a differential equation with variant retarded argument , Applied Mathematics and Computation , 93, 63-71.
- Ayktut A., Celik E., Bayram M. 2003. The modified two sided approximations method and Padé approximants for solving the differential equation with variant retarded argument, Applied Mathematics and Computation , 144, 475-482.
- Ayktut A. 2007. Comparation of Two Methods for a Differential Equation with Variant Retarded Argument, Kastamonu Eğitim Dergisi, 15, 317-322.
- Celik E. , Ayktut A., Bayram M. 2004. The modified successive approximations method and pade approximants for solving the differential equation with variant retarded argument, Applied Mathematics and Computation, 151, 393-400.
- Celik E., Ayktut A., Bayram M. 2003. The ordinary successive approximations method and Pade approximants for solving a differential quation with variant retarded argument, Applied Mathematics and Computation, 144, 173-180
- Norkin S.B. 1972. Differential equations of the second order with retarded argument some problems of theory of vibrations of systems with retardation, A. M. S.
- Mamedov, Ja. D. 1994. Numerical Analysis, Atatürk Üniversitesi.
- Memmedov Y. C. ve Kaçar A. 1993. Lineer Diferansiyel Denklemler İçin Sınır Değer Probleminin Yaklaşık Metodlarla Çözümü, Atatürk Üniversitesi Fen Fakültesi Yayınları, Erzurum.
- Kaçar, A. ve Memmedov Y. C. 1994. Geciken argümentli diferansiyel denklemler için bir özel sınır değer probleminin yaklaşık metodlarla çözümü, Marmara Üniversitesi Yayınları, İstanbul.

Krasnosel'skij M.A., Lifshits, Je. A., and Sobolev A. V. 1989. Positive Linear Systems, Heldermann, Verlag, Berlin.

Myskis A.D. 1946. The free Dirichlet problem, Zukovskii Air Force Engineering Academy, Moscow.
