



C-CLASS FUNCTIONS ON SOME COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED S -METRIC SPACES

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ABSTRACT. In this article, we introduce a new and special type of contraction to prove a common coupled fixed point results in partially ordered S - metric space with the help of auxiliary function. To prove our result, we have utilize the notion of altering distance function and mixed weakly monotone property of maps. To demonstrate applicability of main result, some corollaries are given.

1. INTRODUCTION

Metric spaces are exceptionally incomparable in mathematics and applied sciences. Numerous authors have given reflections of metric spaces in several ways. A space will be an arrangement of unspecified components fulfilling certain axioms and by choosing different set of axioms, we shall obtain different type of spaces. In a partially ordered set, as the name indicates ordering and sequencing is defined between the elements of a set but there are some elements in the set which are not related. A set in which all elements are related is called totally ordered set. In [15] Ralph De Marr defines the concept of convergence in partially ordered metric space. Also, he obtained a relation between metric space and partially ordered metric space, and claimed that the fixed point theorems in metric space are the particular cases of fixed point results in partially ordered metric space.

Definition 1. [15] *A partially ordered space is a set X with a binary relation \leq , which satisfy the three conditions:-*

- (1) $x \leq x$ for all $x \in X$;
- (2) $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$;
- (3) $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$.

There are many results in literature using control function. Initially, Dolbosco [4] gave fixed point results using altering distance function or control function. Altering

Received by the editors: May 21, 2018; Accepted: January 02, 2019.

2010 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H10.

Key words and phrases. Fixed point, S -metric space, mixed weakly monotone property, C -class functions.

distance function measures the distance between two points. In 1984, Khan et al.[11] introduced the following definition of altering distance function.

Definition 2. [11] *An altering distance function is a function $\psi : [0, \infty) \rightarrow [0, \infty)$ which is*

- (1) *monotone, increasing and continuous,*
- (2) *$\psi(t) = 0$ if and only if $t = 0$.*

Some other authors worked on the concept of altering distance between points for self mappings in various spaces for several type of contractions, see [3, 6, 8, 9, 11, 13, 14, 21, 22, 23].

In 1963, Gahler [16] introduced the generalization of metric space and called it 2-metric space. Dhage [2] introduced a new metric named it D -metric. Later, Sedghi and Shobe [17] presented a modified version of D -metric and named it D^* metric space which is more general in literature.

In 2006, Mustafa [26] placed a new metric spaces, called it G -metric space. He defined many concepts like convergence, continuity, completeness, compactness, product of spaces in the setting of G -metric space and stated that every G -metric space is topologically equivalent to a metric space.

Definition 3. [26] *Let X be a non-empty set and $G : X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$,*

- (1) *$G(x, y, z) = 0$ if $x = y = z$;*
- (2) *$0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;*
- (3) *$G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$;*
- (4) *$G(x, y, z) = G(x, z, y) = G(y, z, x) \dots$;*
- (5) *$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.*

Then the function G is called G - metric on X and the pair (X, G) is called a G -metric space.

In 2011, Sedghi and Shobe [18] introduced a more general space namely S -metric space which is generalization of D^* and G -metric space.

Definition 4. [18] *Let X be a non-empty set. A generalized metric or S -metric on X is a function $S : X \times X \times X \rightarrow [0, \infty)$ which satisfies the following conditions for all $x, y, z, a \in X$,*

- (1) *$S(x, y, z) \geq 0$;*
- (2) *$S(x, y, z) = 0$ if and only if $x = y = z$;*
- (3) *$S(x, y, z) \leq S(a, y, z) + S(a, x, x)$.*

Then the pair (X, S) is called a S -metric space.

Example 5. [18] *If $X = R^n$ then we define*

- (1) *$S(x, y, z) = \|y + x - 2z\| + \|y - z\|$.*
- (2) *$S(x, y, z) = d(x, y) + d(x, z)$, here d is the ordinary metric on X .*

Then (X, S) is the S -metric space.

Definition 6. [18] *Let (X, S) be a S -metric space and $A \subset X$,*

- (1) If for every $x \in A$, there exists $r > 0$ such that $B_S(x, r) \subset A$, then subset A is called open subset of X .
- (2) Subset A of X is said to be S -bounded if there exists $r > 0$ such that $S(x, y, y) < r$ for all $x, y \in A$.
- (3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x, x) = S(x, x_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x, x_n, x_n) < \epsilon$ for all $n \geq n_0$.
- (4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_m, x_m) < \epsilon$ for each $n, m \geq n_0$. The S -metric space (X, S) is said to be complete if every Cauchy sequence in X is convergent in X .
- (5) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology induced by the S -metric on X .

In 2012, Sedghi et. al. [19] characterized some properties of S -metric space and proved a fixed point theorem in this space.

Lemma 7. [19] *Let (X, S) be an S -metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

and

$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Theorem 8. [18] *Let (X, S) be a complete S -metric space and let $F, G : R \times X \rightarrow X$ be two functions satisfying the following conditions*

$$S(F(t, x), G(t, y), G(t, y)) \leq k_1 S(x, F(t, x), F(t, x)) + k_2 S(y, G(t, y), G(t, y)) + k_3 S(x, y, y),$$

for every $x, y \in X$, $t \in R$ where $k_i \geq 0$ for $i = 1, 2, 3$ and $0 < k_1 + k_2 + k_3 < 1$. Then F and G have a unique common fixed point.

In 2012, Sedghi et al. [19] also derive a result similar to Banach contraction principle in the setting of S -metric space.

Theorem 9. [19] *Let (X, S) be a complete S -metric space and $F : X \rightarrow X$ be a contraction. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have $\lim_{n \rightarrow \infty} F^n(x) = u$ with*

$$S(F^n(x), F^n(x), u) \leq \frac{2L^n}{1-L} S(x, x, F(x)). \quad (1)$$

In literature, coupled fixed point theorems have been studied by many authors like [5], [20], [25]. Lakshmikantham and Ćirić [24] introduced the concept of mixed g -monotone mapping and proved coincidence fixed point theorems in partially ordered metric space.

Definition 10. [24] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say F has the mixed g -monotone property if F is monotone g -non-decreasing in its first argument and is monotone g -non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \text{ implies } F(x_1, y) \leq F(x_2, y)$$

$$\text{and } y_1, y_2 \in X, g(y_1) \leq g(y_2) \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Theorem 11. [24] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$ and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(t) < t$ and $\lim_{r \rightarrow t^+} \phi(r) < t$ for each $t > 0$. Also suppose $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property satisfying:

$$d(F(x, y), F(u, v)) \leq \phi \left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2} \right),$$

for all $x, y, u, v \in X$, for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Consider $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F . Also suppose either

- (1) F is continuous or
- (2) X has the following property,
 - (a) if a non-decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all n ;
 - (b) if a non-increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

In 2012 Gordji et. al. [10] introduced the concept of the mixed weakly increasing property of mappings and proved a coupled fixed point result.

Definition 12. [10] Let (X, \leq) be a partially ordered set and $f, g : X \times X \rightarrow X$ be mappings. We say that a pair (f, g) has the mixed weakly monotone property on X if for any $x, y \in X$,

$$x \leq f(x, y), y \geq f(y, x) \Rightarrow f(x, y) \leq g(f(x, y), f(y, x)), f(y, x) \geq g(f(y, x), f(x, y))$$

and

$$x \leq g(x, y), y \geq g(y, x) \Rightarrow g(x, y) \leq f(g(x, y), g(y, x)), g(y, x) \geq f(g(y, x), g(x, y))$$

Theorem 13. [10] Let (X, \leq, d) be a partially ordered complete metric space. Let $f, g : X \times X \rightarrow X$ the mappings such that a pair (f, g) has the mixed weakly monotone property on X . Suppose that there exist $p, q, r, s \geq 0$ with $p+q+r+2s < 1$ such that

$$\begin{aligned} d(f(x, y), g(u, v)) &\leq \frac{p}{2}D((x, y), (u, v)) + \frac{q}{2}D((x, y), (f(x, y), f(y, x))) \\ &+ \frac{r}{2}D((u, v), (g(u, v), g(v, u))) + \frac{s}{2}D((x, y), (g(u, v), g(v, u))) \\ &+ \frac{s}{2}D((u, v), (f(x, y), f(y, x))) \end{aligned}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$, where $(X \times X, D)$ be a metric spaces defined as $D((x, y), (u, v)) = d(x, u) + d(y, v)$. Let $x_0, y_0 \in X$ be such that $x_0 \leq f(x_0, y_0)$, $y_0 \geq f(y_0, x_0)$ or $x_0 \leq g(x_0, y_0)$, $y_0 \geq g(y_0, x_0)$. If f or g is continuous, then f and g have a coupled common fixed point in X .

Nguyen [12] generalized the result of Gordji et al. [10] in sense of S -metric space as follows:

Theorem 14. [12] Let (X, \leq, S) be a partially ordered S -metric space and $f, g : X \times X \rightarrow X$ be two maps such that

- (1) X is complete;
- (2) The pair (f, g) has the mixed weakly monotone property on X , $x_0 \leq f(x_0, y_0)$, $f(y_0, x_0) \leq y_0$ or $x_0 \leq g(x_0, y_0)$, $g(y_0, x_0) \leq y_0$ for some $x_0, y_0 \in X$;
- (3) There exist $p, q, r, s \geq 0$ satisfying $p + q + r + 2s < 1$ and

$$\begin{aligned} d(f(x, y), f(x, y), g(u, v)) &\leq \frac{p}{2}D((x, y), (x, y), (u, v)) \\ &\quad + \frac{q}{2}D((x, y), (x, y), (f(x, y), f(y, x))) \\ &\quad + \frac{r}{2}D((u, v), (u, v), (g(u, v), g(v, u))) \\ &\quad + \frac{s}{2}D((x, y), (x, y), (g(u, v), g(v, u))) \\ &\quad + \frac{s}{2}D((u, v), (u, v), (f(x, y), f(y, x))), \end{aligned}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$, where $D((x, y), (u, v), (z, w)) = S(x, u, z) + S(y, v, w)$;

- (4) f or g is continuous or X has the following properties:
 - (a) If $\{x_n\}$ is an increasing sequence with $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
 - (b) If $\{x_n\}$ is an decreasing sequence with $x_n \rightarrow x$, then $x \leq x_n$ for all $n \in \mathbb{N}$.

Then f and g have a coupled common fixed point in X .

Ansari [1] in 2014-15, defined the notion of C -class function as a generalization of Banach contraction principle.

Definition 15. [1] We say $\phi : [0, +\infty) \rightarrow [0, +\infty)$ ultra distance function, if it is continuous and $\phi(0) \geq 0$, and $\phi(t) > 0, t > 0$.

Remark 16. We let Φ_u denote the class of ultra distance functions.

Definition 17. [1] A mapping $F : [0, \infty)^2 \rightarrow \mathbb{R}$ is called C -class function if it is continuous and for all $r, t \in [0, \infty)$

- (F_1) : $F(r, t) \leq r$;
- (F_2) : $F(r, t) = r$ implies that either $r = 0$ or $t = 0$.

For brevity, we denote \mathcal{C} as the family of C class functions. It is also clear that, $F(0,0) = 0$. Some examples of C -class functions are given in [1].

Lemma 18. [7] *Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and*

- (i) $\lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;
- (ii) $\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
- (iii) $\lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

Remark 19. *Clearly from above Lemma 18, we conclude that*

$$\lim_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon \text{ and } \lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon.$$

The aim of this paper is to prove common coupled fixed points results by using altering distance function and mixed weakly monotone property of maps in partially ordered S -metric space with the help of new auxiliary function.

2. MAIN RESULT

Theorem 20. *Let (X, \leq, S) be a partially ordered complete S -metric space and the mappings $f, g : X \times X \rightarrow X$ satisfies the mixed weakly monotone property on X ; $x_0 \leq f(x_0, y_0)$, $f(y_0, x_0) \leq y_0$ or $x_0 \leq g(x_0, y_0)$, $g(y_0, x_0) \leq y_0$ for some $x_0, y_0 \in X$.*

Consider a function $\phi \in \Phi_u$, $F \in C$, such that

$$S(f(x, y), f(x, y), g(u, v)) \leq F\left(\frac{S(x, x, u) + S(y, y, v)}{2}, \phi\left(\frac{S(x, x, u) + S(y, y, v)}{2}\right)\right) \quad (2)$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$.

Also, assume that either f or g is continuous or X has the following property:-

- (i) *If $\{x_n\}$ is an increasing sequence with $x_n \rightarrow x$ then $x_n \leq x$ for all $n \in N$;*
- (ii) *If $\{y_n\}$ is an decreasing sequence with $y_n \rightarrow y$ then $y \leq y_n$ for all $n \in N$.*

Then f and g have a coupled common fixed point in X .

Proof. Given that maps f and g satisfies the mixed weakly monotone property on X , i.e. $x_0 \leq f(x_0, y_0)$ and $y_0 \geq f(y_0, x_0)$.

Let $f(x_0, y_0) = x_1$ and $f(y_0, x_0) = y_1$,

then

$$x_1 = f(x_0, y_0) \leq g(f(x_0, y_0), f(y_0, x_0)) = g(x_1, y_1) = x_2(\text{say})$$

and

$$y_1 = f(y_0, x_0) \leq g(f(y_0, x_0), f(x_0, y_0)) = g(y_1, x_1) = y_2(\text{say}),$$

continuing in this way, we can construct sequences

$$x_{2n+1} = f(x_{2n}, y_{2n}), y_{2n+1} = f(y_{2n}, x_{2n}) \text{ and}$$

$$x_{2n+2} = g(x_{2n+1}, y_{2n+1}), y_{2n+2} = g(y_{2n+1}, x_{2n+1}).$$

Thus, we conclude that $\{x_n\}$ is increasing and $\{y_n\}$ is decreasing sequence.

Similarly, from the condition $x_0 \leq g(x_0, y_0)$ and $y_0 \geq g(y_0, x_0)$, we can say that the sequences $\{x_n\}$ and $\{y_n\}$ are increasing or decreasing.

Now from (2), we obtain

$$\begin{aligned} & S(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})) \\ & \leq F\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}\right)\right) \end{aligned}$$

which implies,

$$\begin{aligned} S(x_{2n+1}, x_{2n+1}, x_{2n+2}) & \leq F\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}\right)\right). \quad (3) \end{aligned}$$

Again, using the similar argument, we have

$$\begin{aligned} S(y_{2n+1}, y_{2n+1}, y_{2n+2}) & \leq F\left(\frac{S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})}{2}\right)\right). \quad (4) \end{aligned}$$

Adding (3) and (4), we get

$$\begin{aligned} & \frac{S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{2} \\ & \leq F\left(\frac{S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})}{2}\right)\right). \end{aligned}$$

Thus

$$\frac{t_{2n}}{2} \leq F\left(\frac{t_{2n-1}}{2}, \left(\frac{t_{2n-1}}{2}\right)\right) \leq \frac{t_{2n-1}}{2}, \quad (5)$$

where $t_{2n} = S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})$.

Interchanging the role of mappings f and g , and using (2), we have

$$\begin{aligned} & S(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(x_{2n+2}, y_{2n+2})) \\ & \leq F\left(\frac{S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{2}\right)\right) \end{aligned}$$

Proceeding as above one can get, $\frac{t_{2n+1}}{2} \leq F\left(\frac{t_{2n}}{2}, \left(\frac{t_{2n}}{2}\right)\right) \leq \frac{t_{2n}}{2}$,
 this gives

$$t_{2n+1} \leq t_{2n}. \tag{6}$$

From (5) and (6), we conclude that $\{t_n\}$ is a decreasing sequence. Therefore there exists some $t \geq 0$ such that

$$\lim_{n \rightarrow \infty} t_n = t.$$

Also, using the properties of function ϕ

$$\frac{t}{2} = \lim_{n \rightarrow \infty} \frac{t_{2n}}{2} \leq F\left(\lim_{n \rightarrow \infty} \frac{t_{2n-1}}{2}, \lim_{n \rightarrow \infty} \phi\left(\frac{t_{2n-1}}{2}\right)\right) = F\left(\frac{t}{2}, \phi\left(\frac{t}{2}\right)\right),$$

therefore by Definition 17, either $\frac{t}{2} = 0$ or $\phi\left(\frac{t}{2}\right) = 0$ and hence $t = 0$.

i.e.,

$$\lim_{n \rightarrow \infty} [S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1})] = 0.$$

Now, we have to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the S -metric space (X, S) .

On the contrary, suppose that at least one of $\{x_n\}$ and $\{y_n\}$ is not a Cauchy sequence in (X, S) , then there exist an $\epsilon > 0$ for which we can find sub-sequences $\{x_{2k(j)+1}\}$, $\{x_{2l(j)+1}\}$ of $\{x_n\}$; $\{y_{2k(j)+1}\}$, $\{y_{2l(j)+1}\}$ of $\{y_n\}$ with $j \leq k(j) \leq l(j)$ for all $j \in \mathbb{N}$ such that

$$\alpha_j = S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+1}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+1}) \geq \epsilon. \tag{7}$$

We may also assume that

$$S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)}) < \epsilon, \tag{8}$$

by choosing $l(j)$ to be the smallest number exceeding $k(j)$ for which above holds.

From (7) and (8), and using the second condition of S -metric, we obtain

$$\begin{aligned} \epsilon &\leq \alpha_j = S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+1}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+1}) \\ &\leq S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)}) + S(x_{2l(j)}, x_{2l(j)}, x_{2l(j)+1}) \\ &\quad + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)}) + S(y_{2l(j)}, y_{2l(j)}, y_{2l(j)+1}) \\ &< \epsilon + S(x_{2l(j)}, x_{2l(j)}, x_{2l(j)+1}) + S(y_{2l(j)}, y_{2l(j)}, y_{2l(j)+1}). \end{aligned}$$

Letting $j \rightarrow \infty$ in the above inequality and using (7), we get

$$\alpha_j = S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+1}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+1}) \rightarrow \epsilon.$$

Again by Lemma 18,

$$\begin{aligned} \alpha_j &= S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+1}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+1}) \\ &\leq 2S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+2}) + S(x_{2l(j)+1}, x_{2l(j)+1}, x_{2l(j)+2}) \\ &\quad + 2S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+2}) + S(y_{2l(j)+1}, y_{2l(j)+1}, y_{2l(j)+2}) \\ &= \gamma_{2l(j)+1} + 2S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+2}) + 2S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+2}) \\ &\leq \gamma_{2l(j)+1} + 2S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2k(j)+2}) + S(x_{2l(j)+2}, x_{2l(j)+2}, x_{2k(j)+2}) \end{aligned}$$

$$\begin{aligned}
& + 2S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2k(j)+2}) + S(y_{2l(j)+2}, y_{2l(j)+2}, y_{2k(j)+2}) \\
& \leq \gamma_{2l(j)+1} + 2\gamma_{2k(j)+1} + S(x_{2l(j)+2}, x_{2l(j)+2}, x_{2k(j)+2}) \\
& + S(y_{2l(j)+2}, y_{2l(j)+2}, y_{2k(j)+2}).
\end{aligned}$$

Using (2) and (3), the above inequality becomes

$$\begin{aligned}
\alpha_j & \leq \gamma_{2l(j)+1} + 2\gamma_{2k(j)+1} \\
& + S(f(x_{2l(j)+1}, y_{2l(j)+1}), f(x_{2l(j)+1}, y_{2l(j)+1}), g(x_{2k(j)+1}, y_{2k(j)+1})) \\
& + S(f(y_{2l(j)+1}, x_{2l(j)+1}), f(y_{2l(j)+1}, x_{2l(j)+1}), g(y_{2k(j)+1}, x_{2k(j)+1})) \\
& \leq \gamma_{2l(j)+1} + 2\gamma_{2k(j)+1} \\
& + 2F\left(\frac{S(x_{2l(j)+1}, x_{2l(j)+1}, x_{2k(j)+1}) + S(y_{2l(j)+1}, y_{2l(j)+1}, y_{2k(j)+1})}{2}\right), \\
& \quad \phi\left(\left[\frac{S(x_{2l(j)+1}, x_{2l(j)+1}, x_{2k(j)+1}) + S(y_{2l(j)+1}, y_{2l(j)+1}, y_{2k(j)+1})}{2}\right]\right) \\
& = \gamma_{2l(j)+1} + 2\gamma_{2k(j)+1} \\
& + 2F\left(\frac{S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+1}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+1})}{2}\right), \\
& \quad \phi\left(\left[\frac{S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)+1}) + S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)+1})}{2}\right]\right).
\end{aligned}$$

It follows that

$$\alpha_j - \gamma_{2l(j)+1} - 2\gamma_{2k(j)+1} \leq 2F\left(\frac{\alpha_j}{2}, \phi\left(\frac{\alpha_j}{2}\right)\right).$$

Taking the limit as $j \rightarrow \infty$, we obtain

$$\frac{\epsilon}{2} \leq F\left(\frac{\epsilon}{2}, \phi\left(\frac{\epsilon}{2}\right)\right),$$

therefore by Definition 17 either $\frac{\epsilon}{2} = 0$ or $\phi(\frac{\epsilon}{2}) = 0$, and hence $\epsilon = 0$. Thus $\alpha_j \rightarrow 0$, which is a contradiction. Using property of S -metric by interchanging the roles of f and g and proceeding along the arguments discussed above, we also obtain that

$$S(x_{2k(j)}, x_{2k(j)}, x_{2l(j)+1}), S(y_{2k(j)}, y_{2k(j)}, y_{2l(j)+1}) \rightarrow 0,$$

and

$$S(x_{2k(j)}, x_{2k(j)}, x_{2l(j)}), S(y_{2k(j)}, y_{2k(j)}, y_{2l(j)}) \rightarrow 0,$$

and

$$S(x_{2k(j)+1}, x_{2k(j)+1}, x_{2l(j)}), S(y_{2k(j)+1}, y_{2k(j)+1}, y_{2l(j)}) \rightarrow 0.$$

Hence, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the S -metric space (X, S) . Since (X, S) is a complete S -metric space, hence $\{x_n\}$ and $\{y_n\}$ are S -convergent. Then there exist $x, y \in X$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$, respectively.

As f is supposed to be continuous, therefore

$$x = \lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} f(x_{2n}, y_{2n}) = f\left(\lim_{n \rightarrow \infty} x_{2n}, \lim_{n \rightarrow \infty} y_{2n}\right) = f(x, y)$$

and

$$y = \lim_{n \rightarrow \infty} y_{2n+1} = \lim_{n \rightarrow \infty} f(y_{2n}, x_{2n}) = f\left(\lim_{n \rightarrow \infty} x_{2n}, \lim_{n \rightarrow \infty} y_{2n}\right) = f(x, y).$$

From (2),

$$\begin{aligned} & S(f(x, y), f(x, y), g(x, y)) + S(f(y, x), f(y, x), g(y, x)) \\ & \leq 2F\left(\frac{S(x, x, x) + S(y, y, y)}{2}, \phi\left(\frac{S(x, x, x) + S(y, y, y)}{2}\right)\right) \\ & \leq 2F(0, \phi(0)) \leq 2 \times 0 = 0, \end{aligned}$$

this implies that

$$S(x, x, g(x, y)) + S(y, y, g(y, x)) = 0.$$

Thus we have, $g(x, y) = x$ and $g(y, x) = y$. Hence (x, y) is coupled common fixed point of f and g . Similarly, the result follows when g is assumed to be continuous. Consider the other assumption that for an increasing sequence $\{x_n\}$ with $x_n \rightarrow x$, we have $x_n \leq x$ and for decreasing sequence $\{y_n\}$ with $y_n \rightarrow y$, we have $y \leq y_n$ for all $n \in N$.

Consider,

$$\begin{aligned} S(x, x, g(x, y)) & \leq 2S(x, x, x_n) + S(f(x, y), f(x, y), x_n) \\ & \leq 2S(x, x, x_n) + S(f(x, y), f(x, y), g(x_{n-1}, y_{n-1})) \\ & \leq 2S(x, x, x_n) + F\left(\frac{S(x, x, x_{n-1}) + S(y, y, y_{n-1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(x, x, x_{n-1}) + S(y, y, y_{n-1})}{2}\right)\right). \end{aligned}$$

Similarly,

$$\begin{aligned} S(y, y, f(y, x)) & \leq 2S(y, y, y_n) + S(f(y, x), f(y, x), y_n) \\ & \leq 2S(y, y, y_n) + S(f(y, x), f(y, x), g(y_{n-1}, x_{n-1})) \\ & \leq 2S(y, y, y_n) + F\left(\frac{S(x, x, x_{n-1}) + S(y, y, y_{n-1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(x, x, x_{n-1}) + S(y, y, y_{n-1})}{2}\right)\right). \end{aligned}$$

Adding the above inequalities, one can get

$$\begin{aligned} S(x, x, g(x, y)) + S(y, y, f(y, x)) & \leq 2S(x, x, x_n) + 2S(y, y, y_n) \\ & \quad + 2F\left(\frac{S(x, x, x_{n-1}) + S(y, y, y_{n-1})}{2}, \right. \\ & \quad \left. \phi\left(\frac{S(x, x, x_{n-1}) + S(y, y, y_{n-1})}{2}\right)\right), \end{aligned}$$

on taking $n \rightarrow \infty$, we get $S(x, x, f(x, y)) + S(y, y, f(y, x)) < 0$, which shows that $f(x, y) = x$ and $f(y, x) = y$. By interchanging the role of functions f and g , we get the same result for g . Thus (x, y) is the common coupled fixed point of f and g . \square

Theorem 21. *Assume that X is totally ordered set in addition to the hypothesis of Theorem (20). Then f and g have unique common fixed point.*

Proof. From Theorem 20, f and g have a coupled common fixed point (x, y) . Let (l, m) be another coupled common fixed point of f and g . Without loss of generality, we may assume that $(x, y) \leq (l, m)$. Then from (2), we have

$$S(f(x, y), f(x, y), g(l, m)) \leq F\left(\frac{S(x, x, l) + S(y, y, m)}{2}, \phi\left(\frac{S(x, x, l) + S(y, y, m)}{2}\right)\right)$$

$$S(x, x, l) \leq F\left(\frac{S(x, x, l) + S(y, y, m)}{2}, \phi\left(\frac{S(x, x, l) + S(y, y, m)}{2}\right)\right).$$

Similarly,

$$S(y, y, m) \leq F\left(\frac{S(y, y, m) + S(x, x, l)}{2}, \phi\left(\frac{S(y, y, m) + S(x, x, l)}{2}\right)\right).$$

On adding the above two inequalities, we have

$$\frac{S(x, x, l) + S(y, y, m)}{2} \leq F\left(\frac{S(x, x, l) + S(y, y, m)}{2}, \phi\left(\frac{S(x, x, l) + S(y, y, m)}{2}\right)\right),$$

therefore by Definition 17, either $\frac{S(x, x, l) + S(y, y, m)}{2} = 0$ or $\phi\left(\frac{S(x, x, l) + S(y, y, m)}{2}\right) = 0$, and hence $S(x, x, l) + S(y, y, m) = 0$. Hence $x = l$ and $y = m$. This proves that the coupled common fixed point of f and g is unique.

Again from (2), we have

$$S(f(x, y), f(x, y), g(y, x)) \leq F\left(\frac{S(x, x, y) + S(y, y, x)}{2}, \phi\left(\frac{S(x, x, y) + S(y, y, x)}{2}\right)\right).$$

$$S(x, x, y) \leq F\left(\frac{S(x, x, y) + S(y, y, x)}{2}, \phi\left(\frac{S(x, x, y) + S(y, y, x)}{2}\right)\right).$$

Similarly,

$$S(y, y, x) \leq F\left(\frac{S(y, y, x) + S(x, x, y)}{2}, \phi\left(\frac{S(y, y, x) + S(x, x, y)}{2}\right)\right).$$

On adding the above two inequalities, we have

$$\frac{S(x, x, y) + S(y, y, x)}{2} \leq F\left(\frac{S(x, x, y) + S(y, y, x)}{2}, \phi\left(\frac{S(x, x, y) + S(y, y, x)}{2}\right)\right),$$

therefore by Definition 17, either $\frac{S(x, x, y) + S(y, y, x)}{2} = 0$ or $\phi\left(\frac{S(x, x, y) + S(y, y, x)}{2}\right) = 0$. Therefore, $S(x, x, y) + S(y, y, x) = 0$. Thus we get $x = y$. \square

Theorem 22. *Let (X, \leq, S) be a partially ordered complete S -metric space and the mappings $f, g : X \times X \rightarrow X$ satisfies the mixed weakly monotone property on X . Consider a function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(t) < t$ and*

$$\lim_{r \rightarrow t^+} \phi(r) < t \text{ for each } t > 0$$

such that

$$\int_0^{S(f(x,y),f(x,y),g(u,v))} \varphi(t) dt \leq \phi\left(\int_0^{\frac{S(x,x,u)+S(y,y,v)}{2}} \varphi(t) dt\right), \tag{9}$$

$\forall x, y, u, v \in X$ with $x \leq u$ and $y \geq v$. Here $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable function which is a summable for each compact R^+ , non-negative and such that for each $\epsilon > 0$, $\int \varphi(t) dt > 0$.

Also, assume that either f or g is continuous or X has the following property:-

- (i) If $\{x_n\}$ is an increasing sequence with $x_n \rightarrow x$ then $x_n \leq x$ for all $n \in N$;
- (ii) If $\{y_n\}$ is a decreasing sequence with $y_n \rightarrow y$ then $y \leq y_n$ for all $n \in N$.

Then f and g have a coupled common fixed point in X .

Proof. As in Theorem 20, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ and using (9), we have

$$\begin{aligned} \int_0^{S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))} \varphi(t) dt \\ \leq \phi\left(\int_0^{\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)} \varphi(t) dt\right), \\ \int_0^{S(x_{2n+1},x_{2n+1},x_{2n+2})} \varphi(t) dt \leq \int_0^{\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)} \varphi(t) dt, \end{aligned}$$

this gives,

$$S(x_{2n+1},x_{2n+1},x_{2n+2}) \leq \left(\frac{S(x_{2n},x_{2n},x_{2n+1}) + S(y_{2n},y_{2n},y_{2n+1})}{2}\right). \tag{10}$$

In the same way, we get

$$S(y_{2n+1},y_{2n+1},y_{2n+2}) \leq \left(\frac{S(y_{2n},y_{2n},y_{2n+1}) + S(x_{2n},x_{2n},x_{2n+1})}{2}\right). \tag{11}$$

On adding the above inequalities and using the properties of function ϕ , we get $\{t_n\}$ be a decreasing sequence and $\lim_{n \rightarrow \infty} t_n = 0$.

Again, by using the properties of S - metric space we observe that $\{x_n\}$ and $\{y_n\}$ are the Cauchy sequences in X . By continuity of function f , we have $x = f(x, y)$ and $y = f(y, x)$.

Now from (9), we get

$$\int_0^{S(f(x,y),f(x,y),g(x,y))} \varphi(t) dt \leq \phi\left(\int_0^{\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right)} \varphi(t) dt\right),$$

which implies $S(x, x, g(x, y)) = 0$ or $g(x, y) = x$. Similarly $g(y, x) = y$. Thus (x, y) is the coupled common fixed point of f and g . Now, assuming the condition

that for increasing sequence $\{x_n\}$ with $x_n \rightarrow x$ then $x_n \leq x$ and for decreasing sequence $\{y_n\}$ with $y_n \rightarrow y$ then $y_n \geq y$ for all $n \in N$,

$$\begin{aligned} \int_0^{S(x,x,f(x,y))} \varphi(t) dt &\leq \int_0^{2S(x,x,x_n)} \varphi(t) dt + \int_0^{S(f(x,y),f(x,y),x_n)} \varphi(t) dt \\ &= \int_0^{2S(x,x,x_n)} \varphi(t) dt + \int_0^{S(f(x,y),f(x,y),g(x_{n-1},y_{n-1}))} \varphi(t) dt \\ &\leq \int_0^{2S(x,x,x_n)} \varphi(t) dt + \phi \left(\int_0^{\left(\frac{S(x,x,x_{n-1})+S(y,y,y_{n-1})}{2}\right)} \varphi(t) dt \right). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we obtain

$$\int_0^{S(x,x,f(x,y))} \varphi(t) dt \leq \int_0^{2S(x,x)} \varphi(t) dt + \phi \left(\int_0^{\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right)} \varphi(t) dt \right),$$

this shows that f has coupled fixed point. By interchanging the role of mappings f and g , we get the coupled fixed point of g . Hence, we conclude that in both cases f and g have coupled fixed point. \square

On taking $f = g$ in Theorem 20, we obtain the following result.

Corollary 23. *Let (X, \leq, S) be a partially ordered complete S -metric space and the mapping $f : X \times X \rightarrow X$ satisfies the mixed weakly monotone property on X ; $x_0 \leq f(x_0, y_0)$, $f(y_0, x_0) \leq y_0$ or $x_0 \leq f(x_0, y_0)$, $f(y_0, x_0) \leq y_0$ for some $x_0, y_0 \in X$. Consider a function $\phi \in \Phi_u$, $F \in C$, such that*

$$S(f(x, y), f(x, y), f(u, v)) \leq F \left(\frac{S(x, x, u) + S(y, y, v)}{2}, \phi \left(\frac{S(x, x, u) + S(y, y, v)}{2} \right) \right) \quad (12)$$

$\forall x, y, u, v \in X$ with $x \leq u$ and $y \geq v$.

Also, assume that either f is continuous or X has the following properties:

- (i) If $\{x_n\}$ is an increasing sequence with $x_n \rightarrow x$ then $x_n \leq x$ for all $n \in N$;
- (ii) If $\{y_n\}$ is a decreasing sequence with $y_n \rightarrow y$ then $y \leq y_n$ for all $n \in N$.

Then f has a coupled common fixed point in X .

By taking $F(s, t) = as$, $0 < a < 1$ in Theorem 20, we have the following Result.

Corollary 24. *Let (X, \leq, S) be a partially ordered complete S -metric space and the mappings $f, g : X \times X \rightarrow X$ satisfies the mixed weakly monotone property on X . Consider $0 < a < 1$ such that*

$$S(f(x, y), f(x, y), g(u, v)) \leq \frac{a}{2} [S(x, x, u) + S(y, y, v)], \quad (13)$$

$\forall x, y, u, v \in X$ with $x \leq u$ and $y \geq v$.

Also, assume that either f or g is continuous or X has the following properties:-

- (i) If $\{x_n\}$ is an increasing sequence with $x_n \rightarrow x$ then $x_n \leq x$ for all $n \in N$;
- (ii) If $\{y_n\}$ is a decreasing sequence with $y_n \rightarrow y$ then $y \leq y_n$ for all $n \in N$.

Then f has a coupled common fixed point in X .

Taking $f = g$ in corollary 24, we obtain the next result.

Corollary 25. *Let (X, \leq, S) be a partially ordered complete S -metric space and the mapping $f : X \times X \rightarrow X$ satisfies the mixed monotone property on X . Consider $0 < a < 1$ such that*

$$S(f(x, y), f(x, y), f(u, v)) \leq \frac{a}{2} [S(x, x, u) + S(y, y, v)], \quad (14)$$

$\forall x, y, u, v \in X$ with $x \leq u$ and $y \geq v$.

Also assume that either f is continuous or X has the following properties:-

- (i) If $\{x_n\}$ is an increasing sequence with $x_n \rightarrow x$ then $x_n \leq x$ for all $n \in N$;
- (ii) If $\{y_n\}$ is a decreasing sequence with $y_n \rightarrow y$ then $y \leq y_n$ for all $n \in N$.

Then f has a coupled common fixed point in X .

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