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Minimality over free monoid presentations

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Abstract

As a continues study of the paper [4], in here, we first state and prove the p-Cockcroft property (or, equivalently, efficiency) for a presentation, say $\mathcal{P}_E$, of the semi-direct product of a free abelian monoid rank two by a finite cyclic monoid. Then, in a separate section, we present sufficient conditions on a special case for $\mathcal{P}_E$ to be minimal whilst it is inefficient.

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1. Preliminaries

Suppose that $\mathcal{P} = [X; \mathbf{r}]$ is a finite presentation for a monoid M. Then the *Euler characteristic* is defined by $\chi(\mathcal{P}) = 1 - |X| + |\mathbf{r}|$. There also exists an upper bound over M which is defined by $\delta(M) = 1 - rk_{\mathbb{Z}}(H_1(M)) + d(H_2(M))$. In fact, as depicted in [2, 3, 4], S. Pride has shown that $\chi(\mathcal{P}) \geq \delta(M)$. With this background, we define the monoid presentation $\mathcal{P}$ to be *efficient* if $\chi(\mathcal{P}) = \delta(M)$ and then M is called *efficient* if it has an efficient presentation. Moreover a presentation $\mathcal{P}_0$ for M is called *minimal* if $\chi(\mathcal{P}_0) \leq \chi(\mathcal{P})$, for all presentations $\mathcal{P}$ of M. There is also interest in finding *inefficient* finitely presented monoids since if we can find a minimal presentation $\mathcal{P}_0$ for a monoid M such that $\mathcal{P}_0$ is not efficient then we have $\chi(\mathcal{P}') \geq \chi(\mathcal{P}_0) > \delta(M)$, for all presentations

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$\mathcal{P}'$ defining the same monoid M. Thus there is no efficient presentation for M, that is, M is not an efficient monoid.

Some of the fundamental material (for instance, *semi-direct products of monoids*, *Squier complex, a trivializer set* of the Squier complex, *p-Cockcroft property, monoid pictures*) which will be needed to construct the main results of this paper have been defined and referenced in detail in [1, 2, 3, 4].

The following theorem also proved by S. Pride which we will use it rather than making more direct computations of homology for monoids. In fact Kilgour and Pride showed the analogous result for groups in [8] and credit an earlier proof by Epstein ([5]).

1.1. Proposition. *Let $\mathcal{P}$ be a monoid presentation. Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

Let A and K be arbitrary monoids with associated presentations $\mathcal{P}_A = [X\,;\,\mathbf{r}]$ and $\mathcal{P}_K = [Y\,;\,\mathbf{s}]$, respectively. Also let $E = K \rtimes_\theta A$ be the corresponding semi-direct product of these two monoids. For every $x \in X$ and $y \in Y$, choose a word, which we denote by $y\theta_x$, on Y such that $[y\theta_x] = [y]\theta_{[x]}$ as an element of K. To establish notation, let us denote the relation $yx = x(y\theta_x)$ on $X \cup Y$ by T_{yx} and write $\mathbf{t}$ for the set of relations T_{yx}. Then, for any choice of the words $y\theta_x$,

$$\mathcal{P}_E = [Y,\, X\,;\, \mathbf{s},\, \mathbf{r},\, \mathbf{t}] \tag{1.1}$$

is a standard monoid presentation for the semi-direct product E. Then a trivializer set, $\mathbf{X_E}$, of the Squier complex $\mathcal{D}(\mathcal{P}_E)$ has been defined in [10] by J. Wang as the set

$$\mathbf{X_A} \cup \mathbf{X}_K \cup \mathbf{C}_1 \cup \mathbf{C}_2$$

(see also [4, Lemma 1.5]) where $\mathbf{X_A}$ and $\mathbf{X_K}$ are the trivializers of the Squier complexes $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, and also the subsets $\mathbf{C}_1$, $\mathbf{C}_2$ consist of the generating monoid pictures $\mathbb{P}_{S,x}$ $(S \in \mathbf{s},\ x \in X)$ and $\mathbb{P}_{R,y}$ $(R \in \mathbf{r},\ y \in Y)$. Hence, by using the set $\mathbf{X_E}$, Çevik proved the following result which will be used to proof of Theorem 2.4 below.

1.2. Theorem. [3, Theorem 3.1] *Let p be a prime or* 0. *Then the presentation $\mathcal{P}_E$ in* (1.1) *is p-Cockcroft if and only if the following conditions hold.*

- *(i)* $\mathcal{P}_A$ *and* $\mathcal{P}_K$ *are p-Cockcroft,*
- *(ii)* $\exp_y(S) \equiv 0\ (mod\ p)$ *for all* $S \in \mathbf{s}$, $y \in Y$,
- *(iii)* $\exp_S(\mathbb{B}_{S,x}) \equiv 1\ (mod\ p)$ *for all* $S \in \mathbf{s}$, $x \in X$,
- *(iv)* $\exp_S(\mathbb{C}_{y,\theta_R}) \equiv 0\ \ (mod\ p)$ *for all* $S \in \mathbf{s}$, $y \in Y$, $R \in \mathbf{r}$,
- *(v)* $\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) \equiv \exp_{T_{yx}}(\mathbb{A}_{R_-,y})\ \ (mod\ p)$ *for all* $R \in \mathbf{r}$, $y \in Y$ *and* $x \in X$.

This paper has been divided into two main parts. In Section 2, we will investigate the efficiency (in fact, by Proposition 1.1, p-Cockcroft property for a prime p) for a standard presentation of the semi-direct product E of a free abelian monoid rank two, say K_2, by a finite cyclic monoid, say A, (see Theorem 2.4 below). Moreover, in Section 3, we will present the minimality of the monoid E while it has an inefficient presentation (see Theorem 3.1 below) by considering a special case.

2. Efficiency

2.1. The semi-direct product of K_2 by A. By the definition, to define a semi-direct product of K_2 by an arbitrary monoid A, we first need to define an endomorphism of K_2. To do that, let us start with $\mathbb{Z}^{+^n}$ which is the free abelian monoid rank n, say K_n. Also let $\mathcal{M}$ be an $n \times n$-matrix with non-negative integer entries. Then we get a mapping

$$\psi_{\mathcal{M}} : K_n \longrightarrow K_n,\ \ v \longmapsto v\mathcal{M},$$

where $v = (v_1, v_2, \cdots, v_n)$. Actually $\psi_{\mathcal{M}} \in End(K_n)$ (and so $\psi_{\mathcal{M}_1}\psi_{\mathcal{M}_2} = \psi_{\mathcal{M}_1\mathcal{M}_2}$). We note that if $\phi \in End(K_n)$ then there exist a matrix $\mathcal{M}$ (depending on ϕ) such that $\phi = \psi_{\mathcal{M}}$. By the mapping $\mathcal{M} \longmapsto \psi_{\mathcal{M}}$, we get an isomorphism from $Mat_n(\mathbb{Z}^+)$ to the monoid $End(K_n)$, where

$$Mat_n(\mathbb{Z}^+) = \{\mathcal{M} \ : \ \mathcal{M} \text{ is an } n \times n\text{-matrix with non-negative integer entries}\}$$

is a monoid under matrix multiplication.

Suppose $\mathcal{P}_{K_n} = [y_i \ (1 \leq i \leq n) \ ; \ y_i y_j = y_j y_i \ (1 \leq i < j \leq n)]$ is a presentation for K_n and $\mathcal{P}_A = [\mathbf{x} \ ; \ \mathbf{r}]$ is a presentation for A. Suppose also that, for each $x \in \mathbf{x}$, we have an endomorphism ψ_x of K. Since $End(K_n) \cong Mat_n(\mathbb{Z}^+)$, the endomorphism ψ_x $(x \in \mathbf{x})$ will be $\psi_{\mathcal{M}_x}$ for some matrix $\mathcal{M}_x$. For any positive word $W = x_1 x_2 \cdots x_n$ on $\mathbf{x}$, let $\mathcal{M}_W$ be the product $\mathcal{M}_{x_1}\mathcal{M}_{x_2}\cdots\mathcal{M}_{x_n}$ of the matrices $\mathcal{M}_{x_i}$, where $1 \leq i \leq n$. Then the mapping $x \longmapsto \psi_x$ $(x \in \mathbf{x})$ induces a homomorphism $\theta : A \longrightarrow End(K_n)$ if and only if $\mathcal{M}_{R_+} = \mathcal{M}_{R_-}$, for all $R \in \mathbf{r}$.

Now let A be the finite cyclic monoid with a presentation $\mathcal{P}_A = [x \ ; \ x^k = x^l]$ where $1 \leq l < k$ and $l, k \in \mathbb{Z}^+$. (We note that the fundamental material about finite cyclic monoids can be found in the book [6]).

2.1. Remark. Recall that the elements of the finite cyclic monoid A represented by equivalence classes $[x^i]$ $(0 \leq i \leq k)$. For $0 \leq i \leq l$, the equivalence class $[x^i]$ just consist of the single element x^i. However for $i \geq l$, the equivalence class $[x^i]$ consist of infinitely many elements which are defined by $[x^i] = \{x^{i+q(k-l)} \ ; \ q = 0, 1, 2, \cdots\}$.

Also let us consider K_2 and let us suppose that ψ is the endomorphism $\psi_{\mathcal{M}}$ of K_2, where

$$\mathcal{M} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

such that the entries α_{ij}'s are the positive integers given by

$$[y_1] \longmapsto [{y_1}^{\alpha_{11}}{y_2}^{\alpha_{12}}] \ \text{ and } \ [y_2] \longmapsto [{y_1}^{\alpha_{21}}{y_2}^{\alpha_{22}}].$$

Hence, by the previous explanation, the mapping $x \longmapsto \psi_x$ $(x \in \mathbf{x})$ induces a well-defined monoid homomorphism $\theta : A \longrightarrow End(K_2)$ if and only if $\mathcal{M}_{[x^k]} = \mathcal{M}_{[x^l]}$, or equivalently,

$$(2.1) \qquad \mathcal{M}^k \equiv \mathcal{M}^l \mod d,$$

where $d \mid (k - l)$.

2.2. Remark. By considering the elements of finite cyclic monoid A with its presentation $\mathcal{P}_A$ as defined in Remark *2.1*, there exits an inequality between the non-negative integers k and l such as $1 \leq l < k$. Thus to define an induces homomorphism $\theta : A \longrightarrow End(K_2)$, that is, to be able to define $K_2 \rtimes_\theta A$, we must take congruence relation between $\mathcal{M}^k$ and $\mathcal{M}^l$ as given in *(2.1)* with the assumption $d \mid (k - l)$.

In fact the kth and lth powers of the matrices can be written as follows. Initially, let us consider the matrices

$$\mathcal{M}^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ \text{ and } \ \mathcal{M}^1 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix},$$

and then, for simplicity, let us rewrite them as the matrices

$$\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \ \text{ and } \ \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix},$$

respectively. Then we clearly get

$$\mathcal{M}^2 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} A_1\alpha_{11} + B_1\alpha_{21} & A_1\alpha_{12} + B_1\alpha_{22} \\ C_1\alpha_{11} + D_1\alpha_{21} & C_1\alpha_{12} + D_1\alpha_{22} \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \text{say}.$$

Therefore the kth ($k \in \mathbb{Z}^+$) power of $\mathcal{M}$ will be

$$\begin{aligned} \mathcal{M}^k &= \begin{bmatrix} A_{k-1} & B_{k-1} \\ C_{k-1} & D_{k-1} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{k-1}\alpha_{11} + B_{k-1}\alpha_{21} & A_{k-1}\alpha_{12} + B_{k-1}\alpha_{22} \\ C_{k-1}\alpha_{11} + D_{k-1}\alpha_{21} & C_{k-1}\alpha_{12} + D_{k-1}\alpha_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \text{say}. \end{aligned}$$

As a similar idea, the lth ($l \in \mathbb{Z}^+$) power of $\mathcal{M}$ will be

$$\mathcal{M}^l = \begin{bmatrix} A_l & B_l \\ C_l & D_l \end{bmatrix}.$$

Now we can present the following lemma which gives the importance of Equation (2.1). In fact this lemma will be needed in the proof of Theorem 2.4 below.

2.3. Lemma. *The function* $\theta : A \longrightarrow End(K_2)$ *defined by* $[x] \longmapsto \theta_{[x]}$ *is a well-defined monoid homomorphism if and only if* $A_k \equiv A_l \mod d$, $B_k \equiv B_l \mod d$, $C_k \equiv C_l \mod d$ *and* $D_k \equiv D_l \mod d$, *where* $d \mid (k - l)$.

Proof. This follows immediately from $\mathcal{M}^k \equiv \mathcal{M}^l \mod d$. □

Now suppose that (2.1) holds. Then, by Lemma 2.3, we obtain a semi-direct product $E = K_2 \rtimes_\theta A$ and have a presentation

(2.2) $\quad \mathcal{P}_E = [y_1, y_2, x \; ; \; S, R, T_{y_1x}, T_{y_2x}],$

as in (1.1), for the monoid E where

$$S : y_1y_2 = y_2y_1, \qquad R : x^k = x^l$$
$$T_{y_1x} : y_1x = xy_1^{\alpha_{11}}y_2^{\alpha_{12}}, \qquad T_{y_2x} : y_2x = xy_1^{\alpha_{21}}y_2^{\alpha_{22}},$$

respectively.

At the rest of this paper, we will assume that Equality (2.1) always holds when we talk about the semi-direct product E of K_2 by A.

We know that the trivializer set of $\mathbf{X_E}$ of $\mathcal{D}(\mathcal{P}_E)$ consists of the trivializer set $\mathbf{X_{K_2}}$ of $\mathcal{D}(\mathcal{P}_{K_2})$, $\mathbf{X_A}$ of $\mathcal{D}(\mathcal{P}_A)$ and the sets $\mathbf{C}_1$, $\mathbf{C}_2$ (see [4, Lemma 1.5]). In our case, $\mathbf{X_{K_2}}$ is equal to the empty set since, for the relator S, we have $\iota(S_+) \neq \iota(S_-)$ (or, equivalently, $\tau(S_+) \neq \tau(S_-)$) and so, by [7], $\mathcal{P}_{K_2}$ is aspherical then p-Cockcroft for any prime p. Newertheless, the trivializer set $\mathbf{X_A}$ of the Squier complex $\mathcal{D}(\mathcal{P}_A)$ is defined as in Figure 1 (cf. [3, Lemma 4.4]).

Finally the subsets $\mathbf{C}_1$ and $\mathbf{C}_2$ contain the generating monoid pictures $\mathbb{P}_{S,x}$ (which contains a non-spherical subpicture $\mathbb{B}_{S,x}$ as depicted in [3]), $\mathbb{P}_{R,y_1}$ and $\mathbb{P}_{R,y_2}$ of the trivializer set $\mathbf{X_E}$. These pictures can be presented as in Figure 2-(a) and (b).

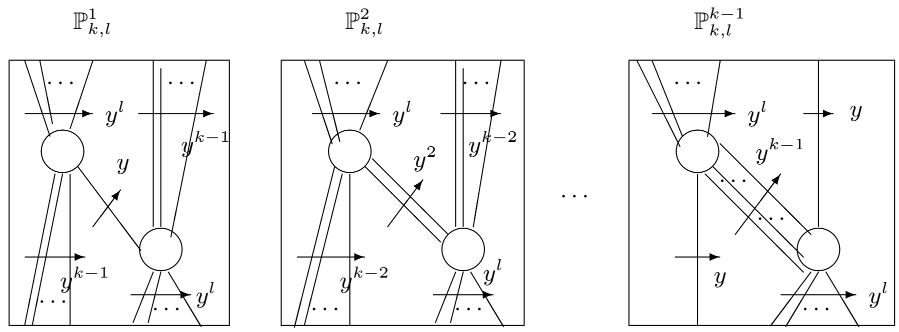
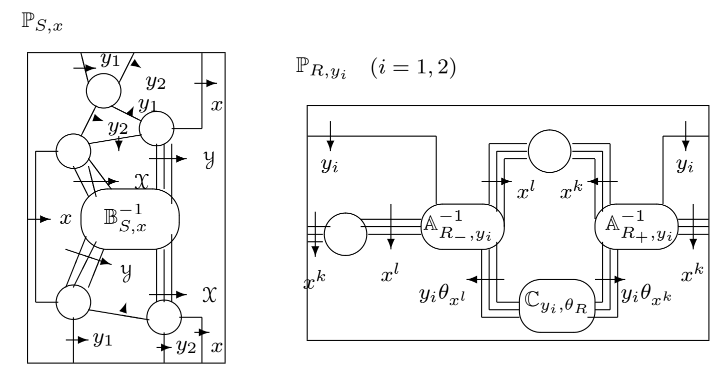


Figure 1. : Generating pictures of finite monogenic monoids



(a) (b)

Figure 2. : In the figure (a), $\mathcal{X} = y_1^{\alpha_{21}} y_2^{\alpha_{22}}$ and $\mathcal{Y} = y_1^{\alpha_{11}} y_2^{\alpha_{12}}$

2.2. The main theorem and its proof. For simplicity, let us replace the sum of coefficients

$$\left.\begin{array}{ll} A_0 + A_1 + \cdots + A_{k-1} \text{ as } \mathcal{A}_k\,, & A_0 + A_1 + \cdots + A_{l-1} \text{ as } \mathcal{A}_l\,, \\ B_0 + B_1 + \cdots + B_{k-1} \text{ as } \mathcal{B}_k\,, & B_0 + B_1 + \cdots + B_{l-1} \text{ as } \mathcal{B}_l\,, \\ C_0 + C_1 + \cdots + C_{k-1} \text{ as } \mathcal{C}_k\,, & C_0 + C_1 + \cdots + C_{l-1} \text{ as } \mathcal{C}_l\,, \\ D_0 + D_1 + \cdots + D_{k-1} \text{ as } \mathcal{D}_k\,, & D_0 + D_1 + \cdots + D_{l-1} \text{ as } \mathcal{D}_l\,. \end{array}\right\} \tag{2.3}$$

Suppose that the positive integer d, defined in (2.1), is equal to a prime p such that $p \mid (k-l)$. Therefore the first main theorem of this paper can be given as in the following.

2.4. Theorem. *Let p be a prime or 0, and consider the replacements in* (2.3). *Then the presentation $\mathcal{P}_E$, as in* (2.2)*, for the monoid $E = K_2 \rtimes_\theta A$ is p-Cockcroft if and only if*

a) $det\mathcal{M} \equiv 1 \mod p$,

b) $\mathcal{A}_k \equiv \mathcal{D}_l \mod p\,,\quad \mathcal{B}_k \equiv \mathcal{C}_l \mod p\,,$
$\mathcal{C}_k \equiv \mathcal{B}_l \mod p\,,\quad \mathcal{D}_k \equiv \mathcal{A}_l \mod p\,.$

Proof. The proof will be given by checking the conditions of Theorem 1.2. By a part of prelimary material of this paper, it is clear that $\mathbf{X_{K_2}} = \emptyset$. Also, since the trivializer set $\mathbf{X_A}$ of the Squier complex $\mathcal{D}(\mathcal{P}_A)$ can be defined as in Figure 1, it is clear that $\mathcal{P}_A$ is p-Cockcroft (in fact Cockcroft). Moreover, by considering the picture $\mathbb{P}_{S,x}$ in Figure 2-(a), we see that $exp_{T_{y_1x}}(\mathbb{P}_{S,x}) = 0 = exp_{T_{y_2x}}(\mathbb{P}_{S,x})$ which is clear by $exp_{y_1}(S) = 0 = exp_{y_2}(S)$. Thus the conditions (i) and (ii) of Theorem 1.2 hold. Furthermore in the picture $\mathbb{B}_{S,x}$, we actually have $\alpha_{11}\,\alpha_{12}$-times positive and $\alpha_{12}\,\alpha_{21}$-times negative S-discs. Thus

$$exp_S(\mathbb{B}_{S,x}) = \alpha_{11}\,\alpha_{12} - \alpha_{12}\,\alpha_{21} = det\mathcal{M}.$$

So to condition (iii) be hold, we must have $det\mathcal{M} \equiv 1 \mod p$, as required.

Let us consider the generating pictures $\mathbb{P}_{R,y_1}$ and $\mathbb{P}_{R,y_2}$ as drawn in Figure 2-(b). We always have $exp_R(\mathbb{P}_{R,y_1}) = 0 = exp_R(\mathbb{P}_{R,y_2})$. Recall that to define a semi-direct product $K_2 \rtimes_\theta A$, we assumed equality (2.1) be held. That means, for each $i \in \{1,2\}$, we must have

$$y_i\theta_{[x^k]} = y_i\theta_{[x^l]}.$$

But we know that this equality be hold if and only if the conditions in Lemma 2.3 are satisfied. Besides of that using the equality of the congruence classes gives us that there will be no $\mathbb{C}_{y_i,\theta_R}$ subpictures. In other words, all arcs in that part will be coincides to each other. So the condition (iv) will be directly held. Let us now consider the subpictures $\mathbb{A}_{R_+,y_i}$ and $\mathbb{A}_{R_-,y_i}$ which consist of only T_{y_ix} discs $(1 \leq i \leq 2)$. Since each of the generating pictures $\mathbb{P}_{R,y_1}$ and $\mathbb{P}_{R,y_2}$ contains a single subpicture $\mathbb{A}_{R_+,y_i}$ and a single subpicture $\mathbb{A}^{-1}_{R_-,y_i}$, we must have

$$exp_{y_i}(\mathbb{A}_{R_+,y_i}) - exp_{y_i}(\mathbb{A}_{R_-,y_i}) = exp_{y_i}(\mathbb{P}_{R,y_i}).$$

Now let us take into account the matrices $\mathcal{M}^0, \mathcal{M}^1, \cdots, \mathcal{M}^{k-1}$. By using the endomorphism $\psi_\mathcal{M}$ of K defined by $[y_1] \longmapsto [{y_1}^{\alpha_{11}}{y_2}^{\alpha_{12}}]$ and $[y_2] \longmapsto [{y_1}^{\alpha_{21}}{y_2}^{\alpha_{22}}]$, a simple calculation shows that the sum of the first row and first column elements in these matrices gives the exponent sum of the T_{y_1x} discs in the subpicture $\mathbb{A}_{R_+,y_1}$. In other words

$$\mathcal{A}_k = exp_{T_{y_1x}}(\mathbb{A}_{R_+,y_1}).$$

Similarly, we also get

$$\mathcal{B}_k = exp_{T_{y_2x}}(\mathbb{A}_{R_+,y_1}), \mathcal{C}_k = exp_{T_{y_1x}}(\mathbb{A}_{R_+,y_2}) \text{ and } \mathcal{D}_k = exp_{T_{y_2x}}(\mathbb{A}_{R_+,y_2}).$$

On the other hand, again by considering the matrices $\mathcal{M}^0, \mathcal{M}^1, \cdots, \mathcal{M}^{l-1}$ with the same idea as above, we obtain

$$\mathcal{A}_l = exp_{T_{y_2x}}(\mathbb{A}_{R_-,y_2}), \quad \mathcal{B}_l = exp_{T_{y_1x}}(\mathbb{A}_{R_-,y_2}),$$
$$\mathcal{C}_l = exp_{T_{y_2x}}(\mathbb{A}_{R_-,y_1}), \quad \mathcal{D}_l = exp_{T_{y_1x}}(\mathbb{A}_{R_-,y_1}).$$

Therefore to p-Cockcroft property be hold, we need

$$exp_{T_{y_ix}}(\mathbb{A}_{R_+,y_i}) \equiv exp_{T_{y_ix}}(\mathbb{A}_{R_-,y_i}) \mod p,$$

for all $1 \leq i \leq 2$.

Conversely let the two conditions $a)$ and $b)$ of the theorem be hold. Then, by using the trivializer of the Squier complex $\mathcal{D}(\mathcal{P}_E)$, we can easily see that $\mathcal{P}_E$ is p-Cockcroft where p is a prime or 0.

Hence the result. □

2.5. Remark. The importance of the assumption $p \mid (k-l)$ seems much clear in the proof of Theorem *2.4*. Otherwise we could not have obtained Equality *(2.1)* and so could not have obtained the exponent sums of the disc T_{y_1x} and T_{y_2x} congruent to zero by modulo p in the subpictures $\mathbb{A}_{R_+,y_i}$ and $\mathbb{A}_{R_-,y_i}$, where $i \in \{1,2\}$, since these sums are directly related to the number of k-arcs and l-arcs, respectively.

2.3. Some applications.

2.6. Example. Let p be an odd prime and suppose that

$$\mathcal{M} = \begin{bmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{bmatrix} \tag{2.4}$$

is a matrix representation for the endomorphism of free abelian monoid K_2 rank two. We then always have

$$\mathcal{M}^{p+1} \equiv \mathcal{M}^1 \mod p$$

and, by Lemma *2.3*, we also have $E = K_2 \rtimes_\theta A$. Hence we get a presentation

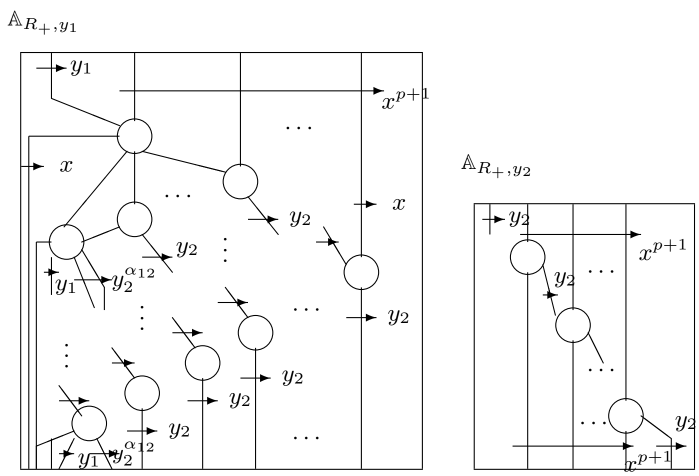
$$\mathcal{P}_E = [y_1, y_2, x \; ; \; y_1y_2 = y_2y_1, x^{p+1} = x, y_1x = xy_1y_2^{\alpha_{12}}, y_2x = xy_2], \tag{2.5}$$

as in *(2.2)*, for the monoid E.

Therefore we can give the following result as a consequence of Theorem 2.4.

2.7. Corollary. *For all odd prime p, the semi-direct product presentation $\mathcal{P}_E$ in* (2.5) *always p-Cockcroft.*

Proof. By considering the subpictures $\mathbb{A}_{R_+,y_1}$, $\mathbb{A}_{R_+,y_2}$, $\mathbb{A}_{R_-,y_1}$ and $\mathbb{A}_{R_-,y_2}$ given in Figures 3 and 4, the proof will be an easy application of Theorem 2.4. In fact the condition



(a) (b)

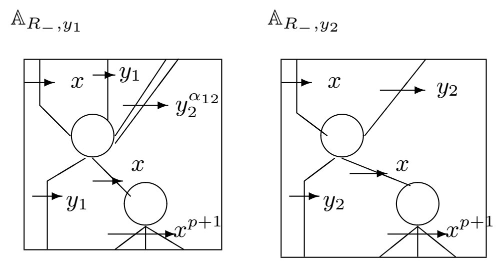
Figure 3

a) of Theorem 2.4 always holds since $det\mathcal{M} = 1$. Moreover we have

$$\begin{aligned} exp_{T_{y_1x}}(\mathbb{P}_{R,y_1}) &= exp_{T_{y_1x}}(\mathbb{A}_{R_+,y_1}) - exp_{T_{y_1x}}(\mathbb{A}_{R_-,y_1}) \\ &= A_0 + A_1 + \cdots + A_p - D_0 = (p+1) - 1 = p, \end{aligned}$$

which is obviously congruent to zero by modulo p, and

$$\begin{aligned} exp_{T_{y_2x}}(\mathbb{P}_{R,y_1}) &= exp_{T_{y_2x}}(\mathbb{A}_{R_+,y_1}) - exp_{T_{y_2x}}(\mathbb{A}_{R_-,y_1}) \\ &= B_0 + B_1 + \cdots + B_p - C_0 \\ &= \alpha_{12}\,\frac{p(p+1)}{2} - 0 = \alpha_{12}\,\frac{p(p+1)}{2} \equiv 0 \mod p. \end{aligned}$$



(a) (b)

Figure 4

Similarly,

$$\begin{aligned} exp_{T_{y_1x}}(\mathbb{P}_{R,y_2}) &= exp_{T_{y_1x}}(\mathbb{A}_{R_+,y_2}) - exp_{T_{y_1x}}(\mathbb{A}_{R_-,y_2}) \\ &= C_0 + C_1 + \cdots + C_p - B_0 \equiv 0 \mod p, \\ exp_{T_{y_2x}}(\mathbb{P}_{R,y_2}) &= exp_{T_{y_2x}}(\mathbb{A}_{R_+,y_2}) - exp_{T_{y_2x}}(\mathbb{A}_{R_-,y_2}) \\ &= D_0 + D_1 + \cdots + D_p - A_0 = (p+1) - 1 = p \equiv 0 \mod p. \end{aligned}$$

Therefore, for all $i \in \{1, 2\}$, $exp_{T_{y_ix}}(\mathbb{P}_{R,y_i}) \equiv 0 \mod p$. (We note that, by the explanation as in the proof of Theorem 2.4, we do not have $\mathbb{C}_{y_i,\theta_R}$ subpictures in $\mathbb{P}_{R,y_i}$). This completes the proof. □

2.8. Remark. In Example *2.6*, if we constructed the matrix $\mathcal{M}$, defined in *(2.4)*, for even prime p while $x^{p+1} = x$ then, by Lemma *2.3*, we would obtain a semi-direct product E for just $\alpha_{12} = 1$ or $\alpha_{12} = 0$ while $\mathcal{M}^3 \equiv \mathcal{M} \mod p$. However, for $\alpha_{12} = 1$, since

$$B_0 + B_1 + B_2 \neq C_0,$$

by Theorem *2.4*, the presentation $\mathcal{P}_E$ in *(2.5)* will be inefficient. Here, by Theorem *2.4*, one can show that $\mathcal{P}_E$ is efficient if and only if $\alpha_{12} = 0$. But $\alpha_{12} = 0$ gives the homomorphism θ is identitiy and so, $K_2 \rtimes_\theta A$ becomes $K_2 \times A$. In fact the efficiency for a presentation of the direct product of arbitrary two monoids has been investigated in [3, Theorem 4.1].

A similar case, as in Example 2.6, can be given by using the matrix

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ \alpha_{21} & 1 \end{bmatrix}.$$

Then we obtain a semi-direct product E with a presentation

(2.6) $\quad \mathcal{P}_E = [y_1, y_2, x\ ;\ y_1y_2 = y_2y_1, x^{p+1} = x, y_1x = xy_1, y_2x = xy_1^{\alpha_{21}}y_2].$

Thus we have the following result, as a consequence of Theorem 2.4, which can be proved quite similarly as in Corollary 2.7.

2.9. Corollary. *Let $\mathcal{P}_E$, as in* (2.6)*, be a presentation for the semi-direct product of K_2 by A. Then, for all odd prime p, $\mathcal{P}_E$ is p-Cockcroft.*

We note that Remark 2.8 is also valid for the above case.

2.10. Example. Suppose that p is a prime and the matrix $\mathcal{M}$ is equal to either $\begin{bmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ \alpha_{21} & 1 \end{bmatrix}$. Then, by applying a simple calculation as in the previous examples, we get an efficient semi-direct product presentation for $k = 2p + 1$ and $l = 1$.

2.11. Example. Let p be any prime and let $\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_{22} \end{bmatrix}$. Hence we get $\mathcal{M}^{2p+1} \equiv \mathcal{M} \mod p$ and, by Lemma *2.3*, we have a semi-direct product $E = K_2 \rtimes_\theta A$ with a presentation

$$\mathcal{P}_E = [y_1, y_2, x \; ; \; y_1y_2 = y_2y_1, x^{2p+1} = x, y_1x = xy_1, y_2x = xy_2^{\alpha_{22}}]. \tag{2.7}$$

As an application of Theorem 2.4, we also have the following corollary.

2.12. Corollary. *The presentation* $\mathcal{P}_E$*, as in* (2.7)*, is p-Cockcroft for all prime* p*, if* $\alpha_{22} = 1 + pt$ *where* $t > 0$.

Proof. In the proof, we will assume $\alpha_{22} = 1 + pt$, $t > 0$, and then just follow the same way as in the proof of Corollary 2.7. It is clear that $det\mathcal{M} \equiv 1 \pmod p$ by the assumption on α_{22}. So the condition *a*) in Theorem 2.4 holds. Now let us consider the subpictures $\mathbb{A}_{R_+,y_1}$, $\mathbb{A}_{R_+,y_2}$, $\mathbb{A}_{R_-,y_1}$ and $\mathbb{A}_{R_-,y_2}$ given in Figure 5. We note that, by fixing these subpictures into the pictures $\mathbb{P}_{R,y_1}$ and $\mathbb{P}_{R,y_2}$ given in Figure 2-(b), we obtain similar $\mathbb{P}_{R,y_i}$ $(1 \leq i \leq 2)$ pictures for this case. Then we have

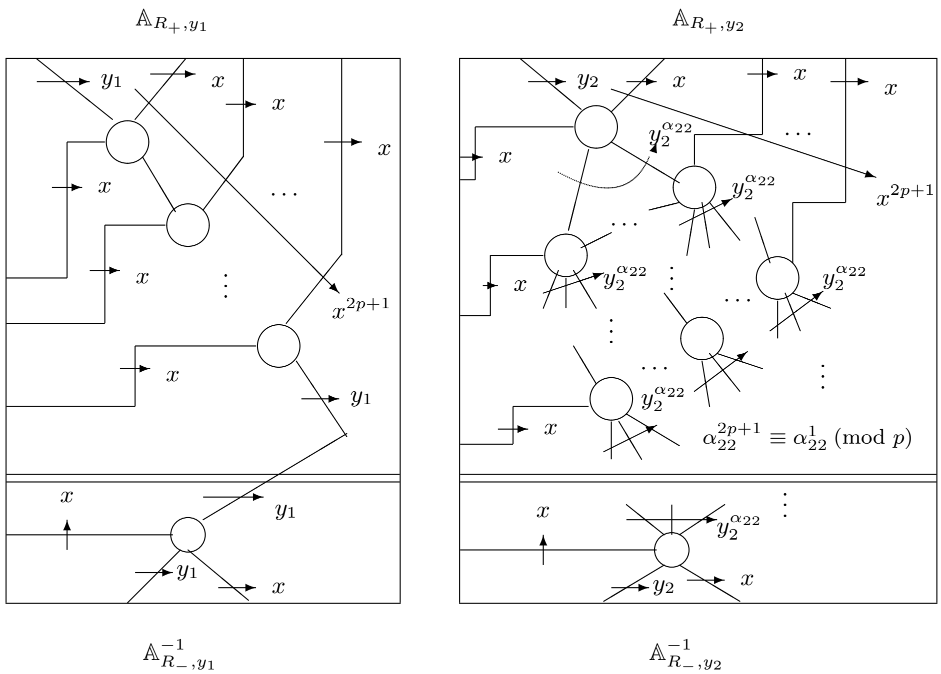


Figure 5

$$\begin{aligned}
exp_{T_{y_1x}}(\mathbb{A}_{R_+,y_1}) - exp_{T_{y_1x}}(\mathbb{A}_{R_-,y_1}) &= (2p+1) - 1 = 2p \equiv 0 \mod p, \\
exp_{T_{y_2x}}(\mathbb{A}_{R_+,y_1}) - exp_{T_{y_2x}}(\mathbb{A}_{R_-,y_1}) &\equiv 0 \mod p \quad \text{and} \\
exp_{T_{y_1x}}(\mathbb{A}_{R_+,y_2}) - exp_{T_{y_1x}}(\mathbb{A}_{R_-,y_2}) &\equiv 0 \mod p.
\end{aligned}$$

Furthermore, since

$$\begin{aligned} exp_{T_{y_2x}}(\mathbb{A}_{R_+,y_2}) - exp_{T_{y_2x}}(\mathbb{A}_{R_-,y_2}) &= 1 + \alpha_{22} + {\alpha_{22}}^2 + \cdots + {\alpha_{22}}^{2p} - 1 \\ &= \frac{{\alpha_{22}}^{2p+1} - 1}{\alpha_{22} - 1} - 1 \\ &= \frac{{\alpha_{22}}^{2p+1} - \alpha_{22}}{\alpha_{22} - 1} \equiv 0 \mod p, \end{aligned}$$

the condition $b)$ of Theorem 2.4 holds.

We should note that $\mathcal{M}^{2p+1} \equiv \mathcal{M} \pmod p$ implies ${\alpha_{22}}^{2p+1} \equiv \alpha_{22} \pmod p$ and this gives us that $\tau(\mathbb{A}_{R_+,y_2}) = \iota(\mathbb{A}^{-1}_{R_+,y_2})$, that is, there is no subpicture $\mathbb{C}_{y_2,\theta_R}$ in the picture $\mathbb{P}_{R,y_2}$ as expressed in the proof of Theorem 2.4. □

By choosing

$$\mathcal{M} = \begin{bmatrix} \alpha_{11} & 0 \\ 0 & 1 \end{bmatrix},$$

for any prime p, we get again $\mathcal{M}^{2p+1} \equiv \mathcal{M} \mod p$ as in Example 2.11, and so we obtain a presentation

$$\mathcal{P}_E = [y_1, y_2, x \ ; \ y_1y_2 = y_2y_1, x^{2p+1} = x, y_1x = xy_1^{\alpha_{11}}, y_2x = xy_2], \tag{2.8}$$

for the semi-direct product $E = K_2 \rtimes_\theta A$. Therefore, by drawing quite similar pictures as in Figure 5, we have the following consequence of Theorem 2.4.

2.13. Corollary. *The presentation $\mathcal{P}_E$, as in* (2.8)*, is p-Cockcroft for all prime p, if $\alpha_{11} = 1 + pt$ where $t > 0$.*

2.14. Remark. The examples and corrollories given in this subsection can also be true for the general case of $k = np + 1$ and $l = 1$ where n is the positive integer.

3. Minimality

3.1. The Main Theorem. Let K_2 be the free abelian monoid rank 2 with a presentation $\mathcal{P}_{K_2} = [y_1, y_2 \ ; \ y_1y_2 = y_2y_1]$ and let A be the finite cyclic monoid with a presentation $\mathcal{P}_A = [x \ ; \ x^{2p+1} = x]$. Also, suppose that ψ is the endomorphism $\psi_\mathcal{M}$ of K, where $\mathcal{M} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ such that (2.1) holds with the assumption $d = p$. Then, by Lemma 2.3, we get a semi-direct product $E = K_2 \rtimes_\theta A$ with a presentation

$$\begin{aligned} \mathcal{P}_E = [\ y_1, y_2, x \quad ; \quad & y_1y_2 = y_2y_1,\ x^{2p+1} = x, \\ & y_1x = xy_1^{\alpha_{11}}y_2^{\alpha_{12}},\ y_2x = xy_1^{\alpha_{21}}y_2^{\alpha_{22}}\ ]. \end{aligned} \tag{3.1}$$

Let us assume that

$$\begin{aligned} &\alpha_{11} = 1,\ \alpha_{12} = \alpha_{21} = 0 \quad \text{and} \quad \alpha_{22} = 1 + pt_1\ \ (t_1 > 0) \quad \text{or} \\ &\alpha_{22} = 1,\ \alpha_{12} = \alpha_{21} = 0 \quad \text{and} \quad \alpha_{11} = 1 + pt_2\ \ (t_2 > 0), \end{aligned}$$

where p is a prime. Then, by Corollary 2.12 or Corollary 2.13, the presentation $\mathcal{P}_E$ in (3.1) is p-Cockcroft for any prime p and so, by Proposition 1.1, it is efficient.

Suppose that p is an odd prime. Then, in particular, $\mathcal{P}_E$ is not efficient if

$$det\mathcal{M} = exp_S(\mathbb{B}_{S,x}) \equiv 0 \ \text{ or } \ p - 1 \mod p.$$

Therefore our another main result in this paper is the following.

3.1. Theorem. *The presentation $\mathcal{P}_E$, as in* (3.1)*, is minimal but inefficient if p is an odd prime and*

$$\text{either} \quad \begin{cases} \alpha_{11} = p-1, \\ \alpha_{12} = \alpha_{21} = 0, \\ \alpha_{22} = 1, \end{cases} \qquad \text{or} \quad \begin{cases} \alpha_{11} = 1, \\ \alpha_{12} = \alpha_{21} = 0, \\ \alpha_{22} = p-1. \end{cases}$$

3.2. Preliminaries for the minimality result. Let M be a monoid with a presentation $\mathcal{P} = [\mathbf{y}; \mathbf{s}]$, and let $P^{(l)} = \bigoplus_{S \in \mathbf{s}} \mathbb{Z}Me_S$ be the free left $\mathbb{Z}M$-module with bases $\{e_S : S \in \mathbf{s}\}$. For an atomic monoid picture, say $\mathbb{A} = (U, S, \varepsilon, V)$ where $U, V \in F(\mathbf{y})$, $S \in \mathbf{s}$, $\varepsilon = \pm 1$, the left evaluation of the positive atomic monoid picture $\mathbb{A}$ is defined by $eval^{(l)}(\mathbb{A}) = \varepsilon \hat{U} e_S \in P^{(l)}$, where $\hat{U} \in M$. For any spherical monoid picture $\mathbb{P} = \mathbb{A}_1 \mathbb{A}_2 \cdots \mathbb{A}_n$, where each $\mathbb{A}_i$ is an atomic picture for $i = 1, 2, \cdots, n$, we then define $eval^{(l)}(\mathbb{P}) = \sum_{i=1}^{n} eval^{(l)}(\mathbb{A}_i) \in P^{(l)}$. Let $\delta_{\mathbb{P},S}$ be the coefficient of e_S in $eval^{(l)}(\mathbb{P})$. So we can write $eval^{(l)}(\mathbb{P}) = \sum_{S \in \mathbf{s}} \delta_{\mathbb{P},S} e_S \in P^{(l)}$. Let $I_2^{(l)}(\mathcal{P})$ be the 2-sided ideal of $\mathbb{Z}M$ generated by the set

$$\{\delta_{\mathbb{P},S} : \mathbb{P} \text{ is a spherical monoid picture}, S \in \mathbf{s}\}.$$

Then this ideal is called the *second Fox ideal* of $\mathcal{P}$.

The fact of the following lemma has also been discussed in [4].

3.2. Lemma. *If $\mathbf{Y}$ is a trivializer of $\mathcal{D}(\mathcal{P})$ then second Fox ideal is generated by the set $\{\delta_{\mathbb{P},S} : \mathbb{P} \in \mathbf{Y}, S \in \mathbf{s}\}$.*

The concept of the second Fox ideals is needed for a *test of minimality* for monoid presentations (see [4]). The group version of this test has been proved by M. Lustig ([9]).

3.3. Theorem. *Let $\mathbf{Y}$ be a trivializer of $\mathcal{D}(\mathcal{P})$ and let ψ be a ring homomorphism from $\mathbb{Z}M$ into the ring of all $n \times n$ martices over a comutative ring L with 1, for some $n \geq 1$, and suppose $\psi(1) = I_{n \times n}$. If $\psi(\lambda_{\mathbb{P},S}) = 0$ for all $\mathbb{P} \in \mathbf{Y}$, $S \in \mathbf{s}$ then $\mathcal{P}$ is minimal.*

3.3. Proof of Theorem 3.1. As previously, let K_2 denotes the free abelian monoid rank two with a presentation $\mathcal{P}_{K_2} = [y_1, y_2, \ ; \ y_1y_2 = y_2y_1]$ and, for an odd prime p, let A denotes the finite cyclic monoid with a presentation $\mathcal{P}_A = [x, \ ; \ x^{2p+1} = x]$. Moreover let $\mathcal{M}$ be the matrix representation of K_2 with the assumption $\mathcal{M}^{2p+1} \equiv \mathcal{M} \mod p$. Then we have a semi-direct product $E = K_2 \rtimes_\theta A$ with a presentation $\mathcal{P}_E$ as in (3.1).

Suppose that $\alpha_{11} = 1$, $\alpha_{12} = \alpha_{21} = 0$ and $\alpha_{22} = p-1$ in $\mathcal{P}_E$.

Let us consider the picture $\mathbb{P}_{S,x}$, as drawn in Figure 2-(a), and also consider the generating set $\{y_1, y_2\}$ of $\mathcal{P}_{K_2}$. For a fixed element y_i in this set, let us assume that $\dfrac{\partial}{\partial y_i}$ denotes the Fox derivation with respect to y_i, and let $\dfrac{\partial^E}{\partial y_i}$ be the composition

$$\mathbb{Z}F(\{y_1, y_2\}) \xrightarrow{\frac{\partial}{\partial y_i}} \mathbb{Z}F(\{y_1, y_2\}) \longrightarrow \mathbb{Z}E,$$

where $F(\{y_1, y_2\})$ is the free monoid on $\{y_1, y_2\}$. Furthermore, for the relator $S : y_1y_2 = y_2y_1$, let us define $\dfrac{\partial^E S}{\partial y_i}$ to be $\frac{\partial^E S_+}{\partial y_i} - \frac{\partial^E S_-}{\partial y_i}$. Thus, for a fixed $y_i \in \{y_1, y_2\}$, the coefficients of $e_{T_{y_i x}}$ in $eval^{(l)}(\mathbb{P}_{S,x})$ is $\dfrac{\partial^E S}{\partial y_i}$. In fact

$$\frac{\partial^E S}{\partial y_1} = y_2 - 1 \quad \text{and} \quad \frac{\partial^E S}{\partial y_2} = 1 - y_1.$$

We then have the following proposition.

3.4. Proposition. *The second Fox ideal $I_2^{(l)}(\mathcal{P}_E)$ of $\mathcal{P}_E$ is generated by the elements*

$$\begin{array}{cc} 1-x(eval^{(l)}(\mathbb{B}_{S,x})), & 1-x^{k-1},\, 1-x^{k-2}, \cdots, 1-x, \\ \dfrac{\partial^E S}{\partial y_1}, & \dfrac{\partial^E S}{\partial y_2}, \\ eval^{(l)}(\mathbb{A}_{R_+,y_1}) - eval^{(l)}(\mathbb{A}_{R_-,y_1}), & eval^{(l)}(\mathbb{A}_{R_+,y_2}) - eval^{(l)}(\mathbb{A}_{R_-,y_2}). \end{array}$$

Proof. Recall that $\mathcal{D}(\mathcal{P}_E)$ has a trivializer $\mathbf{X_E}$ consisting of the sets $\mathbf{X_A}$, $\mathbf{X_{K_2}}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ where $\mathbf{X_A}$ (see Figure 1), $\mathbf{X_{K_2}}$ (which is equal to the empty set) are the trivializer sets of $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_{K_2})$, respectively and $\mathbf{C_1}$, $\mathbf{C_2}$ consist of the pictures $\mathbb{P}_{S,x}$ (see Figure 2-(a) by assuming $\alpha_{11}=1$, $\alpha_{12}=\alpha_{21}=0$, $\alpha_{22}=p-1$), $\mathbb{P}_{R,y_1}$ and $\mathbb{P}_{R,y_2}$ (see Figure 2-(b) by fixing $\mathbb{A}_{R_+,y_i}$ and $\mathbb{A}_{R,y_i}$ given in Figure 5), respectively. Now we need to calculate $eval^{(l)}(\mathbb{P}_{S,x})$, $eval^{(l)}(\mathbb{P}_{R,y_1})$, $eval^{(l)}(\mathbb{P}_{R,y_2})$, and $eval^{(l)}(\mathbb{P}^m_{k,l})$ $(1 \leq m \leq k-1)$. So we have

$$\begin{aligned} eval^{(l)}(\mathbb{P}_{S,x}) &= \delta_{\mathbb{P}_{S,x},S}e_S + \delta_{\mathbb{P}_{S,x},T_{y_1x}}e_{T_{y_1x}} + \delta_{\mathbb{P}_{S,x},T_{y_2x}}e_{T_{y_2x}} \\ &= (1-x(eval^{(l)}(\mathbb{B}_{S,x})))e_S + (\frac{\partial^E S}{\partial y_1})e_{T_{y_1x}} + (\frac{\partial^E S}{\partial y_2})e_{T_{y_2x}} \\ eval^{(l)}(\mathbb{P}_{R,y_1}) &= \delta_{\mathbb{P}_{R,y_1},R}e_R + \delta_{\mathbb{P}_{R,y_1},T_{y_1x}}e_{T_{y_1x}} + \delta_{\mathbb{P}_{R,y_1},T_{y_2x}}e_{T_{y_2x}} \\ &= (1-y_1)e_R + (1+x+x^2+\cdots+x^{2p}-1)e_{T_{y_1x}} + 0e_{T_{y_2x}} \\ &= (1-y_1)e_R + (eval^{(l)}(\mathbb{A}_{R_+,y_1}) - eval^{(l)}(\mathbb{A}_{R_-,y_1}))e_{T_{y_1x}}. \end{aligned}$$

$$\begin{aligned} eval^{(l)}(\mathbb{P}_{R,y_2}) &= \delta_{\mathbb{P}_{R,y_2},R}e_R + \delta_{\mathbb{P}_{R,y_2},T_{y_1x}}e_{T_{y_1x}} + \delta_{\mathbb{P}_{R,y_2},T_{y_2x}}e_{T_{y_2x}} \\ &= (1-y_2)e_R + 0e_{T_{y_1x}} + (1+x+xy_2+x^2y_2^2+\cdots+x^2y_2^{\alpha_{22}-1} + \\ &\quad \cdots + x^{2p}+x^{2p}y_2+x^{2p}y_2^2+\cdots+x^{2p}y_2^{\alpha_{22}{}^{2p}})e_{T_{y_2x}} \\ &= (1-y_2)e_R + (eval^{(l)}(\mathbb{A}_{R_+,y_2}) - eval^{(l)}(\mathbb{A}_{R_-,y_2}))e_{T_{y_2x}}. \end{aligned}$$

Also, for each $1 \leq m \leq k-1$, $eval^{(l)}(\mathbb{P}^m_{k,l}) = \delta_{\mathbb{P}^m_{k,l},R}e_R$, where $\delta_{\mathbb{P}^m_{k,l},R} = 1-x^{k-m}$.

Thus, by Lemma 3.2, we get the result as required. □

Let $aug : \mathbb{Z}E \longrightarrow \mathbb{Z}$, $s \longmapsto 1$ be the augmentation map.

3.5. Lemma. *We have the following equalities.*

1) $aug(eval^{(l)}(\mathbb{B}_{S,x})) = \exp_S(\mathbb{B}_{S,x})$.

2) *i)* $aug(\frac{\partial^E S}{\partial y_1}) = aug(y_2-1) = \exp_{y_1}(S)$,

ii) $aug(\frac{\partial^E S}{\partial y_2}) = aug(1-y_1) = \exp_{y_2}(S)$.

3) $\left.\begin{array}{lcl} i)\ aug(eval^{(l)}(\mathbb{A}_{R_+,y_i})) &=& exp_{T_{y_ix}}(\mathbb{A}_{R_+,y_i}), \\ ii)\ aug(eval^{(l)}(\mathbb{A}_{R_-,y_i})) &=& exp_{T_{y_ix}}(\mathbb{A}_{R_-,y_i}), \end{array}\right\}$ *for* $i \in \{1,2\}$.

4) $aug(eval^{(l)}(\mathbb{P}^m_{k,l})) = aug(1-x^m) = exp_R(\mathbb{P}^m_{k,l}))$, $1 \leq m \leq k-1$.

Proof. Since similar proofs of 1) and 2) can be found in [4], we will only show the remaining conditions.

<u>*Proof of* 3)</u>:

We will just consider $i)$ since the proof of $ii)$ is completely same with the first one. We can write

$$eval^{(l)}(\mathbb{A}_{R_+,y_i}) = \varepsilon_1 W_1 e_{T_{y_i x}} + \varepsilon_2 W_2 e_{T_{y_i x}} + \cdots + \varepsilon_n W_n e_{T_{y_i x}},$$

where, for $1 \leq j \leq n$, $\varepsilon_j = \pm 1$ and each W_j is the certain word on the set $\{y_1, y_2\}$. In the right hand side of the above equality, each term $\varepsilon_j W_j e_{T_{y_i x}}$ corresponds to a single $T_{y_i x}$ disc and, in fact, the value of each ε_j gives the sign of this single $T_{y_i x}$ disc. Therefore, since the $T_{y_i x}$ discs can only be occured in the subpictures $\mathbb{A}_{R_+,y_i}$ and $\mathbb{A}_{R_-,y_i}$, the sum of each ε_j (which is equal to the $aug(eval^{(l)}(\mathbb{A}_{R_+,y_i}))$) must give the exponent sum of the $T_{y_i x}$ discs in the picture $\mathbb{P}_{R,y_i}$, as required.

<u>*Proof of* 4)</u>:

For each $1 \leq m \leq k-1$, since each $\mathbb{P}^m_{k,l}$ contains just two R-discs (one is positive and the other is negative), we write

$$eval^{(l)}(\mathbb{P}^m_{k,l}) = -{W_1}^m e_R + {W_2}^m e_R,$$

where each ${W_j}^m$ is the word on x $(1 \leq j \leq 2)$. As in the previous case, by considering the each term in above equalitiy, we get the sign of this single R-disc. Then the sum of the whole these signs (i.e the augmentation of the evaluation of each picture) must give the exponent sum of R-discs. That is,

$$aug(eval^{(l)}(\mathbb{P}^m_{k,l})) = aug(1 - x^m) = \exp_R(\mathbb{P}^m_{k,l}),$$

as required. Hence the result. □

We note that $det\mathcal{M} = \exp_S(\mathbb{B}_{S,x}) = p - 1$, where p is an odd prime, for the picture $\mathbb{P}_{S,x}$ in Figure 2-(a).

Also let us consider the homomorphism from E onto the finite cyclic monoid $M_{k,l}$ generated by x, defined by $y_1,\ y_2 \longmapsto 1,\ x \longmapsto x$. This induces a ring homomorphism

$$\gamma : \mathbb{Z}E \longrightarrow M_{k,l}[x].$$

Let η be the composition of γ and the mapping

$$M_{k,l}[x] \longrightarrow \mathbb{Z}_p[x], \quad x \longmapsto x,\ \ n \longmapsto \overline{n}\ (n \in \mathbb{Z}),$$

where $\overline{n}$ is n (mod p) and $p \mid (k - l)$.

We note that the restriction of η to the subring $\mathbb{Z}K_2$ of $\mathbb{Z}E$ is just the augmentation map $aug_p : \mathbb{Z}K_2 \longrightarrow \mathbb{Z}_p$ by modulo p. Therefore the following lemma is valid.

3.6. Lemma. *We have the following equalities.*

$i)$ $aug_p(eval^{(l)}(\mathbb{P}^m_{k,l})) \equiv 0\ (mod\ p)$.

$ii)$ $aug_p(\frac{\partial^E S}{\partial y_1}) = aug_p(\frac{\partial^E S}{\partial y_2}) \equiv 0\ (mod\ p)$.

$iii)$ $aug_p(eval^{(l)}(\mathbb{P}_{R,y_1})) \equiv 0\ (mod\ p)$ *and* $aug_p(eval^{(l)}(\mathbb{P}_{R,y_2})) \equiv 0\ (mod\ p)$.

Proof. By Lemma 3.5-4), for $1 \leq m \leq k-1$, since $aug(eval^{(l)}(\mathbb{P}^m_{k,l})) = aug(1 - x^m) = exp_R(\mathbb{P}^m_{k,l}))$ and since, by Figure 1, $exp_R(\mathbb{P}^m_{k,l})) = 0$, it is obvious that the condition $i)$ holds. Similarly, by Lemma 3.5-2), $aug(\frac{\partial^E S}{\partial y_1}) = \exp_{y_1}(S) = 0 = \exp_{y_2}(S) = aug(\frac{\partial^E S}{\partial y_2})$. Then the condition $ii)$ clearly holds.

Let us consider the generating pictures $\mathbb{P}_{R,y_1}$ and $\mathbb{P}_{R,y_2}$, as drawn in Figure 2-(b) (by fixing the subpictures $\mathbb{A}_{R_+,y_i}$ and $\mathbb{A}_{R_-,y_i}$ given in Figure 5 into them). By Lemma 3.5-3), we then have

$$\begin{aligned} aug(eval^{(l)}(\mathbb{P}_{R,y_1})) &= aug[eval^{(l)}(\mathbb{A}_{R_+,y_1}) - eval^{(l)}(\mathbb{A}_{R_-,y_1})]e_{T_{y_1 x}} \\ &\quad + aug(1-y_1)e_R \\ &= exp_{T_{y_1 x}}(\mathbb{A}_{R_+,y_1}) - exp_{T_{y_1 x}}(\mathbb{A}_{R_-,y_1}) + 0 \\ &= (2p+1) - 1 = 2p \end{aligned}$$

which is congruent to zero by modulo p. Moreover

$$\begin{aligned} aug(eval^{(l)}(\mathbb{P}_{R,y_2})) &= aug[eval^{(l)}(\mathbb{A}_{R_+,y_2}) - eval^{(l)}(\mathbb{A}_{R_-,y_2})]e_{T_{y_2 x}} \\ &\quad + aug(1-y_2)e_R \\ &= exp_{T_{y_2 x}}(\mathbb{A}_{R_+,y_2}) - exp_{T_{y_2 x}}(\mathbb{A}_{R_-,y_2}) + 0 \\ &= \frac{\alpha_{22}^{2p+1} - 1}{\alpha_{22} - 1} - 1 = \frac{\alpha_{22}^{2p+1} - \alpha_{22}}{\alpha_{22} - 1} \\ &= \frac{(p-1)^{2p+1} - (p-1)}{(p-2)} \end{aligned}$$

which is congruent to zero by modulo p. Hence the result. $\square$

Thus, by Lemmas 3.5 and 3.6, the image of $I_2^{(l)}(\mathcal{P}_E)$ under η is the ideal of $\mathbb{Z}_p[x]$ that is generated by the element $1 - x(\exp_S(\mathbb{B}_{S,x})) = 1 - \overline{(p-1)}x$ since $\exp_S(\mathbb{B}_{S,x}) = det\mathcal{M} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = p-1$. In other words,

$$\eta(I_2^{(l)}(\mathcal{P}_E)) \quad = \quad < 1 - \overline{(p-1)}x >= I, \text{ say.}$$

3.7. Remark. A simple calculation shows that $I \neq \mathbb{Z}_p[x]$ since $1 \notin I$.

Let ψ be the composition

$$\mathbb{Z}E \xrightarrow{\eta} \mathbb{Z}_p[x] \xrightarrow{\phi} \mathbb{Z}_p[x]/I,$$

where ϕ is the natural epimorphism. Then

$$\begin{aligned} \psi(1 - \hat{x}(eval^{(l)}(\mathbb{B}_{S,x}))) &= \phi\eta(1 - \hat{x}(eval^{(l)}(\mathbb{B}_{S,x}))) \\ &= \phi(1 - \hat{x}\overline{(\exp_S(\mathbb{B}_{S,x})}) \text{ since } \eta \text{ is a ring} \\ &\quad \text{homomorphism and by Lemma } 3.5-1) \\ &= \phi(1 - \hat{x}\overline{(p-1)}) \text{ since } \exp_S(\mathbb{B}_{S,x}) = p-1 \\ &= 0. \end{aligned}$$

Moreover, by Lemmas 3.5 and 3.6, the images of $1 - x^{k-1}$, $1 - x^{k-2}$, $\cdots$, $1 - x$, $\frac{\partial^E S}{\partial y_1}$, $\frac{\partial^E S}{\partial y_2}$, $eval^{(l)}(\mathbb{A}_{R_+,y_1}) - eval^{(l)}(\mathbb{A}_{R_-,y_1})$, $eval^{(l)}(\mathbb{A}_{R_+,y_2}) - eval^{(l)}(\mathbb{A}_{R_-,y_2})$ under ψ are all equal to 0 since the related exponent sums are all congruent to zero by modulo p. That means the images of the generators $I_2^{(l)}(\mathcal{P}_E)$ are all 0 under ψ. Therefore, by Theorem 3.3 (Pride), $\mathcal{P}_E$ is minimal and so $E = K_2 \rtimes_\theta A$ is a minimal but inefficient monoid.

We note that, by using the same method as in this proof, one can see that E is a minimal but inefficient monoid if p is an odd prime and

$$\alpha_{11} = p-1, \; \alpha_{22} = 1 \text{ and } \alpha_{12} = 0 = \alpha_{21}.$$

These all above progress complete the proof of Theorem 3.1. $\diamondsuit$

3.8. Example. Let $p = 3$ and $\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Thus we have $\mathcal{M}^7 \equiv \mathcal{M} \mod 3$ and, by Lemma *2.3*, we have $E = K_2 \rtimes_\theta A$ with a presentation $\mathcal{P}_E = [y_1, y_2, x\ ;\ y_1y_2 = y_2y_1, x^7 = x, y_1x = xy_1, y_2x = xy_2^2]$, as in *(3.1)*, for the monoid E. It is clear that $det\mathcal{M} = 2$ so, by Theorem *2.4*, $\mathcal{P}_E$ is inefficient and also, by Theorem *3.1*, $\mathcal{P}_E$ is minimal. Moreover, by taking the matrix $\mathcal{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, it can also be obtained a minimal but inefficient presentation.

3.9. Remark. *1)* By using same progress as in the proof of Theorem *3.1*, one can see that if $det\mathcal{M} = 0$ then $1 \in I$, that is,

$$\eta(I_2^{(l)}(\mathcal{P}_E)) =< 1 >= I$$

and so $I = \mathbb{Z}_p[x]$ (see Remark *3.7*). In fact this equality holds for any prime p. That means the minimality test (Theorem *3.3*) used in this paper cannot work for this case. Therefore it can be remained as a conjecture whether the presentation obtained by this case is minimal.

2) For $p = 2$, we have $det\mathcal{M} = 0$ or 1. In the case of $det\mathcal{M} = 1$, we know that $\mathcal{P}_E$ is efficient (see Corollary *2.12* or Corollary *2.13*) and so we cannot apply Theorem *3.1*. Furthermore if $det\mathcal{M} = 0$ then we need to turn back condition *1)*.

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