AL-KHWÂRAZMÎ, ʿABDU'L-ḤAMÎD IBN TURK, AND THE PLACE OF CENTRAL ASIA IN THE HISTORY OF SCIENCE AND CULTURE

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Abû Jaʿfar Muḥammad ibn Mûsâ al-Khwârazmî is a truly outstanding personality and a foremost representative of the supremacy of the Islamic World during the Middle Ages in scientific and intellectual pursuits. Medieval Islam was largely responsible for the shaping of the canon of knowledge which dominated medieval European thought. This was the result of a noteworthy process of multidimensional and complex transmission of scientific knowledge enriched at most stages by new contributions and creative activity. Al-Khwârazmî is a symbol of this historical process and a key figure at its early and formative stages which were realized in Islam as well as in its later phases in which the passage of systematic influence from Islam to Western Europe was involved.

Indeed, Al-Khwârazmî's fame and sphere of influence overstepped the boundaries of the World of Islam itself and extended into Western Europe upon the advent of the "Twelfth Century Renaissance". Though his activity ranged clearly over much wider spheres, his main title to fame rested upon his achievements in the fields of arithmetic and algebra, in both of which he had the reputation of being a trailblazer and an innovator. The European word *algebra* was derived from the name of his book entitled "An Abridged Treatise on the Jabr and Muqâbala (Type of) Calculation" *(Al-Kitâb al-Mukhtaṣar fî Ḥisâb al-Jabr wa al-Muqâbala),* while the method of calculation with the so called Hindu-Arabic numerals, or number system, was called *algorismi* or *algoritmi* and its several other variants, derived from the name of Al-Khwârazmî, in Western Europe, in the late Middle Ages, and this was the origin of the modern word *algorithm,* signifying the art of computing in a specific or particular manner or way.

Sarton says, "... the history of science is not simply the history of great scientists. When one investigates carefully the genesis of any dis-

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covery, one finds that it was gradually prepared by a number of smaller ones, and the deeper one's investigation, the more intermediary stages are found. ..."[1] These words are rather sharply reminiscent of the results of scholarly research on Al-Khwârazmî as an innovator in the field of algebra and a trailblazer in his activity of transmitting and spreading the method of calculation with the Hindu-Arabic numerals. But Sarton's words quoted above should at the same time serve to make us feel sure that such elaborations and developments of our knowledge of the history of various subjects should be looked upon as entirely in keeping with the nature of things. Consequently any observation of this kind in connection with Al-Khwârazmî's work should not detract in any way from the greatness of Al-Khwârazmî as an outstanding scientist and teacher of world-wide scope.

Al-Khwârazmî's years of greatest productivity coincided with the reigns of the seventh Abbasid caliph Al-Ma'mûn (813-833 A.D.) and his two successors Al-Muᶜtaṣim and Al-Wâthiq (842-847). He worked in the Bayt al-Ḥikma, or the House of Wisdom, which was founded by Al-Ma'mûn's father Hârûn al-Rashîd and the Barmaks,[2] but developed especially during Al-Ma'mûn's reign. This was a kind of academy and center of systematic translation of scientific, philosophical, and medical works especially from Greek and Syriac into Arabic, and Al-Khwârazmî was associated with it. He was apparently at the head of this institution, as it may be gathered from certain statements of Ibn al-Nadîm and Ibn al Qifṭî.[3]

According to Aristide Marre, Ibn al-Âdamî wrote in his zîj called Naẓm al-ᶜIqd that Al-Ma'mûn, before his accession to the throne of the caliphate, had Al-Khwârazmî prepare for him a compendium or abridged version of the book called Sindhind which had been brought to Baghdad by Manqa during the reign of Al-Manṣûr (754-775).[4] This means that Al-

[1] George Sarton, *The History of Science and New Humanism,* Henry Holt and Company, New York 1931, pp. 35-36.

[2] See, Aydın Sayılı, *The Observatory in Islam,* Turkish Historical Society publication, Ankara 1960, 1988, the Arno press publication, 1981, pp. 54-55.

[3] See, Aydın Sayılı, *The Observatory in Islam,* p. 55.

[4] See, Aristide Marre, *Le Messahat de Mohammed ben Moussa al-Khwârazmî, Traduit et Annoté,* 2ᵉ edition revue et corrigée sur le texte arabe, Rome 1866, p. 2; Abû'l-Qâsim Qurbânî, *Riyâḍîdânân-i Irânî ez Khwârazmî tâ Ibn-i Sînâ,* Tehran 1350 HS., p. 3; Ahmad Saidan, *Al-Fuṣûl fî'l-Hisâb al-Hindî li Abî'l-Ḥasan ibn Ibrâhîm al-Uqlîdisî,* Urdun 1973, p. 8.

Khwârazmî was a scientist with an established fame already sometime before the year 813, in case Aristide Marre's assertion is well-founded.

Al-Khwârazmî is also known to have been the author of a zîj, i.e., a book containing astronomical tables and material of an auxiliary nature. We may assume that this was not the same as the one described as an "abridged version of the Sindhind." Only a version of Al-Khwârazmî's zîj as revised by Maslama al-Majrîṭî (fl.ca. 1000) has come down to us. This book of Al-Khwârazmî contains sine and tangent tables, but the latter function may have been added by Maslama.[5]

It is also known that Al-Khwârazmî wrote not only one but two zîjs, or that he brought out perhaps two editions of his zîj.[6] Ibn al-Nadîm says that people had confidence in Al-Khwârazmî's "two zîjs, the first and the second, and used them, before the observation program and after."[7] E.S. Kennedy assigns Al-Khwârazmî's zîj, i.e., the one of which the Latin translation of the Maslama al-Majrîṭî version has come down to us, to the year 840 approximately, without explaining the justification for this dating.[8]

A justification for Kennedy's dating may possibly be sought in a statement of Ibn Yûnus (d. 1009) reporting that Al-Khwârazmî referred in the introduction, now lost, to his zîj, to astronomical observations made in Baghdad during Al-Ma'mûn's reign for the purpose of determining the obliquity of the ecliptic.[9] Al-Khwârazmî was, it seems, more or less involved in practically all of the scientific work carried out under Al-Ma'mûn's patronage, and we know on the authority of Al-Beyrûnî (d. after 1050) that he was present at least at one solstice observation made in 828 A.D. in Al-Ma'mûn's Shammâsiyya Observatory of Baghdad.[10]

[5] See, George Sarton, *Introduction to the History of Science*, vol. 1, 1927, pp. 563-564. See also, J. Vernet, "Al-Khwârazmî", *Encylopaedia of Islam*, new edition, vol. 4, 1978, pp. 1070-1072.

[6] See, Qurbânî, *op. cit.*, pp. 3, 15; C.J. Toomer, "Al-Khwârazmî", *Dictionary of Scientific Biography*, vol. 7, 1973, pp. 360-361.

[7] Ibn al-Nadîm, *Kitâb al-Fihrist*, ed. Flügel, 1871, p. 274; Bayard Dodge, *The Fihrist of Al-Nadîm*, Columbia University Press, vol. 2, 1970, p. 652.

[8] E.S. Kennedy, "A Survey of Islamic Astronomical Tables", *Transactions of the American Philosophical Society*, New Series, vol. 46, part 2, 1956, pp. 128, 148.

[9] See, Toomer, *op. cit.*, p. 361 and note 18.

[10] See, Aydın Sayılı, *The Observatory in Islam*, p. 56.

There are thus two references at least, one by Ibn Yûnus and one by Ibn al-Nadîm, to a zîj by Al-Khwârazmî which was written after a certain astronomical observation, or observations, carried out under Al-Ma'mûn's patronage. The astronomical observations made in Baghdad during Al-Ma'mûn's reign with the purpose of the determination of the obliquity of the ecliptic to which Ibn Yûnus refers may possibly belong to a time prior to the foundation of the Shammâsiyya Observatory. Ibn al-Nadîm's reference to two zîjs written respectively before and after the "observations", on the other hand, gives the impression that he is thinking of Al-Ma'mûn's observatory building activity and his elaborately conceived and directed astronomical observations carried out in his two observatories, one of Baghdad and the other of Damascus.

In Al-Khwârazmî's zîj, which has come down to our time in the Latin translation of its Maslama version, methods of Indian astronomy are generally used, but Al-Khwârazmî is seen to have also adopted in it some Persian and Ptolemaic procedures and parameters.[11]

The foundation of the Shammâsiyya Observatory of Al-Ma'mûn in Baghdad marks the beginning of the definitive predominence of Ptolemaic astronomy in Islam. Indian astronomy was used by Al-Ma'mûn's astronomers until sometime before the foundation of the Shammâsiyya Observatory. The earliest observation known to have been made from that observatory is in the year 828 A.D. (213 H.).[12] This makes it quite likely therefore that the date 828 must have been some years later than the latest possible date for the composition of Al-Khwârazmî's zîj, i.e., for the composition of the earlier of the two zîjs said to have been prepared by him.

Kennedy writes: "Bîrûnî (in *Rasâ'il,* I, pp. 128, 168) notes the existence of a book by Al-Farghânî, a younger contemporary of Khwârazmî, criticizing the latter's zîj, and Bîrûnî himself demonstrates (in *Rasâ'il,* I,

[11] See, Kennedy, *op. cit.,* pp. 148-151, 170-172; Ahmad Saidan, *op. cit.,* p. 8; Abû'l-Qâsim Qurbânî, *op. cit.,* p. 3; Toomer, *op. cit.,* pp. 360-361, 364-365. See also, Toomer, *ibid.,* for further bibliography on the subject, and, Sukumar Ranjan Das, "Scope and Development of Indian Astronomy", *Osiris,* vol. 2, 1936, p. 205. D.A. King, "Al-Khwârazmî and New Trends in Mathematical Astronomy in the Ninth Century", *The Hagop Kevorkian Center for Near-Eastern Studies, Occasional Papers on the Near East, Number Two,* New York University, 1981; and A.A. Ahmedov, J. Ad-Dabbâgh, B.A. Rosenfeld, "Istanbul Manuscripts of Al-Khwârazmî's Treatises", *Erdem,* vol. 3, number 7, 1987, pp. 163-211.

[12] See, Aydın Sayılı, *The Observatory in Islam,* pp. 79-80, 56-60.

p. 131) an error in Al-Khwârazmî's planetary equation theory. It is curious to note that in spite of the simultaneous existence of tables based on more refined theories, this zîj was used in Spain three centuries after it had been written, and thence translated into Latin."[13]

But this may bespeak the respect inspired or the authority enjoyed by Al-Khwârazmî's person, or a curiosity felt toward Indian astronomical methods, or it may perhaps represent an exceptional case of some kind. For, in the Baghdad intellectual circle of Al-Ma'mûn's time the situation seems to point to the definitive establishment of the idea of the superiority of the Ptolemaic-Greek astronomy during the reign of Al-Ma'mûn, or during the later parts of that period at any rate.

Indeed, Ḥabash al Ḥâsib writes, in the Introduction to his "Damascene" Zîj, as follows:

"And when he (Al-Ma'mûn) found out that such was the situation, he ordered Yaḥyâ ibn Abî Manṣûr al-Ḥâsib to conduct an investigation into the origins of the books on the science of the stellar bodies and to bring together the scholars well versed in that art and the philosophers of his time in order to have them cooperate in investigating the roots of that science and to attempt to make the necessary corrections. For Ptolemy of Pelusium had brought forth proof to the effect that the comprehension of what he had sought to ascertain concerning the science of the heavens was not impossible.

"Yaḥyâ acted in accordance with the orders he had received from Al-Ma'mûn concerning this undertaking and gathered together scholars proficient in the art of calculations on the stellar bodies, and philosophers considered as the foremost authorities of the time. Yaḥyâ and these co-workers launched an investigation into the roots of these books. They examined them carefully and compared their contents. The outcome of this investigation was that they did not find, among all these works, any which was more correct than the book entitled Almagest, of Ptolemy of Pelusium. ...

[13] Kennedy, *op. cit.*, p. 128. According to M.S. Khan, Ṣâ'id al-Andalusî (born in 1029) in his Ṭabaqât al-Umam, criticized "Al-Majrîtî for not correcting the errors while reconstructing the astronomical tables of Al-Khwârazmî." See, M.S. Khan, "Ṭabaqât al-Umam: The First World History of Science", *Islamic Studies*, 30:4, 1991, p. 528. See also, *ibid.*, s. 529.

"They therefore accepted this book as a canon for themselves. They then resorted to the use of instruments with which astronomical observations are made, such as the armillary sphere and others, and in their astronomical observations they followed the methods and rules prescribed by Ptolemy and examined the trajectories of the sun and the moon on different occasions in Baghdad.

"Then, after the death of Yaḥyâ ibn Abî Manṣûr, Al-Ma'mûn, may God be pleased with him, went to Damascus and addressed himself to Yaḥyâ ibn Aktam and Al-ᶜAbbâs ibn Saᶜîd al-Jawharî ... whereupon they chose for him Khâlid ibn ᶜAbd al-Malik al-Marwrûdhî. Al-Ma'mûn ordered him to make ready instruments of the greatest possible perfection and to observe the stellar bodies for a whole year at Dayr Murrân. ..." [14]

Under these circumstances it seems quite clear that Al-Khwârazmî's zîj prepared for Al-Ma'mûn and written, according to Ibn al-Nadîm, after the *raṣad* (observations at the Observatory) should not be the one somewhat revised by Maslama al-Majriṭî. This must have been a zîj, such as that of Ḥabash al-Ḥâsib, based on the work and especially observations carried out in the Shammâsiyya and Qâsiyûn Observatories. The zîj of Al-Khwârazmî, as revised by Maslama, which we possess in its Latin translation must therefore go back to the years before 828. Ibn al-Qifṭî also states briefly that during Al-Ma'mûn's reign Ptolemy came to the forefront as an authority and that this was followed by an activity based on observational work. [15]

The historian Ṭabarî speaks of Al-Khwârazmî, and on one occasion he reports that when the caliph Al-Wâthiq was fatally ill he ordered astrologers to come to his bedside so that he would have them make a prognostication concerning his life span, shortly before his death, and Al-Khwârazmî was among them. But the name Al-Khwârazmî occurs in the form of Muḥammad ibn Mûsâ al-Khwârazmî al-Majûsî al-Quṭrubbullî. Sanad ibn ᶜAlî is also in the group.

This was supposed to refer to Al-Khwârazmî, and it was assumed that some kind of a mistake had somehow crept in. However, in case it is

[14] Aydın Sayılı, "The Introductory Section of Ḥabash's Astronomical Tables Known as the 'Damascene' Zîj", *Ankara Üniversitesi Dil ve Tarih-Coğrafya Fakültesi Dergisi,* vol. 13, 1955, pp. 142-143.

[15] See, Ibn al-Qifṭî, *Ta'rîkh al-Ḥukamâ,* ed. Lippert, Berlin 1903, p. 271.

assumed that the person in question is Al-Khwârazmî, one has to accept that he had the additional epithet al-Quṭrubbullî referring to a district not far from Baghdad. But such an epithet for him is not attested in any other source. He should also be assumed to have some connection with the Zoroastrian religion because of his epithet Al-Majûsî, and he is known to be a devout Moslem.[16]

Apparently this confusion is due merely to the dropping off of the conjunctive particle "and" *(wa),* as aptly pointed out by Roshdi Rashed. Al-Majûsî al-Quṭrubbullî thus refers to another person who was present among the group assembled at the caliph's bedside. There may thus be missing another word such as Muḥammad or ʿAlî, e.g., i.e., the given name of Al-Majûsî al-Quṭrubbullî.[17]

This means that Al-Khwârazmî was still alive in 847 A.D., the date of Al-Wâthiq's death. Indeed, we have another clue indicating that Al-Khwârazmî was still alive at the beginning of that caliph's reign and that he was held in high esteem by that caliph. According to the testimony of the tenth century historian Al-Maqdisî (or Muqaddasî), the caliph Wâthiq sent Al-Khwârazmî, early during his reign, to Tarkhan, king of the Khazars.

There has been some hesitation as to whether the person in question here was Abû Jaʿfar Muḥammad ibn Mûsâ al-Khwârazmî or Muḥammad ibn Mûsâ ibn Shâkir. Dunlop at first tended to agree with Suter in deciding that the person visiting Tarkhan, the king of the Khazars, was probably Muḥammad ibn Mûsâ ibn Shâkir.[18] But later Dunlop is seen to have changed his opinion in the light of certain additional bits of information. He says, "If it is a fact that Al-Khwârazmî visited Khazaria, very likely he did so for scientific purposes." But there is really no good reason for casting this sentence into the conditional form. For Al-Maqdisî openly states this as a fact, and he gives the name of the person sent to Tarkhan as Muḥammad ibn Mûsâ al-Khwârazmî "the munajjim", so that there is

[16] See, Toomer, *op. cit.*, p. 358.

[17] See, Roshdi Rashed, *Entre Mathématique et Algèbre, Recherches sur l'Histoire des Mathématiques Arabes,* Les Belles Lettres, Paris 1984, p. 17, note l. See also, Aydın Sayılı, *The Observatory in Islam,* p. 33.

[18] D.M. Dunlop, "Muhammad ibn Mûsâ al-Khwârazmî", *Journal of the Royal Asiatic Society of Great Britain and Ireland,* 1943, pp. 248-250.

no reason at all to think that the person may have been one of the Banû Mûsâ Brothers. Furthermore, the text has been subjected to no amendation at this point and the editor does not give any relevant variants in the footnotes.[19]

There is no compelling reason either to uphold the supposition that the visit was of a scientific nature. It is said by Maqdisî that the caliph saw in his dream that the Wall of Gog and Magog built by Alexander had been breached and thereupon sent Sallâm on a journey with the specific purpose of ascertaining the actual situation. It is on this occasion that Al-Maqdisî mentions Al-Khwârazmî's visit to the Khazar king which took place somewhat earlier. The visit may have been of a political nature with a religious or commercial background. Although the Jewish religion was accorded an official status, among the Khazars, the Muslim religion too was extensively practiced in the Khazar state,[20] and the Moslem-Khazar trade relations too were of considerable dimensions.[21]

What can be said with greater certainty is that Al-Khwârazmî's visit to the Khazar King has the earmarks of an official visit. Sallâm, who some time later was commissioned with a similar visit, was an interpreter in the court of Al-Wâthiq and dealt especially with the caliph's Turkish correspondence.[22] It may be conjectured, therefore, that the reason why Al-Khwârazmî was commissioned with the visit was partly the circumstance that he knew Turkish, the language of the Khazars. Indeed, this would not be surprising at all for a person like Al-Khwârazmî simply in view of his being a native of Khwârazm. Beyrûnî too, e.g., who was a native of Khwârazm, knew Turkish in his childhood, while, as a child, both the Arabic and the Persian languages were alien to him.[23]

[19] Al-Maqdisî, *Aḥsanu't-Taqâsîm fî Ma'rifati'l-Aqâlîm,* ed. M.J. de Goeje, E.J. Brill, Leiden 1906, p. 362; D.M. Dunlop, *The History of the Jewish Khazars,* Princeton University Press, 1954, p. 190.

[20] See, Dunlop, *op. cit.,* pp. 222 ff.

[21] A.N. Poliak, "The Jewish Khazar Kingdom in the Medieval Geographical Science", *Actes du VII^e Congrès International d'Histoire des Sciences,* Jerusalem 1953, pp. 488-492.

[22] See, Poliak, *op. cit.,* p. 489; Dunlop, p. 191.

[23] See, Max Meyerhof, "Das Vorwort zur Drogenkunde des Bîrûnî", *Quellen und Studien zur Geschichte der Naturwissenschaften und der Medizin,* Berlin 1932, vol. 3, Heft 3, pp. 12, 39-40; Bîrûnî, *Kitâbu's-Ṣaydana,* ed. Hakîm Mohammed Said, Karachi 1973, p. 12; *Al-Biruni's Book on Pharmacy and Materia Medica,* tr. Hakîm Mohammed Said, Karachi 1973, p. 8; Zeki Velidi Togan, "Bîrûnî", *Islam Ansiklopedisi,* vol. 2, 1949, pp. 635-636; Zeki Velidi

The title Al-Khwârazmî should, in these early centuries of Islam, refer to the old city of Khwârazm situated on the mouth of the Oxus River, on Lake Aral.[24] This was just on the border of the land extending between the Caspian Sea and the Aral Lake, a land which the Arab armies bypassed in their conquest of Persia, Khurasan, and Transoxania. It was inhabited by Turks who gradually accepted the Muslim religion by their own free will and who also infiltrated into Khwârazm.[25] It is of interest that Khazar hegemony and political boundary extended at times beyond the Caspian Sea up to the coast of the Aral Lake, i.e., to the vicinity, or the very boundary, of Khwârazm.[26]

The caliph Al-Wâthiq sent Al-Khwârazmî to the Byzantine Empire also, charging him with the task of investigating the tomb of the Seven Sleepers at Ephesos. Toomer is of the belief that the person charged with this function was not Muḥammad ibn Mûsâ al-Khwârazmî, but was Muḥammad ibn Mûsâ ibn Shâkir,[27] i.e., the oldest one among the three Banû Mûsâ Brothers who received their scientific training in the Bayt al-Ḥikma under Al-Ma'mûn's patronage.[28] But apparently the reason why Toomer tends to believe that it was Muḥammad ibn Mûsâ ibn Shâkir who was sent to Byzantium is that he thinks it was likewise Muḥammad ibn Mûsâ ibn Shâkir who was sent by Al-Wâthiq to the Khazar king.

We can thus conclude with some certainty that Abû Jaᶜfar Muḥammad ibn Mûsâ al-Khwârazmî survived the caliph Al-Wâthiq who died in

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Togan, *Umumi Türk Tarihine Giriş*, Istanbul 1946, pp. 420-421; Aydın Sayılı, "Bîrûnî", *Belleten* (Turkish Historical Society), vol. 13, 1948, pp. 56-57.

[24] See, F.A. Shamsî, "Abû al-Rayhân Muḥammad ibn Ahmad al-Bayrûnî", *Al-Bîrûnî Commemorative Volume: Proceedings of the International Congress Held in Pakistan, November 26 Through December 12, 1973*, Karachi 1979, pp. 260-288.

[25] See, W. Barthold, *Turkestan v Epokhu Mongol'skago Nashestviia*, St. Petersburg 1898, I, texts, p. 99; R.N. Frye and Aydın Sayılı, "Turks in the Middle East Before the Seljuqs", *Journal of the American Oriental Society*, vol. 63, 1943, p. 199 and note 56; R.N. Frye and Aydın Sayılı, "Selçuklulardan Evvel Orta Şarkta Türkler", *Belleten*, (Turkish Historical Society), vol. 13, 1948, p. 55 and note 3. R.N. Frye and Aydm Sayılı, "Turks in Khurasan and Transoxania Before the Seljuqs", *Muslim World*, vol. 35, 1945, pp. 308-315.

[26] See, Dunlop, *op. cit.*, pp. 150, 160.

[27] See, G.J. Toomer, "Al-Khwârazmî, Abû Jaᶜfar Muḥammad ibn Mûsâ", *Dictionary of Scientific Biography*, vol. 7, 1973, p. 358. See also, C.A. Nallino, "Al-Khwârazmî e il suo Rifacimento della Geografia di Tolomeo", *Raccolta di Scritti Editi e imediti*, vol. 5, Rome 1944, pp. 463-465 (458, 532).

[28] See, Aydın Sayılı, *The Observatory in Islam*, pp. 92-93.

the year 847. No information has come down to us concerning the year of Al-Khwârazmî's birth.

It would seem reasonable to conjecture that Al-Khwârazmî had a hand in the geodetic measurements carried out during Al-Ma'mûn's reign in order to measure the length of a terrestrial degree and also the distance between Baghdad and Mecca. For this undertaking was organized by the Bayt al-Ḥikma where Al-Khwârazmî was active as a key figure, although there is no justification for a conjecture that he actually took part in any of these expeditions.

The primary objective of these expeditions was to ascertain for the translation of Ptolemy's Almagest carried out at the House of Wisdom (Bayt al-Ḥikma) the value of one stadium, the unit length used by Ptolemy, in terms of the units known and used in Islam at that time.[29]

In his Algebra Al-Khwârazmî uses the arithmetical rule of "false position" and "double false position" combined with the "rule of three" generally for solving equations of the first degree, i.e., for solving algebraical problems without algebra. As to his solutions of quadratic equations, he employs for this purpose simple geometric constructions consisting of squares and rectangles, reminiscent of analytical methods of completing or transforming into squares, or into exact squares. This is indeed equivalent, in a way, to the analytic solution of the equation practiced in our own day.

This geometric way of solution of the quadratic equation is also somewhat similar to the Pythagorean geometry incorporated by Euclid into Book 2 of his Elements. In fact, it would seem that this secures a solid foundation for the solution of the algebraic problems expressed in the form of quadratic equations. In other words, it serves to prop these solutions with the rigor of geometrical knowledge, i.e., it sets this algebra free from the thorny question of avoiding irrational roots, a circumstance which seems quite instructive since it brings to mind the Pythagorean shift from emphasis on pure number to the expedient alternative of the geometric representation of number. Recourse to geometric representation also opens the door for finding two roots for a quadratic equation provided both roots are positive.

[29] Aydın Sayılı, *The Observatory in Islam*, pp. 85-87.

Several writers have pointed to ties between Al-Khwârazmî's geometrical solutions and certain theorems of Book 2 of Euclid's Elements.[30] This tradition goes back to Zeuten in the nineteenth century.[31] Gandz, however, is not of this opinion. On the contrary, as we shall see in somewhat greater detail below, Gandz believes that Al-Khwârazmî's method of geometrical demonstration shows that Al-Khwârazmî remained outside the sphere of Greek influence.[32] Al-Khwârazmî does not seem to have written a separate work on geometry proper. The translation into Arabic of Euclid's geometry in Islam goes back to the time of Al-Manṣûr (754-775 A.D.).

Al-Khwârazmî speaks of rational numbers as "audible" and of surd numbers as "inaudible", and it is the latter that gave rise to the word surd (deaf-mute). The first European use of the word seems to begin with Gerard of Cremona (ca. 1150). It corresponds to the term irrational or incommensurable.[33]

We may dwell here briefly on the words *Jabr* and *muqâbala* occurring in the name of Al-Khwârazmî's book. Reviewing a book of Julius Ruska, Karpinski writes, "So far as the title (hisâb algabr walmuqabalah) is concerned, Ruska shows that Rosen is extremely careless and unscientific in his English translation of the two terms involving the idea of restoration or completion (algabr) and reduction or comparison (almuḳâbalah).

"Both terms are carefully explained by Al-Khwârazmî in connection with algebraic problems. When the Arab arrives at the equation

[30] See, e.g., Salih Zeki, *Âthâr-i Bâqiya*, vol. 2, 1913, pp. 13-14; Julius Ruska, "Review on Karpinski's English Version of Robert of Chester's Translation of the Algebra of Al-Khwârazmî", *Isis*, vol. 4, 1921, p. 504; Solomon Gandz, "Isoperimetric Problems and the Origin of the Quadratic Equations", *Isis*, vol. 32, 1940, p. 114. Hâmit Dilgan, *Muhammed ibn Mûsâ el-Hârezmî*, Istanbul 1957, p. 5; Martin Levey, "Some Notes on the Algebra of Abû Kâmil Shujâʿ", *L'Enseignement Mathématique*, series 2, vol. 4, fascicle 2, April-June 1958, pp. 77-92; A. Sayılı, *Logical Necessities in Mixed Equations by ʿAbd al-Hamîd ibn Turk and the Algebra of his Time*, Ankara 1962, pp. 68-71, 133-138; G.J. Toomer, "Al-Khwârazmî", *Dictionary of Scientific Biography*, vol. 7, 1973, p. 360.

[31] See, A. Seidenberg, "The Origin of Mathematics", *Archive for History of Exact Sciences*, vol. 18, number 4, 1978, pp. 307-308.

[32] Solomon Gandz, "The Sources of Al-Khwârazmî's Algebra", *Osiris*, vol. 1, 1936, pp. 263-277; Gandz, "The Origin and Development of the Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra", *Osiris*, vol. 3, 1938, pp. 405-557. See below, p. 34 and note 95.

[33] See, D.E. Smith, *History of Mathematics*, vol. 2, p. 252.

$10x - x^2 = 21$, he conceives of $10x$ as being incomplete by the amount x^2 which he "completes" with x^2, arriving at $10x = 21 + x^2$; the word used for "completes" is a verb formed from the same stem as *ğabr* (algabr). When the Arab arrives at an equation $50 + x^2 = 29 + 10x$, he "reduces" by casting out 29 from 50, arriving at $21 + x^2 = 10x$; the verb used for "reduces" here is from the same stem as *mukâbalah.*"[34]

Roshdi Rashed translates the terms *jabr* and *muqâbala* as *transposition* and *reduction.*[35]

George A. Saliba speaks of the two meanings of the word *jabara,* one being "to reduce a fracture", and the other "to force, to compel." He then writes:

"We believe ... that the root *jabara* was employed by the medieval algebraists in its second sense, "to compel". In this we follow one of these same algebraists, Abû Bakr Muḥammad ibn al-Ḥusein al-Karajî, quoted below, and a contemporary historian of science. ...

"The science of Algebra differs from Arithmetic ... in that in the first one assumes a set of relations involving the unknown. Certain mathematical operations are then performed until there emerges a value that satisfies the conditions of the problem. This process can be looked upon as forcing out the value of the unknown. And whatever process, or operation, pushes the unknown closer to the domain of the known can be called *jabr.* This is the essence of al-Karajî's definition of *jabr.* On the other hand, in solving an algebraic problem, more often than not, more than one value for the required unknown is obtained. It is only by checking these values against the conditions of the problem that the appropriate one can be chosen. This process of checking is the one intended by the word *muqâbalah* (lit. comparing, posing opposite). This meaning of *muqâbalah* is that intended by al-Samaw'al (d.ca. 1175 A.D.) in his discussion of "Analysis" quoted below."[36]

[34] Review by Louis C. Karpinski of Julius Ruska, "Zur Ältesten Arabischen Algebra und Rechenkunst" (*Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Philosophisch-historische Klasse,* vol. 8, pp. 1-125, 1917), in: *Isis,* vol. 4, 1921 (pp. 67-70), p. 68.

[35] Roshdi Rashed, "L'Idée de l'Algébre Selon Al-Khwârazmî", *Fundamenta Scientiae,* vol. 4, number l, p. 95.

[36] George A. Saliba, "The Meaning of al-Jabr wa'l-muqâbalah", *Centaurus,* vol. 17, pp. 189-190.

Earlier writers, as e.g., Julius Ruska, Solomon Gandz, Aldo Mieli and Carl B. Boyer[37] have also dwelt at some length on the meaning and usage of the terms *al-jabr* and *al-muqâbala*.

Luckey points out that Thâbit ibn Qurra does not use the term *al-jabr* in the sense of "restoration" or "completion", i.e., the operation of getting rid of a negative term. He rather uses the term *al-jabr,* without adding to it the word *al-muqâbala,* simply in the sense of the branch of mathematics designated now by the word algebra.[38]

There are other examples of such usages of the term algebra. But Thâbit ibn Qurra (ca. 834-901) does so consistently and is a quite early example of such usage. It is therefore of special interest. Indeed, it may possibly reveal or constitute, in a way, an earlier tradition going back to the Mesopotamian use of the word.

Gandz says:

"There are still remnants in the mathematical literature suggesting that in olden times the term *al-jabr* alone was used for the science of equations, and the term *al-jabriyyûn* was taken for the masters of algebra. On the other hand, the term *al-muqâbalah* alone, according to its real meaning of "putting face to face, confronting, equation", seems to be the most appropriate name for equations in general. With these difficulties in mind, the writer undertook to search out the real meaning of *jabara* in the related Semitic languages. Now the Assyrian name *gabrû-maḫâru* means to be equal, to correspond, to confront, or to put two things face to face, see Delitzsch, *Asyyrisches Handwörterbuch,* under *gabru* and *maḫâru,* pp. 193, 401, and Muss-Arnolt, *Assyrian Dictionary,* under *gabru* and *maxaru,* pp. 210, 525. From the first of these we have the etymology of the Hebrew *geber* and *gibbôr. Geber* is the mature man leaving the state of boyhood and

[37] Julius Ruska, "Zur Ältesten Arabischen Algebra und Rechenkunst", *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Philosophisch-Historische Klasse,* vol. 8, Jahrung 1917, pp. (1-125) 7-14; Solomon Gandz, "The Origin of the Term 'Algebra'", *American Scientific Monthly,* vol. 33, 1926, pp. 437-440; Aldo Mieli, *La Science Arabe,* E.J. Brill, Leiden 1939 (1966), pp. 83-84; Carl B. Boyer, *A History of Mathematics,* John Wiley and Sons, Inc., 1968, pp. 252-253.

[38] P. Luckey, "Thâbit b. Qurra über den Geometrischen Richtigkeits Nachweis der Auflösung der Quadratischen Gleichungen", *Sachsische Akademie der Wissenschaften zu Leipzig, Mathematisch-Naturwissenschaftliche Klasse,* Bericht 93, Sitzung von 7 Juli 1941, pp. (93-114), 95-96.

being *equal* in rank and value to the other men of the assembly or army. *Gibbôr* is the hero who is strong enough *to fight and overcome his equals and rivals* in the hostile army. *Gabara = jab(ra,* in its original Assyrian meaning, is, therefore, the corresponding name for the Arabic *qâbala* (verbal noun *muqâbalah),* and an appropriate name for equations in general."[39] According to J. Høyrup, however, the origin of the word algebra goes back to the Sumerians.[40]

We are interested here mainly in Al-Khwârazmî's work in the field of algebra. Now algebra which, in its essence and early history, is the art of making the solutions of arithmetical problems less cumbersome than they would ordinarily be in arithmetic proper, was in a sense a new field, although it went back to ancient Mesopotamia, on the one hand, and to Diophantos, on the other. In the form it made its appearance in Islam and as it is represented in Al-Khwârazmî it was closely associated with arithmetic, but some of its essential features, i.e., in the solutions it provided for quadratic equations, it was clearly geometrical. Moreover, as far as the question of its predecessors in Greek mathematics is concerned, its direct or indirect ties with Diophantos' arithmetic and algebra and with Euclid's geometry should certainly be made subject of weighty consideration.[41]

It is generally admitted that Al-Khwârazmî's book on algebra represents the first systematic treatment of the general subject of algebra as distinct from the theory of numbers. This does not mean the first appearance of algebra. For this goes clearly back to the early centuries of the second millenium B.C. in Mesopotamia. This is amply testified by the researches of such scholars as F. Thureau-Dangin, O. Neugebauer, Solomon Gandz, E.M. Bruins, and B.L. van der Waerden.[42]

[39] S. Gandz, "The Origin of the Term 'Algebra'", *American Scientific Monthly,* vol. 33, 1926, p. 439.

[40] See, J. Høyrup, "Al-Khwârazmî, Ibn Turk, and the Liber Mensurationum: On the Origins of Islamic Algebra", *Erdem,* vol. 2, no 5, 1986, p. 476; Melek Dosay, *Kerecî'nin Ilel Hesab el-Cebr ve'l-Mukabele Adlı Eseri,* Ankara 1991, p. 10.

[41] See, Roshdi Rashed, *Entre Arithmétique et Algèbre, Recherches sur l'Histoire des Mathématiques Arabes,* Paris 1984, p. 9.

[42] See, Aydın Sayılı, *Mısırlılarda ve Mezopotamyalılarda Matematik, Astronomi ve Tıp,* Ankara 1966, pp. 246-247; B.L. van der Waerden, "Mathematics and Astronomy in Mesopotamia", *Dictionary of Scientific Biography,* vol. 15, Charles Scribner's Sons, 1981, pp. 667, 668-670.

That the idea that algebra as an independent discipline and as distinct from Arithmetic or the theory of numbers first appeared all of a sudden in Islam, and with Al-Khwârazmî, is a thesis that used to be considered more or less reasonable during the last century, in the absence of a knowledge of Babylonian algebra and in spite of the existence of a considerable amount of knowledge concerning Diophantos. It was especially as a result of the discovery of Mesopotamian algebra that this image has largely disappeared. Notwithstanding the Babylonian and Diophantine achievements in algebra, the thesis that Al-Khwârazmî's share of original contribution to the discipline is quite substantial is recently being revived by Professor Roshdi Rashed.[43]

Florian Cajori, writing shortly before concentrated work on Mesopotamian Algebra had started to give its substantial fruits, said concerning Al-Khwârazmî's algebra, "The work on algebra, like the arithmetic, by the same author, contains little that is original. It explains elementary operations and the solutions of linear and quadratic equations. From whom did the author borrow his knowledge of algebra? That it came entirely from Indian sources is impossible, for the Hindus had no rules like the "restoration" *(jabr)* and "reduction" *(muqâbala)*. They were for instance never in the habit of making all terms positive, as is done by the process of "restoration". Diophantos gives two rules which resemble somewhat those of our Arabic author, but the probability that the Arab got all his algebra from Diophantos is lessened by the consideration that he recognized both roots of a quadratic, while Diophantos noticed only one; and the Greek algebraist, unlike the Arab, habitually rejected irrational solutions. It would seem, therefore, that the algebra of Al-Khwârazmî was neither purely Indian nor purely Greek."[44] As is seen, there is no mention of Babylonian algebra in this text. The perspective was to extensively change as a result of the copious light shed upon the subject by the content of relevant cuneiform tablets.

[43] Roshdi Rashed, *Entre Arithmétique et Algèbre,* p. 9. Jens Høyrup has recently published a critical appraisal of this question where he also gives a survey of the trends with regard to the question of historical continuity in this matter, i.e., in the history of algebra starting with its most ancient and formative phases in Mesopotamia. See, Jens Høyrup, *Changing Trends in the Historiography of Mesopotamian Mathematics —An Insider's View—*, Preprints og Reprints, 1991, Roskilde University Center, Denmark.

[44] F. Cajori, *A History of Mathematics,* The Mac Millan Company, 1931, p. 103.

Algebra can be distinguished in its earlier phase as a study of equations and methods of solving them from modern abstract algebra which is enormously more complex and many-sided. Now, was this earlier phase of algebra as a continued tradition before its transition, in an uninterrupted historical process, into modern algebra, created first in Islam, or did the World of Islam inherit it almost ready made from the past? Moreover, in either case, as Arabic was the language of science in Islam, the first appearance of the subject in Islam had to be in Arabic, regardless of whether it was a brand-new achievement or taken over from a past tradition.

Another question is this: Who wrote the first book in algebra in Arabic? The question seems to have been to some extent controversial, and a short reference to it has come down to us in the words of Ḥâjî Khalîfa. The source statement reproduced in Ḥâjî Khalîfa's text is that of Abû Kâmil Shujâᶜ ibn Aslam. According to him, the mathematician Abû Barza claimed that his ancestor, i.e., possibly his grandfather or great-grandfather, had priority over Al-Khwârazmî in writing a book in algebra and drawing attention to this discipline in the newly emerging intellectual world of Islam.

Abû Kâmil flatly rejected this claim, and he also gave vent to his skepticism concerning Abû Barza's assertion that ᶜAbd al-Ḥamîd ibn Turk was an ancestor of his. This latter assertion of Abû Barza is confirmed, however, by both Ibn al-Nadîm and Ibn al-Qifṭî, and Abû Barza too had the surname Ibn Turk in common with ᶜAbd al-Ḥamîd ibn Wâsiᶜ ibn Turk.

The phraseology of the report concerning this controversy creates the impression that Abû Barza ibn Turk's life span was perhaps somewhat before that of Abû Kâmil. Indeed, Abû Barza died in 910 A.D., according to Ibn al-Qifṭî,[45] while Abû Kâmil seems to have outlived Abû Barza by about two decades. For Roshdi Rashed gives the life span of Abû Kâmil as from 850 to 930 A.D.[46] Adel Anbouba[47] gives Abû Kâmil's year of death as approximately 900 A.D., however. It may be noted in this connection that Ibn al-Nadîm mentions the name of Abû Barza before that

[45] Ibn al-Qifṭî, *Kitâb Ta'rikh al-Ḥukamâ*, ed. Lippert, Berlin 1903, p. 230.

[46] See, Roshdi Rashed, *Entre Arithmétique et Algèbre*, p. 44.

[47] Adel Anbouba, "Al-Karajî", *Études Littéraires*, University of Lubnan, 1959, p. 73.

of Abû Kâmil in his synoptic account of calculators and arithmeticians of the Islamic World.[48]

Only a fragment of several pages of ʿAbd al-Ḥamîd ibn Turk's book on algebra entitled Kitâb al-Jabr wa'l-Muqâbala has come down to our day. Salih Zeki speaks of this treatise, as referred to by Ḥâjî Khalîfa,[49] and Carl Brockelmann, and Max Krause also refer to it.[50]

Ibn al-Nadîm says concerning ʿAbd al-Ḥamîd: "He is Abû'l-Faḍl ʿAbd al-Ḥamîd ibn Wâsiʿ ibn Turk al-Khuttalî (or, al-Jîlî), the calculator, and it is said that he is surnamed Abû Muḥammad, and of his books are *The Comprehensive Book in Arithmetic* which contains six books (chapters?) and *The Book of Commercial Transactions.*"[51] Ibn al-Nadîm is seen not to speak of a book by ʿAbd al-Ḥamîd on algebra. But he does the same thing in speaking of Al-Khwârazmî, although he refers three times, elsewhere in his book, to commentaries written on Al-Khwârazmî's Algebra. We know, on the other hand that ʿAbd al-Ḥamîd too was the author of a book on algebra, on the basis of a reference to such a name (Kitâb al-Jabr wa'l-Muqâbala) in the extant manuscript of a fragment of this book.[52]

Ibn al-Qifṭî, on the other hand, has the following to say about ʿAbd al Ḥamîd: "He is a calculator learned in the art of calculation (ḥisâb) having antecedence in the field, and he is mentioned by the people of that profession. He is known as Ibn Turk al-Jîlî, and he is surnamed also as Abû Muḥammad. In the field of Arithmetic he has well-known and much used publications. Among them is *The Comprehensive Book in Arithmetic,* which comprises six books, and *The Book of Little-Known Things in Arithmetic,* and *The Qualities of Numbers.*"[53]

[48] Ibn al-Nadîm, *Kitâb al-Fihrist,* ed. Flügel, vol. 1, p. 281.

[49] Salih Zeki, *Âthâr-i Bâqiye,* vol. 2, Istanbul 1913, p. 246.

[50] Carl Brockelmann, *Geschichte der Arabischen Literatur,* Supplement vol. 1, p. 383; Max Krause, "Istanbuler Handschriften Islamischer Mathematiker", *Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik, Abteilung B:Studien,* vol. 3, 1936, p. 448. See also, Aydın Sayılı, *Logical Necessities in Mixed Equations by ʿAbd al-Hamîd ibn Turk and the Algebra of his Time,* Ankara 1962, pp. 79-80.

[51] Ibn al-Nadîm, *Kitâb al-Fihrist,* ed. Flügel, vol. 1, 1871, p. 273. See also, Bayard Dodge (editor and translator), *The Fihrist of Al-Nadîm,* Columbia University Press, vol. 2, 1970, p. 664.

[52] See, Aydın Sayılı, *Logical Necessities in Mixed Equations ...,* pp. 145, 162.

[53] Ibn al-Qifṭî, ed. Lippert, Berlin 1903, p. 230. See also, Aydın Sayılı, *Logical Necessities in Mixed ...,* pp. 88-89.

The fragment, or tract, of the book on algebra of ᶜAbd al-Ḥamîd ibn Turk that has come down to us apparently made up one whole chapter. For it bears the specific and distinct title "Logical Necessities in Mixed Equations" and deals in particular with the solution of second degree equations, having terms in x^2 and x, and a term consisting of a constant.

It is clear in the light of the text fragment that has survived that Abû Kâmil is not altogether objective and impartial in his appraisal of Abû Barza and ᶜAbd al-Ḥamîd ibn Turk.

Indeed, this chapter of ᶜAbd al-Ḥamîd's book which has come down to us may with good reason be claimed to be a bit superior to the corresponding or parallel section in Al-Khwârazmî's text. This is apparently the reason why Roshdi Rashed refers to it as an attempt to continue Al-Khwârazmî's work by dwelling upon its theory of equations and the question of the demonstration of its solutions. Roshdi Rashed believes, moreover, that Al-Khwârazmî was in a way the founding father of algebra and that the priority in this respect belonged definitively to Al-Khwârazmî and not to ᶜAbd al-Ḥamîd ibn Turk. Roshdi Rashed backs up this conviction of his with statements of Sinân ibn al-Fatḥ, Al-Ḥasan ibn Yûsuf and Ibn Mâlik al-Dimishqî, who simply and clearly state that Al-Khwârazmî was the first person to write a book on algebra in Islam.[54]

Jens Høyrup, on the other hand, is of the opinion that the appearance of the Khwârazmian algebra was the result of a long and slow pre-Islamic process of development, and he also tentatively points to a clue indicating that perhaps Ibn Turk represents a slightly earlier phase in this process, as compared with Al-Khwârazmi.[55] Kurt Vogel simply sides in favor of the priority of ᶜAbd al-Ḥamîd ibn Turk. He apparently believes that the evidence at our disposal is sufficient for such a decision.[56]

[54] Roshdi Rashed, "La Notion de Science Occidentale", *Proceedings of the Fifteenth International Congress of the History of Science,* Edinburgh, 10-19 August 1977, pp. 48-49; Roshdi Rashed, "L'idée de l'Algèbre Selon Al-Khwârazmî", *Fundamenta Scientiae,* vol. 4, no. l, 1983, p. 88; Roshdi Rashed, *Entre Arithmétique et Algèbre,* 1984, p. 27.

[55] Jens Høyrup, "Al-Khwârazmî, Ibn Turk, and the Liber Mensurationum : On the Origin of Islamic Algebra", *Erdem,* vol. 2, pp. 473-475. See also, below, p. 26 and note 76.

[56] Kurt Vogel, "Die Übernahme das Algebra durch das Abendland", Folkerts Lindgren, Hg., *Mathemata, Festschrift für Helmuth Gericke* (Reihe "Boethius", Bd. 12), Franz Steiner Verlag, Wiesbaden, Gmb H, Stuttgart 1984, p. 199.

Boyer says, "In one respect ʿAbd al-Ḥamîd's exposition is more thorough than that of Al-Khwârazmî, for he gives geometrical figures to prove that if the discriminent is negative, a quadratic equation has no solution. Similarities in the works of the two men and the systematic organization found in them seem to indicate that algebra in their day was not so recent a development as has usually been assumed."[57] Youschkevitch too says that the theory of the equations of the second degree in Ibn Turk is the same as that of Al-Khwârazmî but that the subject is taken up in considerably greater detail by Ibn Turk.[58]

Sanad ibn ʿAlî too is mentioned by Ibn al-Nadîm as the author of a book entitled Kitâb al-Jabr wa'l-Muqâbala.[59] Sanad ibn ʿAlî was a close contemporary of Al-Khwârazmî. He too would seem to have been of quite mature age during the reign of Al-Ma'mûn. And there were others who were nearly contemporary with, though of a bit later date than, Al-Khwârazmî and who wrote books on algebra, so that Boyer's remark would seem to be corroborated by this circumstance too.

It is true that as his Algebra is not mentioned among Al-Khwârazmî's books in the section dealing with Al-Khwârazmî in the Kitâb al-Fihrist, Suter has expressed doubt in the veracity of the assertion that Sanad ibn ʿAlî wrote a book on algebra, thinking that in this way it may be possible to ascribe this book on algebra to Al-Khwârazmî.[60] But it is difficult to deny the authorship of Sanad ibn ʿAlî for such a book on the basis of hypothetical conjectures. It is more reasonable to assume that a source book like the Fihrist should fail to mention a certain book as it does for Al-Khwârazmî's algebra than to imagine its inclusion of a non-existing item. At any rate, we know that Ibn al-Nadîm knew of the existence of Al-Khwârazmî's Algebra, for he refers to commentaries written on it on at least three occasions.[61]

[57] Carl B. Boyer, *A History of Mathematics,* John Wiley and Sons, 1968, p. 258.

[58] Adolph P. Youschkevitch, *Les Mathématiques Arabes,* tr. M. Cazenave, and K. Jaouiche, Vrin, Paris 1976, p. 44.

[59] Ibn al-Nadîm, *Kitâb al-Fihrist,* ed. Gustav Flügel, vol. I, Leipzig 1871, p. 275.

[60] See, Qurbânî, *op. cit.,* p. 7.

[61] Ibn al-Nadîm, *Kitâb al-Fihrist,* ed. Flügel, p. 280 (speaking of ʿAbdullâh ibn al-Ḥasan al-Ṣaydanânî), 281 (speaking of Sinân ibn al-Fatḥ), 283 (speaking of Abû'l-Wafâ al-Buzjânî); *The Fihrist of Ibn al-Nadîm,* edited and translated by Bayard Dodge, Cambridge University Press, 1970, pp. 662, 665, 668; Qurbânî, *op. cit.,* pp. 7-8.

A.S. Saidan says concerning the Kitâb al-Fihrist of Ibn al-Nadîm that it has been unjust to Al-Khwârazmî and he continues with the following remarks:

"It attributes a few works to him, but no algebra and no arithmetic. Yet in other places it refers to the Algebra of Al-Khwârazmî. It has been a circulating fact that Ibn al-Nadîm, the author, had his work written and was in the habit of inserting additions and corrections stuffed around the name concerned.

"With this in mind, we find: 1) That the name which precedes Al-Khwârazmî is that of Sahl ibn Bishr. To him are attributed some books which include no algebra. Yet the statements end pointing out: 'It is said that the Rûm value highly his Al-Jabr wa'l-Muqâbala.' I guess that this statement should go to Al-Khwârazmî. 2) That the name which follows is that of Sanad ibn ʿAlî. To him are attributed works ending with: Ḥisâb al-Hindî, Al-Jamʿ wa't-Tafrîq, and Al-Jabr wa'l-Muqâbala. These are exactly the works missing from Al-Khwârazmî's list. I guess that they must go there. This will do him justice."[62]

Other mathematicians two or three generations later than Al-Khwârazmî too are known to have written such books. And it is important to note that according to the manuscripts at our disposal the little text of ʿAbd al-Ḥamîd ibn Turk which has come down to us is not an independent article, but only one part of a book on algebra. Our sources state also, as we have seen, that ʿAbd al-Ḥamîd was the author of other books, as well.[63]

In dealing with these matters it is undoubtedly of some importance to take into consideration the fact that we are in possession only of one chapter or section of Ibn Turk's book, and that this book is said to have been entitled simply Book on al-Jabr and al-Muqâbala and therefore that, in contrast to Al-Khwârazmî's book, apparently Ibn Turk did not use the word "abridged", or some equivalent expression, when naming his book.

[62] A.S. Saidan, "The Algebra and Arithmetic of Al-Khwârazmî, Muhammad ibn Mûsâ", *Acts of International Symposium on Ibn Turk, Khwârazmî, Fârâbî, Beyrûnî, and Ibn Sînâ*, Ankara, September 9-12, 1990, English and French edition, p. 279. *Uluslararası İbn Türk, Hârezmî, Fârâbî, Beyrûnî, ve İbn Sînâ Sempozyumu Bildirileri*, p. 315.

[63] Aydın Sayılı, *Logical Necessities in Mixed Equations...*, pp. 88-89; Ahmed Ârâm, "Risâleî der Jebr wa'l-Muqâbele", *Sukhan-ı ʿIlmî*, 1343, series 3, number 11-12, pp. 1-23 (offprint).

It may come to mind, therefore, that Ibn Turk's book would be expected to deal in greater detail with the subject taken up in each chapter. In Ibn Turk's book parallelism with that of Al-Khwârazmî would be expected to exist normally to the exclusion of the parts on Mensuration and on Legacies. For, in case such a consequence of the usage of the word abridged is not assumed, it would be difficult to reconcile the situation that although ʿAbd al-Ḥamîd ibn Turk's text is superior in some of the details it takes up, it is at the same time the slightly earlier text; or that since it is the older text it should reasonably be expected to be the slightly more primitive one.

We should be heedful, in short, that, as pointed out by Al-Khwârazmî, his text is an abridged one, that it is a text in which the algorithm called algebra has been presented by way of summary, by somehow abbreviating it. It is worth noting that Al-Khwârazmî did not only put the word abridged in the title of his book, but that he also uses this word abridged or short *(mukhtaṣar)* in the course of his introductory remarks where he states that Al-Ma'mûn encouraged him to compose a book of such a nature *(mukhtaṣar)*.[64]

It is perhaps also worth noting that it is not so easy to consider a text which is characterized by its writer as brief or summarized (or, condensed, compendious, or abridged) to represent at the same time an innovation or a fresh contribution and to constitute something not existing or not known previously, unless one writes down only part of what he has conceived and formulated in his mind. But in such a case too one would be expected to clarify the point and say something more specific about the part that has been omitted though it would have been a fresh contribution, had it been brought to light. Moreover, in such a case it would be unlikely though not impossible for Al-Ma'mûn to request Al-Khwârazmî to write such a book, i.e., to write down in an abridged form such an innovation. But Al-Khwârazmî stresses the fact that Al-Ma'mûn encouraged him *(qad shajjaʿanî)* to write the book in such a way.

Indeed, Al-Khwârazmî states that he has composed his Algebra because Al-Ma'mûn encouraged him to write a short, or abridged, book on algebra, "confining it to what is easiest and most useful in arithmetic,

[64] See, Rosen's edition of the text, p. 2, and his translation, p. 3; Melek Dosay's improved translation: Pakistan Hijra Council, Islamabad 1989, text, p. 4, translation, p. 66.

such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where measuring of land, the digging of canals, geometrical computations, and other objects of various sorts and kinds are dealt with."[65]

Al-Khwârazmî's book on algebra, therefore, seems to have been conceived as a popular handbook on certain subjects with a method that would not be difficult to follow. And its wide influence and popularity among scholars and mathematicians for several centuries may, therefore, be partly explained by this very nature of the book and by the objective assigned for it by the caliph Al-Ma'mûn as well as by Al-Khwârazmî himself.

As we have seen previously, Julius Ruska made a critical study of the terms *al-jabr* and *al-muqâbala*. Ruska carried out also a quite profound study of the nature of the fundamental terms *mâl*, *jadhr* or *jidhr*, (meaning "root"), and *shay* of the algebra of Al-Khwârazmî, as a result of which he comes to the conclusion that *mâl*, which means *wealth* or *possession*, should preferably not be translated as *square*, as it is usually done. For although by translating *mâl* as *square* and *jidhr* as (the unknown) quantity —not to speak of translating it as root — the nature of the relationship between *mâl* and *jidhr* is not changed, the primacy or precedence of *mâl* over *jidhr* is disregarded, or, to be more exact, their order is reversed. Rosen, e.g., translates *mâl* as *square* in Al-Khwârazmî's text, but, as pointed out by Ruska, in seven (in reality, in nine) concrete examples Rosen is seen to have been forced to translate *mâl* as *number* and in three cases as *square root*.

Ruska, therefore, proposes the use of some such non-committal formula as $w+bv=c$, in turning the rhetorical mode of expression of Al-Khwârazmî into a symbolic form. $w+bv=c$ may hardly be considered to satisfy the function expected from such a transformation, however. For it totally ignores the basic relationship which exists between *mâl* and *jidhr*. Ruska may have been willing to settle the issue by considering $x+b\sqrt{x}=c$ as an acceptable alternative. But mathematically the difference between $x+b\sqrt{x}=c$ and $x^2+bx=c'$ is trivial. The difference thus boils down to stating that $\sqrt{\textit{mâl}}=\textit{jidhr}$ instead of saying that $\textit{mâl}=(\textit{jidhr})^2$. But Ruska also points out, or implies, that the question involved here should

[65] Rosen's translation, p. 4; Melek Dosay's translation, p. 3-4.

not be looked upon as a merely philological one and that it could not be satisfactorily taken up as an isolated historical fact.[66]

Ruska draws attention also to the fact that neither of these two fundamental algebraic terms, as well as *shay,* meaning thing, is of a basically geometrical nature, but that, nevertheless, in Al-Khwârazmî's geometrical figures elucidating the solutions of his three "mixed" quadratic equations *mâl* and *jidhr* represent respectively the area and the side of a square. Ruska qualifies, therefore the algebra of Al-Khwârazmî as essentially of an arithmetical nature and looks upon the geometrical figures with the help of which the "mixed" second degree equations are illustrated, and their solutions justified, as superimposed upon the main arithmetical body of these equations and as the "reasons" or "causes" for the proofs given. The word "Grund" which Ruska uses on this occasion is the translation of the word *ʿilla* in Al-Khwârazmî's text.[67] Solomon Gandz translates this word or term *ʿilla* with the word *cause.* "Reason" should be more appropriate, however.[68] Terms such as *proof* and *justification* are the words used more frequently in this context nowadays.

In giving the geometrical explanation of the solutions of his "mixed" equations, Al-Khwârazmî speaks of the *mâl* as represented by squares "with unknown sides", the unknown values of the area and the sides being required to be found. Here, both the area and the sides of these squares are sought, and the order of priority seems to recede to the background. This peculiarity of considering both x and x^2 (or X and $\sqrt{X}$, as Ruska would rather have it), as the two unknowns required to be solved, continued after Al-Khwârazmî too, and it may be speculated that the reason why x^2 too was kept in the foreground may have been the consequence of some concern related to the difficulty of finding the exact values of so many square roots.

This may indeed have been at the bottom of the fact of resorting to the method of geometrical solutions. In this case the origin of Al-Khwârazmî's geometrical solutions may possibly be traceable to the discovery of the irrational numbers by the Pythagoreans. Or it may possibly go back to the Babylonian algebra.

66 Julius Ruska, *op. cit.,* (see above footnote 34), pp. 47-70, especially, pp. 62-64.

67 *Ibid.,* pp. 66-67.

68 See below, p. 34 and note 94.

Gandz says, "Diophantos (c. 275 A.D.) admits of no irrational numbers. The condition or *Diorismus* is always that the term under the root be a square. ... Al-Khwârazmî, however, never mentions such a condition. ..."[69] Thinking in terms of paradigms and tradition shattering scientific work, therefore, Diophantos' *Diorismus*, on the one hand, and recourse to geometry as seen in Al-Khwârazmî, on the other, would both represent repercussions to the discovery of irrational numbers. It would seem possible to conceive, therefore, the Al-Khwârazmian algebra, as in some ways a continuation of a tradition bypassing Diophantos.

There recently have been some very interesting speculations, or, more properly speaking, investigations on the possibility that the so-called geometrical and analytical approaches to algebra may possibly go back to a much more remote past, a hypothetical common origin of Babylonian, Indian, and Greek algebras. Thus, this seemingly dualistic approach in the Khwârazmian algebra may be envisaged as going back to a past much earlier than the crisis arising from the discovery of the irrationals.[70]

Speaking of the quadratic equations of two unknowns represented, e.g., by sets of equations such as $x+y=b$ and $xy=c$, or $x-y=b$ and $xy=c$, Gandz writes as follows:

"Historically, it would perhaps be more proper to speak of rectangular instead of quadratic equations, because it was the problems of the rectangle that gave rise to these questions. In the square, there is only one unknown quantity, x. If one knows the side x, one may find the area x^2, and if one knows the area, he may find the side. In the rectangle, there are two quantities that must be ascertained, the length and breadth, or the flank and the front, as the Babylonians call them (reference is made here to Thureau-D'Angin), x and y in our designation. If one knows both of them, he may find the area, and if one knows the area and one of the sides, ..."[71]

Kurt Vogel dwells in somewhat greater detail on such examples giving evidence of the possible connection of the Babylonian quadratic equa-

[69] S. Gandz, "The Origin and Development of the Quadratic Equations", p. 534.

[70] A. Seidenberg, "The Origin of Mathematics", *Archive for History of Exact Sciences*, vol. 18, 1978, pp. 301-342.

[71] Gandz, *ibid.*, pp. 410-411.

tions with geometry.[72] Examples containing such clues seem to belong generally to the earlier phases of the history of Babylonian algebra. That would seem to explain why the Babylonian algebra in its more classical form is generally regarded to be of an analytic nature.

Martin Levey, who, following Gandz, assumed that Greek geometry and algebra had no direct influence on Al-Khwârazmî, writes as follows:

"... Abû Kâmil utilized not only the ideas of Al-Khwârazmî, the inheritor of Babylonian algebra, but also the concepts of the Greek mathematics of Euclid. The result of this approach was a welding of Babylonian and Greek algebra, the first time such a fusion had ever been attempted. ...

"Euclid, in his book II, gives geometric demonstrations of algebraic formulas, while, on the other hand, the works of the early Muslims are primarily algebraic with geometric explanations, more or less abstract."[73]

The same author also says:

" ... Muslim Algebra seems to parallel the development of Arabic chemistry in that it is a fusion of the practical arts and the more theoretical Greek approach to mathematical thinking. Although there is no conclusive chain of transmission, it is probable that this combining of the two methods also traces back to Alexandrians Heron and others like him, of the second century.

"Abû Kâmil ... utilized the theoretical Greek mathematics without destroying the concrete base of Al-Khwârazmî's algebra and evolved an algebra based on practical realities derived from Babylonian roots and strengthened by Greek theory."[74]

Speaking of Euclid's geometrical algebra and quoting Heath, Levey remarks that "the proofs of all the first ten propositions of Book II are practically independent of each other" and then adds, "Heath then asks

[72] See, Kurt Vogel, "Bemerkungen zu den Quadratischen Gleichungen der Babylonischen Mathematik", *Osiris*, vol. I, 1936, pp. 703-717.

[73] Martin Levey, *The Algebra of Abû Kâmil, Kitâb al-Jabr wa'l-Muqâbala in a Commentary by Mordecai Finci,* The University of Wisconsin Press, 1966, p. 20.

[74] *Ibid.*, p. 4. See also, Martin Levey, "Some Notes on the Algebra of Abû Kâmil Shujâʿ: A Fusion of Babylonian and Greek Algebra", *Enseignement de Mathématique,* vol. 4, fascicle 2, 1958, p. 78.

and answers the question: 'What then was Euclid's intention, first, in inserting some propositions not immediately required, and secondly, in making the proofs of the first ten independent of each other?' Surely the object was to show the power of the method of geometrical algebra as much as to arrive at results."[75]

In Al-Khwârazmî's algebra the word *murabbaʿ* is used in the meaning of square, although in a few examples Al-Khwârazmî adds to this word the adjectives *equilateral* and *equiangular.* In ʿAbd al-Ḥamîd ibn Turk's text, on the other hand the word *murabbaʿ* seems to be used more often in the meaning of equilateral. For while speaking of the geometrical square the word *murabbaʿ* often occurs in his text too with the adjectives *equilateral* and *rectangular,* this word is used without further specification when referring to rectangles, and at times to squares.[76] Could this possibly represent a vestigial or residual evidence of influence coming from the remote past, i.e., old Babylonian algebra? It may be worth trying to investigate this point.

Neugebauer writes: "To say that Greek mathematics of the Euclidean style is a strictly Greek development does not mean to deny a general Oriental background for Greek mathematics as a whole. Indeed, mathematics of the Hellenistic period, and still more of the later periods, is in part only a link in an unbroken tradition which reaches from the earliest periods of ancient history down to the beginning of modern times. As a particularly drastic example might be mentioned the elementary geometry represented in the Hellenistic period in writings which go under the name of Heron of Alexandria (second half of first century A.D.). These treatises on geometry were sometimes considered to be signs of the decline of Greek mathematics, and this would indeed be the case if one had to consider them as the descendents of the works of Archimedes or Apollonius. But such a comparison is unjust. In view of our recently gained knowledge of Babylonian texts. Heron's geometry must be considered merely a Hellenistic form of a general Oriental tradition. The fact, e.g., that Heron adds areas and line segments can no longer be viewed as a novel sign of the rapid degeneration of the so-called Greek spirit, but simply reflects the algebraic or arithmetic tradition of Mesopotamia. On this more elemantary level, the axiomatic school of mathematics had as

[75] Martin Levey, *The Algebra of Abû Kâmil,* p. 20. See also, Roshdi Rashed, "La Notion de Science Occidentale" (see above, note 54), p. 49.

[76] See, Aydın Sayılı, *Logical Necessities in Mixed Equations...,* p. 84.

little influence as it has today on surveying. Consequently, parts of Heron's writings, practically unchanged, survived the destruction of scientific mathematics in late antiquity. Whole sections from these works are found again, centuries later, in one of the first Arabic mathematical works, the famous "Algebra" of al-Khwârazmî (about 800 to 850). This relationship can be especially easily demonstrated by means of the figures. In order to make the examples come out in nice numbers, the figures were composed from a few standard right triangles. One of these standard examples is shown in figure 21 which appears in Heron as well as in al-Khwârazmî. Two right triangles with sides 8, 6 and 10 are combined into an isosceles triangle of altitude 8 and base 12."[77]

There is some evidence showing that this dichotomy into more theoretical and more practical in mathematics went back to Mesopotamia, and to Elam and Susa, which in turn means that it was also practiced by the Sumerians. Indeed, the concept of *napkharu* seems to indicate that these men wished to avoid the fallacy of misplaced precision.

On a previous occasion I have made, in connection with ʿAbd al-Ḥamîd ibn Turk's Logical Necessities in Mixed Equations, the following remark:

"In our present text x^2 is seen to come to the foreground as an unknown, almost as prominently as x, and this observation may be said to be applicable to Al-Khwârazmî as well. It almost seems as if ʿAbd al-Ḥamîd thinks in terms of an equation of the form $X+b\sqrt{X}=c$, rather than $x^2+bx=c$, X being the real unknown and $\sqrt{X}$ the square root of the unknown."[78]

Martin Levey says, "Al-Khwârazmî explained a total of forty problems in his algebra compared with Abû Kâmil's sixty-nine. The latter greatly expanded Al-Khwârazmî's algebra with the addition of different types of problems and also varied solutions for these problems. Abû Kâmil's work represented innovations in algebraic method such as in the solution directly for x^2 instead of for x, since the latter was frequently not desired by Islamic mathematicians." Martin Levey has here a footnote for this last remark of his, and the footnote is "J. Tropfke, *Gesch. d. Elementar-*

[77] Otto Neugebauer, *The Exact Sciences in Antiquity*, Brown University Press, 1957, pp. 146-147.

[78] Aydın Sayılı, *ibid.*, pp. 84-85.

Mathematik, 3, 74-76, 80-82. (Berlin 1937); see also the important chapter in J. Weinberg, Dissertation."[79]

A question of the type we are here confronted with, viz., why should a second degree equation be conceived to have two solutions, one of x and one of x^2, is often a question of the order of historical background, a question of ascertaining the relevant historical setting, and it can be answered only by placing the question successfully within its appropriate historical perspective. It may not often have much meaning as a question detached from its historical background. In other words, this peculiarity of form or structure can be answered only in terms of its past history. It cannot be accounted for merely as a development, as an appearance out of nothingness. This appearance or development may partake of the attributes of a transformation, of reorganization of some related stockpiles of knowledge and constitute a revolution. It may be the result of a break in some past trend, but even then its appearance needs to be made intelligible within the framework of the principle of historical continuity. Regardless, therefore, of whether Al-Khwârazmî was an innovator or a relatively passive follower of past tradition, his achievement stands in need of being made intelligible by placing it into relation with its past history.

Ruska believed as we have seen, that the Al-Khwârazmian algebra was arithmetical in nature and that the geometrical scheme of solution was superimposed upon it. Let us take a look at an example from ʿUmar Khayyâm. In ʿUmar Khayyâm the solution of equations is based upon geometry just as in the case of Al-Khwârazmî. Again the terms *mâl* and *jidhr* are used by ʿUmar Khayyâm exactly in the same manner as they occur in Al-Khwârazmî. The word for cubic, however, is *kaʿb,* i.e., a geometrical term in ʿUmar Khayyâm. Moreover, ʿUmar Khayyâm's geometry coming into play in the solutions of cubic equations cannot be qualified by any means as primitive or elementary. It is of great interest also that in solving a simple example such as $x^3 + cx^2 = bx$, the procedure employed by ʿUmar Khayyâm to reduce this equation to $x^2+cx = b$ is of a clearly geometrical nature,[80] so that it is not in conformity with Ruska's verdict that Al-Khwârazmî's approach to the quadratic equations

[79] Martin Levey, *op. cit.,* p. 18.

[80] F. Woepcke, *L'Algèbre d'Omar al-Khayyâmî,* Paris 1851, Arabic text, p. 15, French translation, pp. 25-26.

is of an essentially non-geometrical nature; it does not constitute a parallel to Ruska's conjecture.

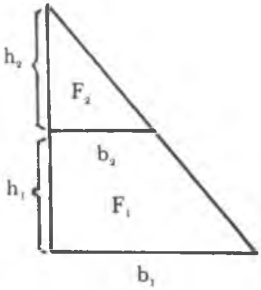
The Mesopotamian tablets dealing with algebra usually contain solutions of equations. These solutions are systematic, the solutions for each individual problem being presented step by step, but no explanations are explicitly given for these solutions. The method of recource to auxiliary unknowns is seen to have been quite general, however. Thus in solving the pair of equations $x+y=b$, $xy=c$, e.g., it may be concluded that they use an auxiliary unknown such as $2z=x-y$. Consequently $2x=b+2z$ and $2y=b-2z$. Consequently $xy=(b/2)^2-z^2=c$, or $z^2=(b/2)^2-c$ and $z=(x-y)/2=\sqrt{(b/2)^2-c}$. Therefore $(x+y)/2=b/2$ and $(x-y)/2=\sqrt{(b/2)^2-c}$. The quadratic equation in two unknowns is thus transformed into a pair of first degree equations in two unknowns. Thus, $x=b/2+\sqrt{(b/2)^2-c}$ and $y=b/2-\sqrt{(b/2)^2-c}$. Now, there is evidence suggesting that in solutions of this nature algebraic identities come into play. Thus the identity $xy=[(x+y)/2]^2-[(x-y)/2]^2=c$. Therefore $[(x-y)/2]^2=[(x+y)/2]^2-c$, and $(x-y)/2=(b/2)^2-c$. Hence, again, $x=b/2+\sqrt{(b/2)^2-c}$ and $y=b/2-\sqrt{(b/2)^2-c}$.[81]

Thus the solutions of quadratic equations in Babylonian algebra would seem to be of a purely analytical nature. The following interesting example shows, however, that this may not have been an exclusive feature or a thoroughly predominent characteristic of the Babylonian algebra as regards their treatment of quadratic equations. This example belongs, properly speaking, to their geometry. But as their geometry was an algebraic geometry it serves to shed light on the question we are dealing with at this point.

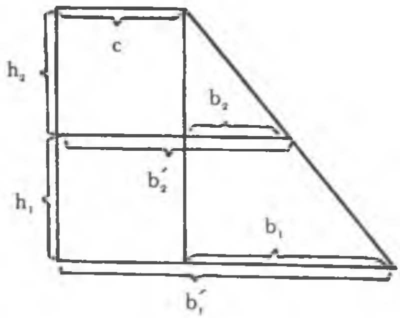
Our example is in the tablet Vat. 8512 and has been studied by O. Neugebauer in his Mathematische Keilschrift-Texte, I.[82] The problem is this: A rectangular triangle is divided by a line, parallel to the base, into

[81] Solomon Gandz, "The Origin and Development of Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra", *Osiris*, vol. 3, 1938, pp. 444, 418-419, 423-424, 447-448, 499; O. Neugebauer, *The Exact Sciences in Antiquity*, Brown University Press, 1957, p. 41; E.M. Bruins, "Neuere Ergebnisse über Babylonische Algebra", *Praxis der Mathematik*, year I, Heft 6, 15 September 1959, pp. 148-149; E.M. Bruins, "Neuere Ergebnisse zur Babylonische Arithmetik", *Praxis der Mathematik*, year I, Heft 4, 15 July 1959, pp. 92-93; Aydın Sayılı, *Mısırlılarda ve Mezopotamyalılarda Matematik, Astronomi ve Tıp*, Ankara 1966, pp. 206-232.

[82] *Quellen und Studien zur Geschichte der Mathematik*, A 3, Berlin 1935, pp. 340 ff.



two parts, a trapezium and the top triangle. The text contains no figure. In Gandz's words, the formulas are rather complicated, but they are pretty well secured by the text. $F_1 - F_2 = D$ and $h_2 - h_1 = d$. The value of b_1 too is known. It is required to find b_2, h_1, h_2, F_1, F_2. The solution formula given for b_2 in the tablet is $b_2 = \sqrt{\frac{1}{2}[(\frac{D}{d}+b_1)^2+(\frac{D}{d})^2]} - \frac{D}{d}$.



This formula exhibits some strange deviations from what would be normally expected to be found. The solution proposed by Neugebauer leads to the formula $b_2 = \sqrt{(D/d)^2 + (D/d)b_1 + (1/2)b_1^2} - D/d$. The two formulas are equivalent and they may be derived one from the other, but the deviation of the text formula from that found by Neugebauer could not be accounted for until Peter Huber discovered a very unexpected geometrical scheme for the derivation of the formula of the tablet for b_2. This is achieved by adding a rectangle to the triangle as seen in the figure presented.[83]

Though taken from algebraic geometry, this example would seem conducive to make us think that in Mesopotamian algebra in its so-to-say classical form too geometry may at times have played some part in conceiving schemes helpful to find the solutions of quadratic equations. Peter Huber refers at the end of his article to the following statement of Neugebauer and Sachs (O. Neugebauer and A. Sachs, Mathematical Cuneiform Texts, New Haven 1945): "Although these problems are sometimes accompanied by figures ... and although their terminology is geometrical, the whole treatment is strongly algebraic," and remarks that this statement stands therefore in need of a bit of modification, although the gen-

[83] Peter Huber, "Zu Einem Mathematischen Keilschrifttext (Vat 8512)", *Isis*, vol. 46, pp. 104-106. For more details on the problem, see S. Gandz, "The Origin and Development of the Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra", *Osiris*, vol. 3, 1938, pp. 475-479. See also, Aydın Sayılı, *Mısırlılarda ve Mezopotamyalılarda Matematik, Astronomi ve Tıp*, Turkish Historical Society Publication, Ankara 1966, pp. 232-236.

eral character of the totality of the Babylonian mathematics is naturally unaffected by such examples.[84]

Jens Høyrup writes:

"A close investigation of the Old Babylonian second degree algebra shows that its method and conceptualization are not arithmetical and rhetorical, ... Instead, it appears to be based on a "naive" geometry of areas very similar to that used by Ibn Turk and Al-Khwârazmî in their justification of the algorithms used in *al-jabr* to solve the basic mixed second degree equations.

"This raises in a new light the question whether the early Islamic use of geometric justifications was a graft of Greek methods upon a "sub-scientific" mathematical tradition, as often maintained, or the relation of early Islamic algebra to its sources must be seen differently.

"Now, the *Liber Mensurationum* of one Abû Bakr, known from a twelfth century Latin translation, refers repeatedly to two different methods for the solution of second-degree algebraic problems: A basic method may be identified as "augmentation and diminution" *(al-jamʿ wa'l-tafrîq?)* and another one labelled *al-jabr* which coincides with Al-Khwârazmî's use of numerical standard algorithms and rhetorical reduction. Since the *Liber Mensurationum* coincides in its phrasing and in its choice of grammatical forms with Old Babylonian texts, and because of peculiar details in the terminology and the mathematical contents of the text, it appears to represent a direct sub-scientific transmission of the Old Babylonian naive-geometric algebra, bypassing Greek as well as late Babylonian (Seleucid) algebra as known to us. This, together with internal evidence from Al-Khwârazmî's *Algebra* and Thâbit's Euclidean justification of the algorithms of *al-jabr,* indicates that Ibn Turk and Al-Khwârazmî combined two existing sub-mathematical traditions with a "Greek" understanding of the nature of mathematics, contributing thereby to the reconstruction of the subject as a scientific mathematical discipline."[85]

Again, Jens Høyrup says: "Since the discovery some fifty years ago that certain cuneiform texts solve equations of the second degree, the

[84] Peter Huber, *ibid.,* p. 106. On this point see also, A. Seidenberg, "The Origin of Mathematics", *Archive for History of Exact Sciences,* vol. 18, number 4, 1978, pp. 308-310.

[85] Jens Høyrup, "Al-Khwârazmî, Ibn Turk, and the Liber Mensurationum: On the Origins of Islamic Algebra", *Erdem,* vol. 2, pp. 445-446. See also, Jens Høyrup, *Algebra and Naive Geometry, An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought,* 3. Raekke, Preprints og Reprints, 1987, Nr. 2, passim.

ideal has been close at hand that the early Islamic algebra known from Al-Khwârazmî and his contemporary Ibn Turk continues and systematizes an age-old tradition. More recently, Anbouba has also made it clear that the two scholars worked on a richer contemporary background that can be seen directly from their extant works. In fact, the same richer tradition can be glimpsed, e.g., from some scattered remarks in Abû Kâmil's *Algebra* — cf. below, section VI."[86]

Gandz, who did work of fundamental importance on Babylonian and early Islamic Algebra gives the following list of the types of second degree equation found in the cuneiform tablets:

1) $x+y=b;\ xy=c,$	2) $x-y=b;\ xy=c,$
3) $x+y=b;\ x^2+y^2=c,$	4) $x-y=b;\ x^2+y^2=c,$
5) $x+y=b;\ x^2-y^2=c,$	6) $x-y=b;\ x^2-y^2=c,$
7) $x^2+bx=c,$	8) $x^2-bx=c,$
9) $x^2+c=bx.$[87]	

Types 1 and 2 lead directly, 3 and 4 with change in the constant, to the types 7, 8, and 9; types 5 and 6 become transformed into first degree equations when reduced to one unknown. It is observed that types 7, 8, and 9 are those found in Al-Khwârazmî and ʿAbdu'l-Ḥamîd ibn Turk.

According to the conclusions reached by Gandz, in a first stage, i.e., in the "old Babylonian school", the first six types of equations in two unknowns seen in the above list were the types in use.[88] Later on, the remaining three types of equation with one unknown also came into use, but the type $x^2+c=bx$ was avoided.[89] Gandz considers a new school to have developed directly out of this second stage found in Babylonian algebra. The place and time of its appearance is not known, and its earliest representative known is Al-Khwârazmî, according to Gandz. The outstanding characteristic of this new school of algebra is its practice of excluding the six old Babylonian types and of using the three "mixed" equ-

[86] Adel Anbouba, "Acquisition de l'Algèbre par les Arabes et Premiers Developpements, Aperçu Général", *Journal for the History of Arabic Science*, vol. 2, 1978, pp. 66-100. See, Jens Høyrup, *op. cit.*, p. 447. See also, Jens Høyrup, *The Formation of "Islamic Mathematics", Sources and Conditions*, May 1987, Preprints og Reprints, Roskilde University Centre, p. 20.

[87] S. Gandz, "The Origin and Development of Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra", *Osiris*, vol. 3, 1938, pp. 515-516.

[88] Gandz, *op. cit.*, pp. 417-456.

[89] Gandz, *op. cit.*, pp. 470-508.

ations, in one unknown, i.e., equations having terms in x^2 as well as in x and in constants. In Gandz' opinion the old Babylonian attitude is thus seen to have been completely reversed.[90]

The reasons for the disappearance of the avoidance of, or the hesitation felt toward, the type $x^2+c=bx$ are not accounted for in these views advanced by Gandz. In Al-Khwârazmî's algebra the equations $x^2+bx=c$ and $x^2=bx+c$ have one solution each, while $x^2+c=bx$ has two solutions or roots. Now, type 1 in the above list gives $x^2+c=bx$ and also $y^2+c=by$, while type 2 gives $x^2=bx+c$ for x and $y^2+by=c$ for y. Therefore, the two solution for $x^2+c=bx$ may be interpreted as the solutions for x and y in type 1 from which $x^2+c=bx$ may be considered to have originated, while for $x^2+bx=c$ and $x^2=bx+c$ such a roundabout interpretation is not necessary.

According to Gandz this explains why the Babylonians tried to avoid the $x^2+c=bx$ type and preferred to deal with the $x+y=b$; $xy=c$ type instead.[91] But the fact that the acceptance and free usage of the type $x^2+c=bx$ was accompanied, as Gandz says, by an aloofness toward the old Babylonian types and methods suggests that the interpretation of the double root of $x^2+c=bx$ exclusively with the help of the pair of equations $x+y=b$ and $xy=c$ should not constitute an explanation that could be prevalent and current in the time of Al-Khwârazmî. It is of great interest, therefore, that the explanation of the double solution of $x^2+c=bx$ without recource to the pair $x+y=b$ and $xy=c$ is clearer and fuller in ʿAbdu'l-Ḥamîd ibn Turk than in Al-Khwârazmî.[92]

To sum up, Gandz claims that the question of the four roots of the three "mixed" equations of Al-Khwârazmî's algebra cannot be made intelligible unless we consider them in the light of their distant Babylonian origins. But this certainly does not seem to be true. The algebra of Al-Khwârazmî was apparently quite self-sufficient in explaining away the question of the number of roots of the "mixed" quadratic equations. Moreover, as Gandz also asserts, strict dependence upon geometrical reasoning was a prominent feature of this algebra, and this feature has to be brought well into prominence.

[90] Gandz, *op. cit.*, pp. 509-510. See also, Aydın Sayılı, *Logical Necessities in Mixed Equations by ʿAbd al-Hamîd ibn Turk and the Algebra of His Time*, pp. 103-105.

[91] Gandz, *op. cit.*, pp. 412-416.

[92] See, Sayılı, *Logical Necessities...*, pp. 99-104, 107-109.

Gandz says, " ... Al-Khwârazmî tries hard to break away from algebraic analysis and to give to his geometric demonstrations the appearance of a geometric independence and self-sufficiency. They are presented in such a way as to create the impression that they are arrived at independently without the help of algebraic analysis. It seems as if geometric demonstrations are the only form of reasoning and explanation which is admitted. The algebraic explanation is, as a rule, never given."[93] It may be added here that, in Gandz's words, Al-Khwârazmî closely associates the "cause" of an equation and its geometrical figure.[94]

Speaking of geometrical demonstrations and comparing Euclid and Al-Khwârazmî, Gandz says, "Euclid demonstrates the antiquated old Babylonian algebra by an highly advanced geometry; Al-Khwârazmî demonstrates types of an advanced algebra by the antiquated geometry of the ancient Babylonians.

"The older historians of mathematics believed to find in the geometric demonstrations of Al-Khwârazmî the evidence of Greek influence. In reality, however, these geometric demonstrations are the strongest evidence against the theory of Greek influence. They clearly show the deep chasm between the two systems of mathematical thought, in algebra as well as in geometry."[95]

As to the relationships between Babylonian algebra and the algebras of Diophantos and Al-Khwârazmî, Gandz says, "Both, Al-Khwârazmî and Diophantos, drew from Babylonian sources, but whereas Diophantos still adheres to old Babylonian methods of solution, Al-Khwârazmî rejects those old methods and introduces the more modern methods of solution."[96]

Both Gandz and Høyrup thus evaluate Al-Khwârazmî's geometrical solutions with roughly equivalent or similar approaches, but while Gandz believes the Babylonians to have more generally used analytical procedure, Høyrup concludes that the Babylonian algebra too was based upon geometrical conceptualizations. In this latter respect Høyrup's judgment seems to rest upon more concrete source evidence.

[93] Gandz, *op. cit.*, pp. 514-515.

[94] Gandz, *op.cit.*, p. 515; Aydın Sayılı, *Logical Necessities...*, p. 107.

[95] Gandz, "The Origin and Development...", pp. 523-524.

[96] Gandz, *ibid.*, p. 527. See also, Gandz, "The Sources of Al-Khwârazmî's Algebra", *Osiris*, vol. I, 1936, pp. 263-277, on the historical foundations of Al-Khwârazmî's algebra.

As to the question of the value judgments on geometrical proofs or demonstrations of Al-Khwârazmî's solutions of his second degree equations, it is seen that already immediately following Al-Khwârazmî there were attempts to cast his solutions into forms conforming to the spirit of Euclidean geometry.[97] Ivonne Dold-Samplonius informs us, on the other hand, that Professor B.A. Rosenfeld of Moscow stated in a letter to her that in his opinion Al-Khwârazmî's geometrical "illustrations" are geometrical proofs.[98] In connection with the solutions of his quadratic equations, all that Al-Khwârazmî had to do, was to prove, or to show, that the said solutions were correct; he was not trying to prove theorems. It would be unreasonable not to accept Al-Khwârazmî's geometrical solutions as valid justifications or arguments establishing the veracity of the solution formulas on the basis of entirely acceptable geometrical evidence.

Indeed, it would very likely be wrong to think that Al-Khwârazmî was not conversant with Euclid's geometry. In his elaborate work on the comparison of Al-Khwârazmî's Bâb al-Masâha with Mishnat ha-Middot too, Gandz speaks of his conviction that Al-Khwârazmî was not familiar with Euclidean geometry or that he stayed aloof from it.[99]

On this occasion William Thomson says:

"The fact that Al-Khwârazmî's book on mensuration shows little or no sign of influence from the side of Greek theoretical mathematics does not prove either his ignorance or his dislike of that mathematics. The only legitimate inference is that he did not use it, or find it useful, for his purpose. ..." On this occasion William Thomson enumerates a few examples of parallelism in geometrical terminology used by Al-Khwârazmî and his older contemporary Al-Hajjâj ibn Yûsuf who had made his translation of Euclid before Al-Ma'mûn became caliph.[100]

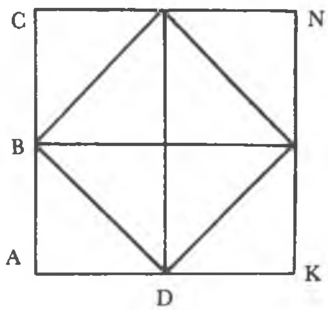
[97] Yvonne Dold-Samplonius, "Developments in the Solution of the Equation $cx^2+bx=a$. From Al-Khwârazmî to Fibonacci", *From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E.S. Kennedy,* ed. David King and George Saliba, The New York Academy of Sciences, New York 1987, pp. 71-87.

[98] *Ibid.,* p. 85, note 4.

[99] Solomon Gandz, "The Mishnat ha Middot and the Geometry of Muhammad ibn Mûsâ al-Khwârazmî", *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung A: Quellen,* vol. 2, 1932, pp. 64-66.

[100] William Thomson's review of Gandz's Quellen und Studien article. See, *Isis,* vol. 20, 1933, pp. 278, 279.

Cantor, on the other hand, has pointed out that the letters accompanying Al-Khwârazmî's geometrical figures serving to prove his solutions of the mixed equations correspond to the letters of the Greek alphabet, and Julius Ruska considers this as strong evidence for the existence of some kind of Greek influence on these Al-Khwârazmian proofs. Gandz, however, is not of this opinion.[101]



Aristide Marre reproduces a proof given by Al-Khwârazmî for the Pythagorean theorem which applies only to the special case of an equilateral right triangle. It is proved here that the square on the diagonal BD is equal to the sum of the squares drawn on BA and AD by showing that the square drawn on BD is equal to the sum of four of the equal triangles into which the square ACNK is divided, while the squares on AB and AD are equal each to two such triangles, their sum therefore being equal to four such triangles. Aristide Marre then remarks that this proof is thus addressed to the type of reader whom Plato would not have admitted to his classes. Then he adds that this example serves to show that Al-Khwârazmî was not presenting in his book the whole of what he knew but was trying to vulgarize the knowledge he dealt with by simplifying it and to place it at the reach of even the youngest readers.[102]

It is interesting to see that Plato ascribes the same kind of proof to Socrates, but this time the proof is being given for a still more special case. For AB here is equal to two feet, this example being connected with $\sqrt{2}$. This passage is in the dialogue Meno of Plato, and in it Socrates is trying to show "that teaching is only reawakening in the mind of the learner the memory of something. He illustrates by putting to the slave a carefully prepared series of questions, each requiring little more than 'yes' or 'no' for an answer, but leading up to the geometrical construction of $\sqrt{2}$. ... Socrates concludes with the words: 'The Sophists call this

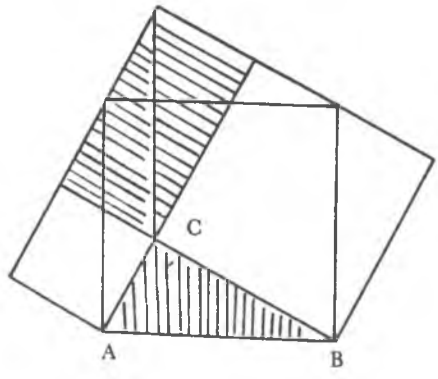
[101] See, Julius Ruska, "Zur Ältesten Arabischen Algebra und Rechenkunst", pp. 69-70; S. Gandz "The Sources of Al-Khwârazmî's Algebra", *Osiris*, vol. I, 1936, pp. 276-277.

[102] Aristide Marre, *Le "Messahat" de Mohammed ben Moussa al-Khwârazmî*, extrait de son Algèbre, traduit et annoté par A. Marre, I[e] édition revue et corrigée sur le texte arabe, Rome 1866, pp. 6-7.

straight line (BD), the diameter (diagonal); this being its name, it follows that the square which is double (of the original square) has to be described on the diameter.'"[103]

This example is quite interesting in that it conforms to Aristide Marre's suggestion that pedagogical concerns aiming to place a book within the reach of even children of small age would make a learned person like Al-Khwârazmî utterly simplify the material presented to his readers. But at the same time it contradicts Marre's other verdict by showing that Plato too was not against such simplifications even if he should not be willing to admit to his classes the readers to which such texts are supposed to address more specifically.

Thâbit ibn Qurra (826-901) was requested by a friend of his who was not satisfied with the "Socratic proof" of the Pythagorean theorem to give a general proof for it. Thâbit conceived this requested proof as one giving a general proof which would be of the same nature or method as the "Socratic special proof". Thus, the fact that Euclid's Elements contains a general proof of the theorem does not make the question superfluous, and Thâbit ibn Qurra gives two different proofs of an appropriate kind.



One of these proofs is shown in the figure presented here. ABC is a right triangle and all the other triangles seen in the figure are equal to it. Now if from the total figure the three shaded triangles are deducted the squares on the right sides of ABC are obtained, while the square on the hypothenuse AB results when from the total figure the three triangles on the corners are subtracted. The sum of the two former squares is therefore equal to the latter square.

Thâbit compliments his friend for seeking a comprehensive knowledge of things and adds that the generalization achieved by the proofs he gives may not be considered sufficient. One could wish, e.g., to generalize

[103] Thomas Heath, *A History of Greek Mathematics,* vol. I; From Thales to Euclid, Oxford 1921, pp. 297-298.

the theorem to any triangle whatsoever, and the figures drawn on the sides may be any similar figures similarly placed upon the sides. But it is noteworthy that although Thâbit ibn Qurra gives two proofs for the simpler and widely known theorem, he merely says that the proof could easily be found on the basis of Euclid's Elements and does not feel the need of proving this more general and somewhat more complicated theorem which apparently constitutes his original contribution to the subject.

Thâbit ibn Qurra also remarks that our knowledge is perfect when it combines the most general and comprehensive with the special and particular. For, he says, in our purely general knowledge the knowledge of the particular cases exists only potentially. He also states that in the course of instruction one has to follow a procedure in which there is a gradual increase in generalization and comprehensiveness, and he adds that the reason why Socrates mentioned only the proof of a special case of the Pythagorean theorem was that the person he was teaching was a beginner in the subject and not an advanced student.[104] It is to be noted that this statement of Thâbit ibn Qurra corroborates the verdict given by Aristide Marre concerning the reason why Al-Khwârazmî preferred an easy proof of a special case to a more comprehensive general proof. It may also be added that this assertion of Thâbit ibn Qurra represents a pedagogical procedure generally practised in the medieval Islamic World.

Thâbit ibn Qurra is, moreover, a highly gifted mathematician who had a thorough appreciation of the spirit of Greek mathematics, and one who happens to have shown a special interest in supplying the Khwârazmian solutions of the second degree equations with thorough geometrical proofs. Thâbit ibn Qurra bases his proofs of the solutions of Al-Khwârazmî for the equations $x^2+bx=c$ and $x^2=bx+c$ on proposition II 6 of Euclid's Elements and the proof of the solution of the equation $x^2+c=bx$ on Euclid's proposition II 5.[105]

[104] See, Aydın Sayılı, "Thâbit ibn Qurra's Generalization of the Pythagorean Theorem", *Isis*, vol. 51, 1960, pp. 35-37. See also, Aydın Sayılı, "Sâbit ibn Kurra'nın Pitagor Teoremini Tamimi", *Belleten* (Turkish Historical Society), vol. 22, 1958, pp. 527-549.

[105] P. Luckey, "Thabit b. Qurra über dem Geometrischen Richtigkeitsnachweis der Auflösung der Quadratischen Gleichungen", *Sächsische Akademie der Wissenschaften zu Leipzig, Mathematisch-Naturwissenschaftliche Klasse,* Bericht 93, Sitzung von 7 Juli, 1941, pp. (93-114) 95, 105-112; J.L. Berggren, *Episodes in the Mathematics of Medieval Islam,* Spinger-Verlag, pp. 104-106.

It may be said that the establishment of the relationships between the geometrical solutions of Al-Khwârazmî and the said propositions of Euclid does not stand in need of an undue forcing of the imagination, but whereas it may be claimed that these are in a way implicit in Al-Khwârazmî they are explicitly set forth and formally established in Thâbit ibn Qurra. Moreover, it is to be noted that Thâbit ibn Qurra does not present these proofs or the establishment of these relationships clearly as an original personal contribution of his own. The possibility that he may be speaking in line with a tradition going back to times before Al-Khwârazmî cannot therefore be entirely excluded on the basis of Thâbit ibn Qurra's text.[106]

Thâbit ibn Qurra's justifications for the solutions of the "mixed" second degree equations are undoubtedly more sophisticated than those of Al-Khwârazmî. But, as we have seen, Thâbit ibn Qurra too, at times, seems to have been satisfied with more down-to-earth and simple geometrical demonstrations, leaving to the reader the more complicated ones. It is reasonable to think, therefore, that Al-Khwârazmî's simple geometrical justifications for his solutions of quadratic equations, and his simple pratical approach in the section on mensuration in his Algebra, do not, in any way, mean that he was unfamiliar or antagonistic to the Euclidean approach to classical synthetic geometry.

At the threshold of modern era in science we witness the discovery of the law of refraction of light. On the subject Cajori writes as follows:

"The law of refraction was discovered by Willebrord Snell (1591-1626), professor of mechanics at Leyden. He never published his discovery, but both Huygens and Isaak Voss claim to have examined Snell's manuscript. He stated the law in the inconvenient form as follows: For the same media the ratio of the cosecants of the angle of incidence and of refraction retains always the same value. As the cosecants vary inversely as the sines, the equivalence of this to the modern form becomes evident. As far as known, Snell did not attempt a theoretical deduction of the law, but he verified it experimentally. The law of sines, as found in modern books, was given by Descartes in his *La Dioptrique,* 1637. He does not mention Snell, and probably discovered the law independently. (1. Various opinions have been held on this point. ...) Descartes made no experi-

[106] *Ibid.*, pp. 95, 106, 107, 110, 111.

ments, but deduced the law theoretically from the following assumptions: (1) the velocity of light is greater in a denser medium (now known to be wrong); (2) for the same media these velocities have the same ratio for all angles of incidence; (3) the velocity component parallel to the refracting surface remains unchanged during refraction (now known to be wrong). The improbability of the correctness of these assumptions brought about attacks upon the demonstration from the mathematician Fermat and others. Fermat deduced the law from the assumption that light travels from a point in one medium to a point in another medium *in the least time*, and that the velocity is less in the denser medium."[107]

It is very interesting that Abû Ṣaᶜd al-ᶜAlâ ibn Sahl of the last quarter of the tenth century, in his geometric study of lenses, arrived at a conclusion of a constant ratio of certain distances and that this is equivalent to Snellius' law of refraction. Here the idea of the physical factor of the denseness of transparent media, i.e., the index of refraction, does not occur as a factor that should be taken into consideration in accordance with the media coming into play. Moreover, as this study of dioptrics concerns the burning quality of lenses, it is tied up with the idea of focus. Thus, Ibn Sahl is led to deal with the conic sections, i.e., to restrict himself to such configurations, and as his work is based on empirical study of the phenomenon of concentration of light on a single point, he is guided by experimental data. This secured the correctness of the results he arrived at and thus made him, at least partly, a forerunner of Snellius, or Snell, at a date even before the time of Ibn al-Haytham.[108]

We see here three contemporary and independent proofs of the same law of physics. This was a law sought for a long time by many outstanding scientists such as Ptolemy and Ibn al-Haytham without success. How did it happen to be established in three different manners within relatively short intervals? One of these later on proved to rest on wrong premises. Fermat's proof of the correctness of the law is entirely theoretical and hypothetical, while that of Snell is based on observation and experiments. It

[107] Florian Cajori, *A History of Physics*, The Macmillan Company, New York 1935, p. 83.

[108] See, Roshdi Rashed, "A Pioneer in Anaclastics. Ibn Sahl on Burning Mirrors and Lenses", *Isis*, vol. 81, 1990, pp. 464-491. On the question of the discovery of the law of refraction, see also: Antoni Malet, "Gregorie, Descartes, Kepler, and the Law of Refraction", *Archives Internationales d'Histoire des Sciences*, vol. 40, no 125, 1990, pp. 278-304.

is perhaps not far-fetched to see a parallelism between these and the proof of Al-Khwârazmî's solution formulas for second degree algebraic equations. Just as Snell need not be antagonistic toward theoretical proofs of a law of physics or Fermat toward an experimental proof, so it is not reasonable to conclude that Al-Khwârazmî was against Euclid's geometry or ignorant of it, because in a tract of his meant for practical men without theoretical training he did not proceed in a formal synthetic geometrical approach. Such an assumption of ignorance or antagonizm on the part of Al-Khwârazmî would, moreover, seem entirely out of tune with the intellectual orientation of the institution in which he seems to have occupied a prominent place and with the cultural policy of the caliph who had great confidence in his knowledge and scholarship.

In 1932 Solomon Gandz published a paper in which he claimed that the Bâb al-Masâḥa part of Al-Khwârazmî's Algebra was borrowed from a Hebrew book by the name of Mishnat ha Middot which, in his estimate, had been written in about 150 A.D.[109] William Thomson reviewed this work in Isis.[110] He notes that Hermann Schapira "was the first to perceive the extraordinary likeness between the *Mishnat ha-Middot* and one section of the algebra of Muḥammad ibn Mûsâ al-Khwârazmî."[111] William Thomson may be said to summarize, in its trenchant lines, his impression of Gandz's work in the following paragraph:

"The whole literary and historical background of the problem presented by the book is discussed by Gandz with great acumen and scholarly simplicity in his introduction to the Hebrew and Arabic texts, and the problem is laid bare in such a masterly fashion and the facts stated so candidly that it is possible for a scholar to draw his own conclusions, if he does not agree with those of Gandz. The emendations and reconstructions of the Hebrew text proposed by Gandz are the fruits of ripe scholarship and based on genuine philological grounds, many of his notes are nothing short of essays on the historical development of mathematical terminology, and so far as the texts and translations are concerned, the edition is as definite as can well be expected. In the statement of his thesis,

[109] Solomon Gandz, "The Mishnat ha Middot and the Geometry of Muhammad ibn Mûsâ al-Khwârazmî", *Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik, Abteilung A: Quellen*, vol. 2, 1932, pp. 1-96.

[110] *Isis*, vol. 20, 1933, pp. 274-280.

[111] *Ibid.*, p. 275.

however, there is some confusion, and the evidence on which he relies to demonstrate it will not be accepted in toto without further proof."[112]

Concerning the date 150 A.D., which Gandz advances for the Mishnat ha-Middot, William Thomson writes as follows:

"The crux of the matter lies in the authorship, and it should be pointed out that the name, Nehemiah, occurs only twice, and both times in the Bodleian fragment only, ... Moreover, the connexion of this name with the Rabbi Nehemiah of the second century C.E. is, of course, a conjecture, resting for the most part on the fact that he appears to have been interested in mathematical computation."[113]

Further on, William Thomson says, "Moreover, the comparative table on page 85 does not prove that the bulk of Al-Khwârazmî's geometry was taken from the *Mishnat ha-Middot.* The order of the sections is entirely different. In one section the Arabic has another text altogether, and another section is not represented in the Hebrew at all. Sometimes the Hebrew is fuller, at others the Arabic. In some sections the Arabic arranges the material quite differently from the Hebrew, in others it adds proofs that appear to be of a more developed type than those given in the *Mishnat ha-Middot,* not to speak of phrases and sentences that are occasionally of vital import and which Gandz on two occasions at least (cf. p. 29, note 38) inserts into the Hebrew text with no other justification than that the author of the *Mishnat ha-Middot* shows in another section that he knew the required formula, a plausible argument, if we overlook the fact that the *Mishnat ha-Middot* has probably had a history of its own. These facts do not point to a direct dependence of the one book upon the other, but only to a family resemblance, and Al-Khwârazmî's chapter on mensuration is probably a more advanced type of a common school text, of which an earlier type served as basis for the *Mishnat ha-Middot.*"[114]

William Thomson's reference to family resemblance brings to mind Hero of Alexandria, one of the most outstanding representatives of the tradition of practical mathematics or the mathematics of mensuration. He flourished around the year 62 A.D. The otherwise vague chronology of his life span was ingeniously tied to that year by Otto Neugebauer, who

[112] *Ibid.,* p. 277.
[113] *Ibid.,* p. 277.
[114] *Ibid.,* p. 278.

discovered that an eclipse of the moon described by Hero corresponds to an eclipse in A.D. 62 and to none other during some five hundred years extending around that time reference point.[115] Concerning Hero of Alexandria, Marshall Clagett writes:

"We have already suggested that Galen and Ptolemy were not the only authors of the early Christian era who represented Greek science at its highest level. Hero of Alexandria also belongs to that select group. We have already discussed his *Mechanics* as being the culminating effort of mechanics in late antiquity (see Chap. Six) and as containing both theoretical and applied mechanics. His writings, particularly the *Metrica,* which included many formulae, and his commentary on Euclid's *Elements* (of which parts remain in Arabic) reveal him as an excellent mathematician."[116]

Michael S. Mahoney speaks as follows concerning Hero's mathematics:

"The historical evaluation of Hero's mathematics, like that of his mechanics, reflects the recent development of the history of science itself. Compared at first with figures like Archimedes and Apollonius, Hero appeared to embody the "decline" of Greek mathematics after the third century B.C. His practically oriented mensurational treatises then seemed to be the work of a mere "technician", ignorant or neglectful of the theoretical sophistication of his predecessors. As Neugebauer and others have pointed out, however, recovery of the mathematics of the Babylonians and greater appreciation of the uses to which mathematics was put in antiquity have necessitated a reevaluation of Hero's achievement. In the light of recent scholarship, he now appears as a well-educated and often ingenious applied mathematician as well as a vital link in a continuous tradition of practical mathematics from the Babylonians, through the Arabs, to Renaissance Europe.

"The breadth and depth of Hero's mathematics are revealed most clearly in his *Metrica,* a mensurational treatise in three books. ... The pro-

[115] A.G. Drachmann, "Hero of Alexandria", *Dictionary of Scientific Biography,* ed. Charles Coulston Gillispie, Charles Scribner's Sons, New York 1972, vol. 6, p. 310.

[116] Marshall Clagett, *Greek Science in Antiquity,* Abelard-Schuman, Inc., New York 1955, p. 117.

logue to the work gives a definition of geometry as being, both etymologically and historically, the science of measuring land. It goes on to state that out of practical need the results for plane surfaces have been extended to solid figures and to cite recent work by Eudoxus and Archimedes as greatly extending its effectiveness. Hero meant to set out the "state of the art", and the thrust of the *Metrica* is thus always toward practical mensuration, with a resulting ambiguity toward the rigor and theoretical fine points of classical Greek geometry. ...

... ...

"Hero's work enjoyed a wide audience. This is clear not only from what has been said above, but also in that fragments of his works can be found in the writings of several Arab mathematicians, including al-Nayrîzî and al-Khwârazmî." [117]

Gad B. Sarfatti, writing in 1968, has estimated, according to Roshdi Rashed, that the date of composition of the Mishnat ha-Middot was later than that of Al-Khwârazmî's book on algebra. [118]

Previously Julius Ruska had advanced the thesis that the Bâb al-Masâḥa was inspired by Indian works. [119] Aristide Marre speaks of parallels of the Bâb al-Masâḥa with certain Indian books and also with Hern. [120] Examples similar to those given by Al-Khwârazmî and Thâbit ibn Qurra, in line with "Socrates' proof" which is called the method of "reduction and composition" by Thâbit ibn Qurra, are not rare in the history of mathematics. The origin of proofs based on this method is sometimes traced to late ninth century Indian mathematicians. [121] But the fact that it can be traced back to Plato indicates clearly that its origins must be sought in much earlier times.

[117] Michael S. Mahoney, "Hero of Alexandria: Mathematics", *Dictionary of Scientific Biography*, vol. 6, 1972, pp. 314, 315.

[118] Roshdi Rashed, *Entre Arithmétique et Algèbre*, p. 19, note 7.

[119] Julius Ruska, "Zur Ältesten Arabischen Algebra und Rechenkunst", *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Philosophisch-Historische Klasse*, 1917, pp. 1-125.

[120] Aristide Marre, *op. cit.*, pp. 2-14.

[121] W. Lietzmann, *Der Pythagorische Lehrsatz*, Stuttgart 1953, p. 24, Harriet D. Hirschy, "The Pythagorean Theorem", *Historical Topics for the Mathematics Classroom*, Thirty-first Yearbook, National Council of Teachers of Mathematics, Washington, D.C., 1969, pp. 215-218.

Such details found in widely separated sources clearly show that Al-Khwârazmî's and ᶜAbdu'l-Ḥamîd's geometrical schemes of verification or justification for their solutions of second degree equations were far from being irreconcilable with Greek classical synthetic geometry and constituting merely "naive" and primitive approaches unworthy of one steeped in Euclidean axiomatic geometry which secured and supplied a clearly thought-out notion of "proof". The Pythagoreans "proved" the irrationality of $\sqrt{2}$ in an irrefutable manner, and, likewise, the theorem $a^2+b^2=c^2$ for a right angled triangle, and Archytas conceived his masterly solution of the duplication of the cube long before Euclid. These should therefore be classified in the group as perfectly satisfactory proofs of pre-Euclidean geometry achieved at a time when the notion of proof was not as yet sufficiently clear and sophisticated or rigorous. Modern mathematicians too have now and then felt quite free to give the status of axiom to widely differing items of knowledge, and this is reminiscent of the pre-Euclidean proofs of Euclidean geometry.

All in all, it would seem perfectly reasonable therefore to qualify the geometric justifications of the solutions of second degree equations seen in Al-Khwârazmî and ᶜAbdu'l-Ḥamîd ibn Turk as geometric proofs or demonstrations although the simplicity of the geometry underlying them may tend to create the impression that they should not deserve such a pretentious name.

The third part of Al-Khwârazmî's Algebra deals with the algebra of inheritance. This part (the Kitâb al-Waṣâyâ) is seen to occupy almost half of the whole book, so that we may conclude that Al-Khwârazmî must have attached great value to this part of his Algebra in particular. This part occupies pp. 65-122 in the Arabic text of 122 pages, as published by Rosen, and pp. 86-174 in Rosen's translation. In fact, as we have seen, and as pointed out by Gandz, Al-Khwârazmî emphasized in his Introduction to his Algebra that he has written his book in order to serve the practical needs of the people in their affairs of inheritance, legacies, partition, lawsuits, commerce, etc. In the Kitâb al-Waṣâyâ (Book on Legacies) inheritance and legacies are mentioned first, thus also indicating that here was the most important part of his work.[122] The algebra of inheritance

[122] S. Gandz, "The Algebra of Inheritance, A Rehabilitation of Al-Khwârazmî", *Osiris*, vol. 5, 1938, p. 324.

part of his book may constitute the most original contribution of Al-Khwârazmî in his book on algebra.

In the Ḥisâb ad-Dawr (Computatic ı of Return) section of the Kitâb al-Waṣâyâ,[123] in his introductory note Rosen criticizes Al-Khwârazmî's treatment of the problems presented, and this criticism is seen to have been accepted in its general outlines by such outstanding authors as Cantor and Wieleitner, until Gandz appeared on the scene and showed that the misunderstanding was due to deficiency of a knowledge of the Islamic laws of inheritance on the part of Rosen, who did the pioneering work on Al-Khwârazmî, and of his followers such as Cantor and Wieleitner.

Al-Khwârazmî did pioneering work in such important fields as arithmetic, algebra, cartography, and the publication of trigonometric and astronomical tables in the World of Islam. In case he has to share the glory due to him in these domains with some fellow scientists and scholars, this should not detract from the credit due to him. It seems, as we have seen, that he has to share some of this glory with ʿAbdu'l-Ḥamîd ibn Turk in algebra and arithmetic, in spite of Abû Kâmil Shujâʿ ibn Aslam's verdict.

Both in the field of algebra and the positional number system Mesopotamia and the Sumerians in particular occupy a very fundamental place in world history. The Mesopotamian influence may be viewed also in a much broader perspective resulting from Neugebauer's wide-reaching researches. David Pingree writes: "The fundamental conclusions which he (Neugebauer) reached is that, almost without exception (the Chinese and the Mayans are the exceptions), the various civilizations of the world have all depended on the Babylonians for their basic understanding of mathematical astronomy, though each has reshaped what they received, directly or indirectly from Babylon, to suit its own traditions and requirements."[124]

We shall see in this paper, shortly hereafter that the Babylonian mathematical astronomy may possibly have influenced China also, and that Central Asia, the home of both Al-Khwârazmî and ʿAbdu'l-Ḥamîd ibn Turk, may have served as intermediary in the passage of this influ-

[123] Rosen's translation, pp. 133-174; Melek Dosay's translation, pp. 106-137.

[124] David Pingree, "Neugebauer, 1899-1990", *Archives Internationales d'Histoire des Sciences,* vol. 40, no 124, Juin 1990, p. 83.

ence. Thus, Neugebauer's impression (or, at least, that of David Pingree) that China was an exception will have turned out to be wrong, i.e., if such an influence can be fully ascertained or substantiated.

It is, moreover, likely also that such an influence played a part, and in more than one way, in the process of the spread of the notion of the system of numerals and calculations on the basis of the positional system. Here too Central Asia and China seem to have possibly come into play to a certain extent, as we shall see. And this particular aspect of the problem also interests us very much both from the standpoint of Al-Khwârazmî and ʿAbdu'l-Ḥamîd ibn Turk.

There is evidence, moreover, as we have seen, that Al-Khwârazmî knew Turkish and that he belonged to the Turkish sector of the population of Khwârazm, just like Al-Beyrûnî.[125] Indeed, there would be little chance or occasion for a non-Turk living in Baghdad to master the Turkish language, and Professor Akmal Ayyubî speaks of him as "one of the greatest Turkish minds of the medieval Islamic age" and says that "he was Turk by nationality but Arab in language."[126] ʿAbdu'l-Ḥamîd ibn Turk too was obviously a Turk by firsthand authentication and open acknowledgment.

Now, Al-Khwârazmî and ʿAbdu'l-Ḥamîd ibn Turk wrote an algebra at a relatively early date, i.e., about two generations before the book of Diophantos on arithmetic, which played a very important part in the history of algebra, was translated into Arabic. The question automatically comes to mind as to what was the source of their knowledge. It is extremely interesting therefore that they were both Turkish, or, more generally, speaking Central Asian.

Ibn Khaldûn (d. 1406), in his well-known Muqaddima states in a categorical manner that in the fields of science and learning (intellectual endeavors) the part played by the Arabs was a very minor one, while that

[125] See above, pp. 8-9 and notes 20-28.

[126] See, N. Akmal Ayyubi, "Contributions of Al-Khwârazmî to Mathematics and Geography", *Bulletin of the Institute of Islamic Studies*, vol. 17-21, 1984-1988, published by The Institute of Islamic Studies, Aligarh Muslim University, Aligarh, p. 82; N. Akmal Ayyubi, "Contributions of Al-Khwârazmî to Mathematics and Geography", *Acts of the International Symposium on Ibn Turk, Khwârazmî, Fârâbî, Beyrûnî, and Ibn Sînâ* (Ankara, 9-12 September 1985), Ankara 1990, pp. 213-214.

of the non-Arabs, i.e., *ajams* was very substantial and outstanding.[127] There is no doubt that Ibn Khaldûn exaggerates the little importance he attributes to the contribution of the Arabs to intellectual pursuits of medieval Islam. But it is equally true that he makes a remarkably apt observation when he emphasizes the part played by non-Arabs of Eastern Islam and Central Asia in an unequivocal manner.

Indeed, it is a fact that Central Asia was the home of a great majority of the most outstanding Islamic thinkers and scientists such as ʿAbdu'l-Ḥamîd ibn Turk, Farġhânî, Fârâbî, Ibn Sînâ, Abû'l-Wafâ, Beyrûnî, Gazâlî, ʿUmar Khayyâm, and Nasîru'd-Dîn Ṭûsî. Certain scholars are wont to tie up this situation solely with the Persian elements of the population of Central Asia, or, those that may be considered as relatives of the Persians. But this attitude is not sufficiantly reasonable. In fact, Persia itself was not, as a region, so much in the forefront of the countries giving rise to the production of scientists and thinkers, when compared to Central Asia, i.e., to the countries in the east and northeast of Persia itself.

These remarks of Ibn Khaldûn are somehow indicative of a basic circumstance that must have been predominent in the medieval Islamic World, and it is worth to attempt to disentangle the various elements involved in this state of affairs. At any rate, it is not a matter to be taken lightly. Indeed, the distinguished German Orientalist Ignas Goldziher writes:

"... Under Islam the Arabization of non-Arab elements and their participation in the scholarly activities of Muslim society advanced rapidly, and there are few examples in the cultural history of mankind to rival this process. Towards the end of the first century there is a grammarian in Madîna named Bushkest, a name which sounds quite Persian. ... The fathers and grandfathers of many others who excelled in politics, science, and literature, had been Persian or Turkish prisoners of war who became affiliated to Arab tribes and who by their completely Arabic *nisbes* almost made people forget their foreign origin. But on the other hand it was not impossible for such Arab *mawâlî* to retain a memory of their foreign descent, though it was not very common." The famous Arab poet Abû Isḥâq Ibrâhîm al-Sûlî was a descendant of Sol Tigin, a Khorasanian Turkish prince who was defeated by Yazîd ibn Muhallab toward the end of the

[127] See, Franz Rosenthal's translation, vol. 3, 1958, pp. 311-315.

first quarter of the eighth century and had lost his throne. Converted to Islam, he became one of the most zealous partisans of his conqueror. He is said to have written upon the arrows he sent against the Caliphs troops: "Sol is calling upon you to follow the book of God and the *Sunna* of His Prophet." [128]

An item of information that an influence of algebraic astronomy came from Central Asia, or from parts of China on the borderlands of Central Asia, to the Chinese astronomers in the eighth century is of great interest in this context, i.e., in view of the fact that both Al-Khwârazmî and ʿAbdu'l-Ḥamîd ibn Turk were from Central Asia. Shigeru Nakayama writes:

"The solar equation of center was the most important problem with which professional mathematical astronomers in ancient times had to deal. Western astronomers traditionally treated it with geometry and trigonometry, while the Chinese generally relied on an entirely different pragmatical and empirical tradition, namely numerical interpolation between values of the midday gnomon-shadow length observed at, say, ten-day intervals. There is, however, another tradition using an algebraic function of second order (degree) that seems to have originated in Central Asia sometime around the eighth century. This third approach was discovered by the present writer in 1964 and briefly described in English. ...

"The Futian calendarical system (that is, the step-by-step methods for computing the ephemeris) has been known as one of the unofficial calendars compiled in A.D. 780-783 in China. The compiler, Cao Shiwei ... originated in the western part of China. One conjectures that he or his family originally came from Samarkand.

"No part of the content of the Futian calendar has survived in China to this day. Tradition says that it was based on an Indian calendar and speaks of it as having entirely altered the old Chinese method. ... Another innovation of the Futian calendar is its use of decimals rather than traditional fractions. ...

"H. Momo has shown that the Futian calendar was the major tool of the Buddhist school of astronomy, the productions of which competed with the official Chinese-style ephemerides made by Japanase court as-

[128] Goldziher, *Muslim Studies*, English translation by S.M. Stern, London 1967, pp. 108-109.

tronomers. He has also proven that two extant twelfth-century Japanese horoscopes had been calculated with the Futian calendar. ...

"In 1963, the late J. Maeyama found a text entitled 'Futenreki nitten sa rissei' (The Futian calendar table of the solar equation of center, in 1 volume) in the Tenri Library. It was analyzed astronomically by the present writer. ..."

Tatara Hoyu "edited several collections, one of them entitled 'Tenmon hisho' (Esoteric works of astronomy) which includes a fragment of the Futian calendar. ...

"The text consists of a short illustration of calculation and a table of the solar equation of center for each Chinese degree (defined as the mean daily solar motion). Though the explanation of the computational method is somewhat clumsy, analysis of the table clearly showed that the data given are all calculated from the formula $x=(182 - y)\, y/3300$, where x is the equation of center and y is the mean solar anomaly, both expressed in Chinese degrees.

"This formula employed in the Futian calendar resembles neither the traditional Chinese empirical (interpolation between observational data) nor the Hellenistic-Indian geometrical or trigonometrical approach. It is an algebraic calculation of second order (degree).

"Whether such an algebraic method is superior to empirical or geometrical techniques is hard to judge. It has the advantage of being easily calculated on a counting board, especially in a culture such as China where decimal calculation was widespread. ... This algebraic function became a regular feature of Chinese calendar calculation. It was also employed later for the same purpose in the Uygur calendar.

"... The traditional approach required empirical data for the solar equation of center on any given day, that is, day-to-day observation of the position of the sun. ...

"... The algebraic expression introduced into calendarical calculation in the eighth century provided an alternative method simpler, easier and more convenient for calendar calculators." [129]

[129] Shigeru Nakayama, "The Emergence of the Third Paradigm for Expressing Astronomical Parameters: Algebraic Function", *Erdem*, vol. 6, (no 18), 1992, pp. 877-884.

Much more knowledge of concrete detail would of course be desirable on this question. But one item of information is quite clear, and this is that a knowledge of algebra, and in particular concerning second degree equations, was apparently available in Central Asian regions neighboring China on its western boundries, i.e., neighboring Chinese Turkistan, or Chinese Turkistan itself, during the eighth century. Roughly speaking, this is the vast area extending between Iran and China, including perhaps the western parts of China itself; Khwârazm also and Khuttal, and Gilân (or Jilân), i.e., the homes or birth places of Al-Khwârazmî and ʿAbd al-Ḥamîd ibn Turk, are located within this geographical region.

Again, we know for sure that this knowledge was available more specificially at a time which corresponds to the beginning of Al-Khwârazmî's life span, and it is also very likely that the life span of ʿAbdu'l-Ḥamîd ibn Turk was roughly the same as that of Al-Khwârazmî, if not somewhat earlier.

All this is clear, and we may therefore conclude that this explains why Al-Khwârazmî and Ibn Turk were in a position to write for the first time in the World of Islam a book on algebra, and more specifically on second degree equations. And we may therefore conclude that it was not due to a mere coincidence that both these mathematicians were natives of Central Asia.

Sanad ibn ʿAlî's name also appears in the Fihrist of Ibn al-Nadîm as the author of a book on algebra. He too was a contemporary of the caliph Al-Ma'mûn and of Al-Khwârazmî. The question arises therefore whether he too was of Central Asian origin. I have not gone into a detailed work on Sanad ibn ʿAlî's place of birth and his life, of which not much seems to be known, however. For A.S. Saidan's observation that these words of Ibn al-Nadîm fit in very well with the works of Al-Khwârazmî, and that as they are not corroborated elsewhere, i.e., other sources on Islamic scholars and scientists, Sanad ibn ʿAlî is not considered here as author of a book on algebra.[130]

Of great interest to our subject would also seem to be Shigeru Nakayama's statement concerning the Chinese tradition of day-to-day observation of the position of the sun. Ḥabash al-Ḥâsib, a contemporary of Al-Ma'mûn, states that the one-year program of observation in Al-Ma'mûn's

[130] See above, p. 20 and note 62 for Saidan's remarks on this question.

Qâsiyûn Observatory at Dayr Murrân was fully accomplished and that these astronomical observations were made every day.[131]

This program of astronomical work was set up just after the decision of Al-Ma'mûn and his astronomers that Ptolemy's astronomy constituted the definitely superior knowledge of the time and that it should be adopted in preference to methods of Indian astronomy. We also know that Al-Ma'mûn was personally involved in the taking of this decision.[132]

Day-to-day observation was probably very rarely practiced in Islam and Western Europe up to the time of Tycho Brahe. We know very little about the type of work carried out in the Islamic observatories of the Middle Ages, but there is no evidence at all that such a method of daily observation became established as a tradition in these institutions. The Qâsiyûn example is just about the first serious and systematic attempt to establish a fruitful scientific research program. And it was decided at this juncture that Greek astronomy was superior to that of India. One may wonder therefore whether there was also an influence deriving from the Chinese empirical tradition upon Islam at such a relatively early date. For the Chinese had astronomical bureaus with imperial astronomers and astrologers. These bureaus were equipped with staffs, and regular observations of stellar bodies were carried out in these bureaus. They may be likened to primitive astronomical observatories or to the Islamic *muvakkit* offices,[133] and Al-Ma'mûn was the first to found an astronomical observatory in Islam.

A.S. Saidan doubts the existence of any Chinese influence on Islamic mathematicians (and astronomers) before the foundation of the Marâgha Observatory in the second half of the thirteenth century.[134]

Cultural contact between China and the Islamic World before the spread and establishment of the Moslem religion in Central Asia must have been relatively insignificant due to the vast distances between the

[131] See, Aydın Sayılı, *The Observatory in Islam,* p. 57 and note 37.

[132] See, Aydın Sayılı, *The Observatory in Islam,* pp. 79-80, and also above, p. 4 note 12.

[133] Colin A. Ronan, *The Shorter Science and Civilization in China: 2,* Cambridge University Press, 1981, pp. 75-77.

[134] Ahmad Saidan, *Al-Fuṣûl fî'l-Ḥisâb al-Hindî li Abî'l-Ḥasan Aḥmad ibn Ibrâhîm al-Uqlîdisî, History of Arabic Arithmetic,* vol. 2, Urdun 1977, p. 251; see also, A.S. Saidan, *The Arithmetic of al-Uqlîdisî,* D. Reidel Publishing Co., 1978, pp. 466-485.

two Worlds. There was the Silk Route causing some cultural contact between China and the Near East, and Central Asia. But we are more interested here with serious and weighty scientific and intellectual contacts, which may at times be casual and rather personal, from relatively early dates on and particularly before the advent of the Seljuqs, and it is clear that Central Asia acted as intermediary between China and the bulk of the Islamic World, as it did, through Buddhism in particular, between India and China.

As is well known, there is a *hadith,* i.e., a saying attributed to the Prophet Muḥammad in which the Moslems are recommended to search knowledge ("*ᶜilm,*" i.e., scientific knowledge, or, at least, knowledge including the scientific) even in China: *uṭlubû'l-ᶜilme walaw bi's-Sîn.* This saying is not found in the six basic and classical *hadith* collections, and this indicates that, very likely, it is not a true *hadith.* Abû'l-Ḥasan ᶜAlî al-Hujwîrî (d. 1072) mentions it,[135] and he may be among the early examples of the persons who propagated it.

Hujwîrî was from the southern extention of Central Asia whose interest in such cultural contacts should naturally be of significance, and the very fact that such a saying was put into circulation would indicate that China was considered as of some importance from this standpoint. Indeed, examples of fruitful contacts of great significance were, as we shall see, already in existence. It was natural therefore that their continuation should be considered profitable in intellectual centers.

Arab conquests in Central Asia brought the boundaries of the Muslim World closer to China. But direct contact between Arab and Chinese forces was rare, and the Battle of Talas in 751 A.D. marked the end of such rare direct military encounters. The Turkish element of the population of Central Asia acted as intermediary between whatever contact the Moslem World had with China. Whether Moslem or non-Moslem, Turks appear as the major element of Central Asia's population. Modern scholarship seems to have exaggerated the importance of the Indo-European elements of Asia's population.

From the start of the Arab conquests beyond the northern and eastern boundaries of Persia on, the Arab armies came into contact with

[135] See, Aydın Sayılı, *The Observatory in Islam,* pp. 13-14.

Turks practically everywhere in Central Asia, including the southwestern regions of Central Asia, i.e., northern India. The same situation seems to be true from the standpoint of cultural contact too, including what we may characterize as major scientific and intellectual ones.

The picture created by Firdawsî's Shâhnâma, i.e., the world of the Turans as confronting that of Iran, or Persia, seems to turn out to be quite realistic. The same impression is corroborated by the accounts of Moslem travellers in non-Moslem regions of Central or Inner Asia too. In this connection the picture created by Nizâmî of Ganja in his couplet "Zi Kûh-i Hazar tâ bi Deryâ-yi Çîn — Heme Turk bar Turk Bînî Zemîn" [i.e., from the Khazar Mountain (Caucasian Mountains) to the Sea of China (Pacific Ocean) — All the way through, you came accross regions populated by Turks, one after another] seems to constitute a correct image of the situation if one excepts China itself, i.e., if one thinks, beyond the Chinese Wall, of the lands to the north of China.[136]

Central Asia is a vast region, and its boundaries may be established of course by convention, but they are and should be based on historical as well as geographical considerations. The northern India of the Middle Ages, i.e., Afganistan and the present Pakistan, may conveniently be included within the bounds of Central Asia.

The Hephthalites extended their conquests to Northern India in this sense. They were apparently of Turkish origin, and the Tukyus inherited the lands within the Hephthalite Empire. As a consequence of this, when the Arab armies penetrated these lands shortly after the termination of their conquest of Persia, they met, in these regions, with Turkish rulers and princes such as Rutbil of Kabul and the Turkish Shahîs and several other rulers and princes of Turkish stock all over the different regions of Central Asia. The rulers and armies as well as a considerable part of the population of these regions were Turkish, and the Arab conquerors consequently left these local rulers in power as tributaries of their Khorasan governors.[137]

[136] See, Nizâmî-i Ganjawî, *Iskendernâme,* Sharafnâme section, *Kulliyât-i Hamsa-i Hakîm Nizâmî-i Ganjawî,* Emîr-i Kebîr edition, 1344 HS (1965) Tehran, p. 1100.

[137] See, H.A.R. Gibb, *The Arab Conquest in Central Asia,* The Royal Asiatic Society, 1923; H.A.R. Gibb, *Orta Asya Fütuhatı* (M. Hakkı çevirisi), Evkaf Matbaası, Istanbul 1930; Richard N. Frye and Aydın Sayılı, "Turks in the Middle East Before the Seljuqs", *Journal*

This general ethnic picture of Central Asia, as far as its Turkish element of population is concerned, is apparently capable of extention quite a long way back, through extrapolation and interpolation – as a matter of fact, to times close to the dawn of history in Mesopotamia.

As revealed by cuneiform tablets of the Sumerians dating back to 2500-2200 B.C., the titles of the kings of the Gutians of Mesopotamia are seen to be in a Turkish very close to that of the Orhun inscriptions of Central Asia belonging to the Tukyus from the first half of the eighth century A.D.[138] A bone amulet, carved in the shape of a deer, on which is written "white meral", i.e., white deer, in runic letters, i.e., the letters of the Orhun alphabet, shows, in the words of Altay Amonjolov, that the runic alphabet "was the script of the early Turkic speaking peoples, the alphabetic script of the Sakas ... in the fifth century (previous to our era) in South Siberia and Kazakhistan. ..." The same author speaks also of a silver bowl found in an excavation near the city of Esik at the foot of the mountains in the environs of the Ili River. On this, again, stands a short inscription in Turkic belonging to the Saka period (the fifth and fourth centuries B.C.). Altay Amonjolov writes, concerning this archeological find, as follows:

"The great value of this writing is that it once again concretely proves that the language of the Saka peoples, who settled in the territory of Kazakhistan in early times was the ancient Turkic language. Furthermore, ... it testifies to the fact that Turkic speaking peoples of 2500 years ago knew

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of the American Oriental Society, vol. 63, 1943, pp. 194-207; Richard N. Frye and Aydın Sayılı, "Selçuklulardan Evvel Orta Şark'ta Türkler", *Belleten* (Türk Tarih Kurumu), vol. 10, 1946, pp. 97-131; R.N. Frye and Aydın Sayılı, "Turks in Khurosan and Transoxania Before the Seljuqs", *Muslim World,* vol. 35, 1945, pp. 308-315; Zeki Velidi Togan, "Eftalitlerin Menşei Meselesi", see below, note 156. Aydın Sayılı, "The Nationality of the Hepthalites", *Belleten* (TTK), vol. 46, 1982, pp. 17-33; N.A. Baloch, "An Evaluation of Bîrûnî's References to the Turk Rulers of Kabul and Peshawar Region in the Light of Historical Perspective of the Turkish States and Principalities During the 7th-10th Centuries A.D.", *Acts of the International Symposium of Ibn Turk, Khwârazmî, Fârâbî, and Ibn Sînâ,* Ankara 1990, An Atatürk Culture Center Publication, pp. 23-32; same article in Turkish translation by Esin Kâhya, *Uluslararası Ibn Turk, Hârezmî, Fârâbî, Beyrûnî, ve Ibn Sînâ Sempozyumu Bildirileri,* Ankara 1990, pp. 26-34.

[138] See, Kemal Balkan, "Relations Between the Language of the Gutians and Old Turkish", and its Turkish: "Eski Önasya'da Kut (veya Gut) Halkının Dili ile Eski Türkçe Arasındaki Benzerlik", *Erdem,* no 16, 1992, pp. 1-125.

alphabetic writing and made use of it widely."[139] It has been known, on the other hand, since the last decade or so of the last century that the language of the Sumerians, who occupy an altogether extraordinary place at the origins of the history of our present-day Western civilization, was an agglutinative tongue similar in various respects to the Turkish language.[140]

The similarity of Turkish and the Sumerian language, and the probability therefore that the "Sumerians were a Turkish-related people" has recently been attested, on a special occasion, by Samuel Noah Kramer, one of the greatest Sumerologists of our era.[141]

The Sumerians are believed to have come from Central Asia to Mesopotamia about 3500 or 4000 B.C., i.e., a millenium or 1500 years before the Gutians. All this goes to show that Turks were a constituent part of the population of Central Asia since times immemorial and that Turkish is one of the most ancient languages of history.

In Arabic there is a special word, *qirṭâs,* for paper, but the word *kâghad,* or *kâghaż,* is more widely used, and not only in Arabic but also in Persian, Turkish, Urdu, and the languages of southeast Asia. Several etymological origins have been suggested for this word by various authors. Berthold Laufer believes it to be of Uyghur Turkish origin.[142]

Philip K. Hitti writes: "Worthy of special note is the manufacture of writing paper, introduced in the middle of the eighth century into Samarqand from China. The paper of Samarqand which was captured by the Moslems in 704, was considered matchless. Before the close of that century Baghdad saw its first paper-mill. Gradually others for making paper followed."[143]

[139] Altay Amonjolov, "The Words of Ancestors", *Erdem,* vol. 5, no 15, 1991, pp. 794, 795. See also, Semih Tezcan, "En Eski Türk Dili ve Yazını", *Bilim, Kültür ve Öğretim Dili Olarak Türkçe,* ed. Aydın Sayılı, Ankara 1978, p. 282. Semih Tezcan warns us that the conclusions to be drawn from the Esik excavation must be handled with caution.

[140] For the place of the Sumerians in the world intellectual history, see, e.g., Samuel Noah Kramer, *History Begins at Sumer,* London 1961, or, *From the Tablets of Sumer, 25 Firsts in Man's Recorded History,* 1965, and, more specifically, for their contributions to the exact sciences and medicine, see, Aydın Sayılı, *Mısırlılarda ve Mezopotamyalılarda Matematik, Astronomi ve Tıp,* Ankara 1966, 1992.

[141] Mübahat Türker-Küyel, "Atatürk'ün Çivi Yazılı Kültür Araştırmalarına İlişkin Katkıları Hakkında Üç Tarihsel Belge Daha", *Erdem,* no 16, Ankara 1992, pp. 294-297.

[142] B. Laufer, *Sino-Iranica,* Chicago 1919, pp. 557-559, see, p. 557, note 6.

[143] Philip K. Hitti, *History of the Arabs,* Macmillan 1940, p. 347.

Rag paper was supposed to have been made for the first time in Europe in relatively modern times. But research made in the later years of the last century and the early parts of the present century showed that the manufacture of rag paper went back, in Turkistan as well as in China, almost to the very period when paper was invented. Thomas Francis Carter says: "Examination of paper from Turkistan, dating from the second to the eighth centuries of our era, shows that the materials used are the bark of the mulberry tree; hemp, both raw fibers and those which have been fabricated (fish nets, etc.); and various plant fibers, especially China grass (Brehmeria Nicea), not in their raw form but taken from rags." [144]

Concerning the passage of paper from China and Central Asia to the World of Islam, Emel Esin writes:

"According to information contained in various Islamic sources, Chinese prisoners captured by the Moslems in the Battle of Talas (751 A.D.) or Uyghur Turks taken as prisoners of war by the Amîr of Samarqand during the reign of the Abbasid caliph Al-Mahdî (775-788), taught the manufacture of paper to the people of Samarqand. It is possible that both these assertions are meant to refer to Uyghurs (Toguz-Guzz). For in that era the term Chin (China) did not refer exclusively to the China of our day. In those days China proper was called "Mâchîn" which was, apparently, a distorted form of Maha-Chîn (Great China). The region of East Turkistan, Kashgar, and the lands of the Uyghurs, which are all in the boundary district of the China of our time, were called "Chîn", i.e., China, in those days. Moreover, the sovereignty of the Uyghur Empire extended in the west to the region of Farghâna and could become involved in warlike activities with the Islamic realm. The likelihood or possibility that these artisan or artist war prisoners were of Uyghur extraction is enhanced by the circumstance that the Uyghurs were familiar with the manufacture of paper which they called "kegde" and they were well known for the production of their renowned arms, swords in particular. Likewise, Laufer's conviction to the effect that the word *kâgaž* (kagıd) was derived or borrowed in Arabic and Persian not from the Chinese language but from Turkish, i.e., from the Turkish word "kagash", meaning the bark of a tree, also confirms the thesis that the artisan prisoners of

[144] See, Thomas Francis Carter, *The Invention of Printing in China,* Columbia University Press, 1931, pp. 1, 4, and 4-6.

war in question were Uyghurs and not Chinese. The Uyghurs decorated their swords by inlaying them with darkened steel. This method of ornamentation was further developed later on in Damascus."[145]

It would undoubtedly be worthwhile to mention some of the sources from which Emel Esin gleaned this information. Concerning the question whether the artisan prisoners were Chinese or Turkish these sources are: V. Minorsky, "Tamim ibn Bahr's Journey to the Uyghurs", *Bulletin of the School of Oriental and African Studies,* vol. 12, 3-4, London 1948; Marwazî (Sharaf al-Zamân Ṭâhir), *Marwazî on China The Turks and India,* ed. Minorsky, London 1942. Concerning the words *kâgaẓ* and *kegde:* O. Franke, *Geschichte des Chinesischen Reiches,* Berlin 1925, vol. 3, p. 392.[146]

Turks of Central Asia seem, indeed, to have had a great share of contribution in the cultural development realized and accomplished in medieval Islam. Apparently this was especially conspicuous in intellectual pursuits. This can best be illustrated in the light of concrete examples conducive to making assessments and value judgements, as much as possible in the state of our rather chary state or sort of information. Examples relating to Turkish influences in the fields of decorative art and architecture in the relatively formative eras of Islamic civilization may not be out of place at all here. This brings us back to Emel Esin. Emel Esin writes:

"A Turkish monarch, perhaps Kül Tigin, was represented on the murals of Kusair Amra amongst the world kings vanguished by the Caliph. Influences of the art of Western and Eastern Turkistan are already notable at the Palace of Mafjar and other Omayyad castles. These influences must have been further introduced by personalities such as the yabgu of Tokharistan and the 'Son of the Turkish *Khâgân'* who were taken prisoners in Khorasan and brought to Damascus in the reign of the Caliph Hishâm (735-742), the builder of the Palace of Mafjar. But the bulk of the Turkish contribution to Islamic art began in the ninth century. ... Al-Yaᶜqûbî who wrote his description of Sâmarrâ fifty-five years after the

[145] Emel Esin, "Türklerin İslâm'a Girişi, İlk Devir: VIII. -X. Yüzyıllar", *İslâmiyetten Önceki Türk Kültürü Tarihi ve İslâm'a Giriş, Türk Kültürü El-Kitabı II,* cilt lb'den ayrıbasım, Edebiyat Fakültesi Matbaası, Istanbul 1978, pp. 155-156, p. 259, note 81-82; see also, *op. cit.,* p. 319. In connection with the artisans taken prisoner at the Battle of Talas, see also, below, p. 68 note 172.

[146] See, *ibid.,* pp. 259, 315, 313. I owe my acquaintance with this remarkable work of Emel Esin to the kind interest of Professor Mübahat Türker-Küyel.

construction of the city (started in 836), attributes the erection of several monuments to Turks, Khazars, and Central Asians. 'It happened', said Al-Yaᶜqûbî, that most of the Turkish were then of the 'ᶜajam category'. These were carefully isolated from Muslims, even of the slave class and allowed intercourse only with the people of Farghâna, who were equally *'ajam*. It was a group of such 'Turk al-*ᶜajam*' (non-Muslim Turks) who under the leadership of the Muslim Turkish dignitary ᶜUrtunj (Artug in Tabarî) Abû'l-Fatḥ ibn Khâqân (another son of the Khâqân built the Khâqân Palace of Sâmarrâ (al-Jawsaq al-Khâqânî) celebrated for its paintings. Al-Yaᶜqûbî states clearly that these non-Muslims had not contact with their environment."[147]

Oktay Aslanapa writes, "It would appear, from the limited works and records that have survived, that a very advanced art of miniature painting and book production had been reached by the Uyghur Turks as early as the eighth century. These miniatures, together with the Bezeklik and Sorchuk frescoes that were brought to light in the Turfan excavations, show that there existed a characteristic Central Asian Turkish style of painting that, even at first glance, is quite distinct from Chinese art."[148]

This example serves to show that although Central Asian medieval Turkish culture and civilization could be expected to show signs of strong influence from China, it had characteristics that were quite independent from China. As we shall see[149] Beyrûnî classified Turkish culture and civilization as that of the East together with China and India, and individual traits of it seem to corroborate and justify such a classification. But on the other hand, Turks belonged to a vast area in Central Asia, and it would seem natural if "Turkish culture" should show a notable range of variation within its own bounds.

The custom of building mausoleums did not exist in Islam in Umayyad times. The earlier Abbasids too, and the Muslims in general of that era, were not anxious at all to have buildings erected over their grave as later Muslims were. The first exception to this rule occurred with the Abbasid caliph Al-Muntaṣir (862). His Greek mother obtained permission to

[147] Emel Esin, "Central Asian Turkish Painting Before Islam", *Türk Kültürü El-Kitabı*, vol. 2, part la, Istanbul 1972, pp. 262-263.

[148] Oktay Aslanapa, *Turkish Art and Architecture*, London 1976, p. 308.

[149] See below, p. 71 and note 177.

have a mausoleum built for him. This edifice was called Qubba al-Ṣulaybiyya. It was in Sâmarrâ and was located on a hill. The caliphs Al-Muᶜtazz (866-867) and Al-Muhtadî (869-870) also were subsequently buried in it. The plan is octogonal, and it is covered by a dome. It consists mainly of two octogons with an ambulatory in between, and the central chamber is square shaped.[150]

Otto Dorn writes concerning this first Islamic Mausoleum, "The whole thing is entirely non-Islamic and has apparently come into being under foreign influence. The origin of the domed octogon with an interior ambulatory is very clear. As with the Dome of the Rock (Qubba aṣ-Sakhra) in Jerusalem, here too, certainly the general plan of the early Christian martyr churches of Syria and Palestine have been of influence, though this fact has heretofore remained unnoticed. ...

"Disregarding, however, the special type of the plan of this sepulchral monument and concentrating on the fact that we have here a first exemplification of the making of the burial places outwardly visible, then it becomes reasonable to suppose that an old Central Asian tradition was responsible for this innovation, namely the tradition of the tent tomb and mound or tumulus (kurgan) which was extremely well known and alive among the Turks settled in Sâmarrâ and which, ... made a deep impression on Abbasid art. When viewed from the standpoint of this complex background upon which we shall dwell in greater detail in connection with the Seljuqid türbe (i.e., mausoleum) this burial monument has a fundamental significance aside from the fact that it is the forerunner of all the later monumental sepulchral edifices, which from the eleventh century on, and in an unbroken sequence have contributed to the fixing of the usage in Islam, especially under the dynasties set up by the steppe peoples, beginning with the Seljuqs. ..."[151]

Within these veins there are other points of importance to our main topic which could be taken into consideration. Jean-Paul Roux, e.g., says that during the Wei Dynasty, i.e., the period of Turkish To-ba or Tabgach rule, China reached an acme of its achievements in the field of

[150] K.A.C. Creswell, *A Short Account of Early Muslim Architecture,* Pelican-Penguin Books, 1958, pp. 286-289, 320; Katharina Otto-Dorn, *Kunst der Islam,* Baden-Baden 1964, pp. 71-72.

[151] Katharina Otto-Dorn, *op. cit.*, p. 72.

sculpture with the works of art found in the Yun Kang and Lung-Men Caves.[152]

All this shows that Turkish art was of considerable importance from relatively early times on in Islam. The chronology of the Central Asian influence on Islam is of much importance, and we also note that such Central Asian influences were not always traceable to Chinese origins either. Moreover, this early chronology of influences in art is parallel to the Central Asian influences in such fields as algebra and chemistry and much prior to the period of the establishment of Seljuk political supremacy.

The hospital in medieval Islam was, unlike the Greek asklepion and Byzantine institutions of charity in which medical care was available, a specialized institution devoted to the cure of the sick and having recourse to scientific medicine exclusively. It was thus in Islam that the true prototype of modern hospital came into being. It went through a relatively speedy process of development, it seems, which was realized within a span of time of about three centuries.

The first Islamic hospital was built at the very beginning of the eighth century in Damascus. Barmak who was the head of the Buddhist temple of Balkh in Central Asia when the Arabs conquered that city, may possibly have had a hand in its foundation, though its establishment may also have been influenced by certain sayings of the Prophet concerning medicine and contagious skin diseases and, quite likely the Byzantine nosocomia also may have served as a model or prototype for it.

The second Islamic hospital was located in Cairo, and practically nothing is known concerning it. The third hospital in the order of chronology was built in Baghdad by the Barmakids, late in the eighth century. It was under Indian medical influence. The fourth hospital was founded by Hârûn al-Rashîd with the aid of Jundisapur physicians, and it therefore represented Greek medicine. The fifth hospital was built in Cairo by Fatḥ ibn Khâqân, Turkish general and statesman, who was minister of the caliph Al-Mutawakkil. And the sixth hospital, in chronological order, was brought into existence by Aḥmad ibn Ṭulûn, famed Turkish statesman. This hospital seems to bare traces of Indian influence.

[152] Jean Paul-Roux, *Histoire des Turcs*, Fayard 1984, p. 38.

Aḥmad ibn Ṭulûn's hospital was built in Cairo in 872-874, and it was supplied with *waqf* revenues, the first to be so endorsed, so far as is known. This was not only a guarantee for its longevity but also a sign or agent of a more thorough integration of the hospital with the Moslem religious culture.

If we extend this list of early Islamic hospitals so as to include the next four hospitals endowed with *waqf,* thus reaching the date 967 approximately, we note that out of these latter four three were built by Turks. Two of the earlier six too owed their existence to Turks. This means that out of the first ten hospitals of Islam five were founded by Turks.[153]

It is of great interest, on the other hand, that by the side of the sources which trace the genealogy of the Barmak family back the Sasanians,[154] there is a parallel trend in the sources, or, rather, there is one which can be disentangled from the sources, as established by Zeki Velidi Togan and which is deemed by him as much more trustworthy, according to which the ancestry of the Barmaks goes back to the Epthalites[155] and this means that they were, very likely, Turkish.[156] The geneology connecting the Barmaks with the Sasanians had previously been deemed suspicious by Barthold especially because it represents the Barmaks of Umayyad times as fireworshippers, whereas the Barmak whom the Arabs met for the first time and who was of ripe age at the beginning of the eighth century was at the head of the Buddhist temple of that city.

[153] See, Aydın Sayılı, "The Emergence of the Prototype of the Modern Hospital in Medieval Islam", *Belleten* (Turkish Historical Society), vol. 44, 1980, pp. 279-286; Aydın Sayılı, "Central Asian Contributions to the Earlier Phases of Hospital Building Activity in Islam", *Erdem,* vol. 3, no. 7, 1987, pp. 149-162, Turkish translation by Ahmet Cevizci, *ibid.,* pp. 135-148.

[154] See, *Encyclopedia of Islam* (Turkish), vol. 2, (article "Bermekiler"), 1949, pp. 560-563.

[155] Zeki Velidi Togan, "Bermekî ve Sâmânîlerin Menşe'i ile İlgili Kayıtlar", appendix to note 48 of: Nazmiye Togan, "Peygamberin Zamanında Şarkî ve Garbî Türkistanı Ziyaret Eden Çinli Budist Rahibi Hüen-Çang'ın Bu Ülkelerin Siyasî ve Dinî Hayatına Ait Kayıtları", *İslam Tetkikleri Enstitüsü Dergisi,* vol. 4, part 1-2, Istanbul 1964, pp. 61-64.

[156] See, Zeki Velidi Togan, "Eftalitlerin Menşei Meselesi", appendix to note 41 of: Nazmiye Togan, "Peygamberin Zamanında Şarkî ve Garbî Türkistanı Ziyaret Eden Çinli Budist Rahibi Hüen-Çang'ın Bu Ülkelerin Siyasî ve Dinî Hayatına Ait Kayıtları", *İslam Tetkikleri Enstitüsü Dergisi,* vol. 4, part 1-2, Istanbul 1964, pp. 58-61; Aydın Sayılı, "The Nationality of the Epthalites", *Belleten* (Turkish Historical Society), vol. 46, 1982, pp. 17-33.

Now, if the Barmaks were Turks, then not only five out of the first ten hospitals considered above were founded by Turks, but the third Islamic hospital in chronological order of construction too would fall into this category. There is, in addition, the question of the probable part played by the above-mentioned father Barmak, who was the head of the Buddhist temple of Balkh at the beginning of the eighth century, in the construction of the first Islamic hospital, i.e., the Walîd ibn ʿAbdu'l-Malik Hospital of Damascus. And if this was the case, it would then mean that out of the first ten Islamic hospitals seven were built by Turks.[157] And this is almost incredible, at the first sight at least. But if the Barmaks are not added to the five mentioned above, still, five out of ten of these institutions owed their existence to Turks, and still seven to the people of Central Asia, and this is quite remarkable.

The Baghdad hospital of the Barmaks and also probably the Cairo hospital and dispensary of Aḥmad ibn Ṭulûn, as well as the Damascus hospital of Walîd ibn ʿAbd al-Malik perhaps, show the existence of Indian and more particularly Buddhist influence on the early hospitals of Islam, and Turks appear to have played a major part in the transmission of this influence to Islam.[158] It is of great interest therefore that we are in possession of fragmentary evidence of a relatively clear and entirely independent nature which can serve to lend further credence to this impression.

Indeed, the propagation of Buddhism among the Turks, beginning not later than the sixth century, brought them into contact with Indian medicine. In the Buddhist Turkish monastries the physician monks *(otaci bakshi)* healed the sick. The term *iglig yatgu ev,* i.e., dormitory for the ailing, shows the existence of hospitals in Buddhist Turkish monastries.[159]

[157] See, Aydın Sayılı, *ibid., Belleten,* vol. 44, and *Erdem,* vol. 3, no. 7. The chronological list of early Islamic hospitals founded later than the Aḥmad ibn Ṭulûn Hospital of Cairo, as referred to here, is based on the information given by A. Issa Bey in his *Histoire des Bimaristan (Hôpitaux) à l'Epoque Islamique* (Cairo 1929) and *Tarîkh al-Bîmâristânât fi'l-Islam,* (Damascus 1939).

[158] Aydın Sayılı, *ibid., (Belleten),* pp. 284-286; Aydın Sayılı, *ibid.,(Erdem),* pp. 155-161, 139, 142-143, 146-147.

[159] Emel Esin, "Otaci' Notes on Archaeology and Iconography Related to the Early History of the Turkish Medical Science", *Proceedings of the First International Congress on the History of Turkish-Islamic Science and Technology,* 14-18 September 1981, vol. 2, pp. 11, 13.

A reference to such a hospital is in the Uyghur Turkish work Maytrisimit which belongs perhaps to the ninth or the eighth century at the latest, but there is much uncertainty concerning the date.[160] The wording of the passage here is in the form of "building hospitals as an act of benevolence."[161] This may be taken as an indication that such places for hospitalizing the sick were not rare. Moreover, as this is traceable to Buddhistic influences, it should be reasonable to conjecture that they existed also in earlier centuries and among the Turkish Buddhist pre-Islamic inhabitants of Transoxania, Tokharistan, and environs.

It is also of interest in this connection, on the other hand, that in spite of the rapid dissemination of the Islamic religion among the Turks, we witness the survival of Buddhistic medicine still in later centuries in east Central Asia. In fact we have a document written in Uyghur Turkish attesting the existence of a medical school in a Buddhist monastary in the twelfth century A.D.[162]

Since this is a situation tied up to Buddhism, it may be reasonable to conjecture that the tradition was in existence in earlier centuries too. Indeed, this makes us understand better the circumstance that the head of a Buddhist monastary in Central Asia should be invited to Damascus to cure a member of a royal family.

The hospital and medical instruction in Islam seem to have stamped certain characteristic features of theirs upon the Renaissance hospitals of Europe, of the sixteenth and the seventeerth centuries. Moreover, clinical medical instruction was established in Islam by a Turkish ruler, Nûr al-Dîn Abû'l-Qâsim Zangî, Atabak of Halab and Damascus (1118-1174), in the Damascus Hospital bearing his name. This hospital served as model for the finest hospitals of Islam, and, among them, for the Qalawun Hospital of Cairo, a sort of acme among such institution in the Middle Ages. This hospital was founded by the Turkish Mamluks of Egypt. It was apparently a great source of inspiration for European sixteen and seventeenth century hospitals not only in architectural planning and decoration

[160] See, Şinasi Tekin, *Uygurca Metinler II, Maytrisimit*, Ankara 1976, pp. 28-29, note 53.

[161] Şinasi Tekin, *ibid.*, pp. 109, 229.

[162] See, Halim Baki Kunter, "Türk Vakıfları ve Vakfiyeleri Üzerine Mücmel Bir Etüd", *Vakıflar Dergisi*, vol. I, Ankara 1938, pp. 117-118. Halim Baki Kunter's source for this information is W. Radloff and S. Malow, *Uygurische Sprachdenkmäler*, Leningrad 1928.

but also from the standpoint of the clinical method of medical instruction. This method was adopted in Padua and Leiden, and from these centers it was disseminated in other parts of Western Europe. But we are not in possession of sufficient evidence to trace back these all-important developments to Central Asia. [163]

An extremely interesting example of contributions of Turks to more strictly scientific pursuits in medieval Islam may be chosen from the field of chemistry. Here too the part played by the Turks seems to have close ties with Chinese culture. That is, in this example Turks act also as intermediaries between Chinese culture and the culture of the World of Islam. But their independent achievement seems also to be of considerable magnitude and importance.

Just as in the field of algebra, in the Medieval Islamic World, chemistry or alchemy too began its growth and development two or three generations before Islamic contacts with Greek scientific, medical, and philosophical texts made a clear start. Here we may consider Jâbir ibn Ḥayyân as-Ṣûfî, in the second half of the eighth century as representing the beginning of this important activity in chemistry.

The tendency of generalizing the concept of cure so as to encompass different kinds of improvement and betterment has an interesting exemplification in the idea of elixir in the history of chemistry. Elixir is not mentioned explicitely in Hellenistic or Alexandrian alchemy. Jâbir, however, has recourse to the method of using elixirs, and he employs this concept in the sense of "curing" the "sick" metals, i.e., the deficient or imperfect ones, in order to convert them up to the status or perfection of silver and gold. But the Chinese had such a conception. They believed the base metals could be transformed into the noble ones by treating them with certain "medicines". [164] The trend of generalizing the concept of cure may therefore have originated in China. This brings to mind the probability that Jâbir received influence from Chinese chemistry.

[163] See, Aydın Sayılı, "Certain Aspects of Medical Instruction in Medieval Islam and its Influences on Europe", *Belleten* (Turkish Historical Society), vol. 45, 1981, pp. 9-21.

[164] See, Henry M. Leicester, *The Historical Background of Chemistry,* 1956, pp. 65, 67, 68. See also, Joseph Needham, *Science and Civilization in China,* vol. 5, part 2, Cambridge University Press 1974, pp. 71, 235, 236.

Another item or consideration, the stress on sal ammoniac, or, more specifically, ammonium carbonate, in the works of Jâbir, may serve to shed additional light on this question. This substance was not known to the Greeks. It was introduced into the Islamic world under the Persian name *nushâdur,* suggesting that it represents an influence on Jâbir's chemistry received from Persia or via Persia from somewhere further east.[165]

But the origin of this word is in need of some clarification. *Nushâdur* was found in Persia, Khurasan, and especially in West Turkistan. *Nushâdur* is a loan word in the Persian language. It has been supposed to be of Soghdian origin. But this suggestion leaves the ending *tur* unexplained. Its Chinese is *nao-sha,* so that the theory of Chinese origin for the word does not help to entirely clarify the question either. There is, moreover, some evidence that this Chinese term is also a foreign loan word.[166]

The Turkish word for Nishâdur is *chatur.*[167] The ending *tur* exists therefore in the Turkish name of this substance, so that the word *nishadur* apparently ows its Persian form to some influence from the Turkish language.

There is, moreover, some evidence that sal ammoniac was highly prized among the Turks of Central Asia. For it apparently figured as an item among the objects sent to the Chinese emperor as gift by an Uyghur king in the tenth century.[168]

In the year 981 a Chinese ambassador to the Uyghurs speaks of hills in the vicinity of the city of Beshbalıq, which he saw during his journey and where ammonia *(kang-sha)* (NH3) was produced. He says that smoke and flames rose from these hills and that the men who worked there wore shoes with wooden soles in order to protect their feet from heat.[169]

[165] See, Henry M. Leicester, *op. cit.,* p. 65. Nishadur as known to the chemists of Islam is of two kinds. One is ammonium carbonate, $(NH_4)_2CO_3$, an organic substance which is easily distilled. This was the substance occurring in Jâbir for the production of elixirs. The other variety of Nuşadur is sal ammoniac properly called. It is a crystalline volatile salt, its chemical formula being ammonium chloride (NH_4Cl). It was found, or prepared, near the temple of the Egyptian god Ammon. Hence the name given to it later on in Europe.

[166] Berthold Laufer, *Sino-Iranica,* pp. 503-508.

[167] See, Clauson, *An Etymological Dictionary of Pre-thirteenth Century Turkish,* p. 403.

[168] See, Laufer, *op. cit.,* p. 306.

[169] Özkan İzgi, *Çın Elçisi Wang Yen-Te'nin Uygur Seyahatnamesi,* Turkish Historical Society, Ankara 1989, pp. l, 66 and pp. 63, 64, 65, note 179.

Joseph Needham writes:

"... a medieval Persian writer of a history of China attributed the invention of chemistry to a Chinese named Hua Jen, or Changer; while (at first sight) the Persian's Chinese source regarded him as a man from the Far West. Rashîd al-Dîn al Hamdânî, in his history of China finished in 1304, speaking of the time of the High King Mu of the Chou, mentions the exploits of the legendary charioteer Tsao Fu, and then goes on to say:

"'At that time there lived a man called Ḥwâr.n (Hua Jen). He invented the science of chemistry and also understood the knowledge of poisons, so well that he could change his appearance in an instant of time. ...'

"Here there is no suggestion that Hua Jen was anything but a Chinese.

"In order to clarify Rashîd al-Dîn's source one has to know two things expounded by John Franke; ... 'The oldest of these', says Franke, 'was ...;' but the closest to Rashîd al-Dîn's history was the work of a monk named Nien-Chang. ...

"The statements of Nien-Chang about 'Changer' are as follows:

"'In king Mu's time a Changer appeared from the Furthest West. He could overturn mountains and reverse the flow of rivers, he could remove towns and cities, pass through fire and water, and pierce metal and stone – there was no end to the myriad changes and transformations (he could effect and undergo). ...'

"The story echoes familiarity. ... Its original intention had probably been to suggest that the visible world was like a dream or a magician's illusion, and Changer was certainly not a historical person, but the chemical artisans of the Middle Ages did not appreciate such fine distinctions, so it was wholly natural that Changer should have become in due course the technic deity and patron saint of the art, craft and science of chemical change.

"As for the 'Furthest West' in *Lieh Tzu* and the *Fo Tsu Li Tai Thung Tsai*, it never meant Europe or the Roman Empire, but rather the legendary land of the immortals, thought of as somewhere near Tibet or Sinkiang, where reigned the Great Queen Mother of the West, Hsi Wang Mu, nothing short of a goddess. King Mu of Chou paid her a celebrated visit, the main theme of the ancient book *Mu Thien Tzu Chuan,* and also referred to in *Lieh Tzu.* When centuries later the story came to the knowledge of real Westerners like the group around Rashîd al-Dîn all this was omitted, and they took Changer (Hua Jen) to have been a Chinese with marvellous chemical knowledge. The significant fact that early in the fourteenth century they were quite ready to do this is the only justification of these paragraphs." [170]

Joseph Needham says:

"... in contrast to the flood of Greek scientific books which poured into Arabic we do not so far know of one Chinese work which was translated into that language until a very late date. Of Persian writings there were many and of Sanskrit more than a few, but because Chinese books remained behind the ideographic-alphabetic barrier that is no reason whatever for thinking that Chinese ideas also did. Indeed, seminal concepts divested of verbiage might be all the more compelling." [171]

Speaking of the Battle of the Talas river, the same author writes, "... the Chinese were defeated but the Arabs so mauled that they could press no further. Soon afterwards, because of the rebellion of An Lu-Shan, the Chinese withdrew from the whole of Turkistan (Sinkiang) leaving a vacuum as it were between the two civilizations; and very soon afterwards al-Manṣûr was to be seem despatching (in +756) a contingent of Muslim troops to help the young emperor Su Tsung regain control after An Lu-Shan's revolt. Thus it came about that no Arab army ever crossed the Chinese border in hostility. And already a closeness of cultural contact had appeared, for many Chinese artisans taken prisoner at the Talas River setteled with their arts and crafts in Baghdad and other Arabic cities, some returning home in +762 but others (like the paper-makers and weavers) staying to exert permanent effects — very likely some workers with chemical knowledge were among them, especially as painters and gilders are mentioned. We even know their names." [172]

[170] Joseph Needham, "Contributions of China, India, and the Hellenistic-Syrian World to Arabic Alchemy", *Prizmata, Festschrift für Willy Hartner,* ed. Y. Maeyama and W. G. Saltzer, Steiner Verlag, Wiesbaden 1977, pp. 250-251.

[171] *Ibid.,* p. 250.

[172] *Ibid.,* p. 252.

This last paragraph serves well to indicate the complexity of the question of cultural relations between China and the World of Islam. But it seems to omit the place of Central Asia almost wholly, and we are much interested in this particular subject. The following additional item of knowledge bearing on the possible role played by Central Asia is therefore very welcome indeed.

A. Waley writes:

"T'ao Hung-ching (Giles, Biographical Dictionary, No 1896) who was born in 451 or 452 and died in 536, was a prolific writer on Taoist subjects, and was in later times regarded as an important alchemist. But in his existing writings there are only fleeting allusions to alchemy. There is, however, in one of his books (the Teng Chēn Yin Chuch, Wieger, No 418) an interesting reference to foreign astrology: ... 'These exotic methods (speaking of certain loose methods of determining a man's destiny by the date of his birth) are all much the same as the astronomical notions of the Hsiung-nu (Huns) and other foreign countries.' Alchemy in China as elsewhere is closely bound up with astrology, and if the Chinese were in the fifth century in contact with foreign astrology they were, it may be assumed, in a position to be influenced by foreign alchemy.

"For the centuries that follow (sixth to ninth, the period covered by the Sui and T'ang dynasties) we have plenty of anecdotes, but an almost complete lack of datable literature. It is strangely enough, in Buddhist literature (Takakusu Triptika, vol. xlvi; p. 791, column 3, Nanjia, 1576) that we find our most definite landmark. Hui-ssu (517-77), second patriarch of the T'iem-t'ai sect, prays that he may succeed in making an elixir that will keep him alive till the coming of Maitreya. ...

"The wizard Ssu-ma Chēng-chēn, who lived at an advanced age c. 720, had a great reputation as an alchemist; but his surviving works deal with other subjects. One of the few works on alchemy which may with certainty be accepted as belonging to the T'ang Dynasty is the *Shih Yar Erh Ya* (Wieger, No 894), a dictionary of alchemical terms, by a certain Mei Piao. Internal evidence, such as the mention of Ssu-ma Chēng-chēn, shows that the book is at least as late as the eighth century. I should feel rather inclined from the general tone and style, to place it in the ninth. Several obviously foreign terms are given. ... There is also a reference by an alchemical treatise called ... 'Treatise of the Hu (Central Asian) king

Yakat (Yakath or the like)'"[173] The same author continues somewhat later with the following remarks: "The Central Asian king Yakat (Yakath or the like)" to whose treatise I have already referred remains an enigma. It is probable, but not quite certain, that he proves the existence of a pre-Mohemmedan alchemy in Central Asia. As to the nationality the name does not, to my knowledge, give us any clue. He may have been Eastern Iranian (Sogdian) or Turk. But after the Arabic conquest the influence was, I believe, all from East to West. Further examination of Arabic alchemy will show, I am convinced, that it contains a vast element which it owes to China rather than to the Greek world. In particular the idea of 'philosopher's stone' as an elixir of life is a contribution of the Chinese."[174]

The variant Yakar, if such a variant is permissible, is a Turkish word, and in the Middle Ages it may have been used as a personal name.[175] Waley refers to the possibility of variants besides Yakar and Yakath, and several such variants may be said to sound like Turkish words. Yakak and Dukak, e.g., are personal names in Turkish. They are mentioned as the name of the father of Seljuq, the founder of the Seljuq Empire.[176]

This fits pretty neatly together with what we have noted about sal ammoniac and the evidence it brings to light concerning the part played by Central Asia and its Turkish elements of population, in creating the novelties Jâbir ibn Ḥayyân eṣ-Ṣûfî brought to medieval alchemy, in their own right or as an element acting as intermediary between China and the World of Islam. And we have seen that a similar situation exists in relation to the knowledge of algebra and its propagation in the World of Islam through Al-Khwârazmî and 'Abd al-Ḥamîd ibn Turk, and also with respect to cultural contact between Central Asia and China. We also know that Central Asia and India had a pretty lively cultural contact perhaps mainly through the influence of Buddhism in China.

[173] A. Waley, "Notes on Chinese Alchemy", *Bulletin of the School of Oriental Studies London Institution,* vol. 6, 1930-1932 (pp. 1-24), p. 14.

[174] *Ibid.,* pp. 23-24.

[175] See, Clauson, *An Etymological Dictionary of Pre-thirteenth Century Turkish,* pp. 896-897.

[176] See, Ṣadru'd-Dîn Abû'l-Ḥasan ʿAlî ibn Nâṣir ibn ʿAlî al-Ḥusaynî, *Akhbâru'd--Devleti's-Selchûqiyya,* ed. Muhammad Iqbâl, Lahore 1933, p. l, ed. Z. Bunyatov, Moskova 1980, facsimile, p. lb; Turkish translation by Necati Lugal, Ankara 1943, p. l; Besim Atalay, *Türk Büyükleri ve Türk Adları,* Istanbul 1935, p. 133; Mehmet Altay Köymen, *Büyük Selçuklu İmparatorluğu Tarihi,* vol. l, Ankara 1979, pp. 6-9.

Al-Beyrûnî seems to be well informed in these matters. According to Zeki Velidi Togan, Beyrûnî considered the civilized world to be composed of two major parts, the East and the West. According to him the Chinese, the Turks, and the people of India made up the Eastern civilization, and the World of Islam was a continuation of the Western civilization which was based on Greek civilization. He was of the opinion that the acceptance of the Moslem religion by the Turks brought a great expansion to the Western civilization, and this was a great gain for humanity as a whole and especially for the cause of science.[177]

Coming back to Al-Khwârazmî, we have to take here into consideration, first and foremost, his work and influence in the field of arithmetic. Unlike Al-Khwârazmî's algebra, his place in the spread of the so-called Hindu-Arabic numerals and calculation with zero and the positional or place-value numeration system seems to have its vague points in certain respects. Notwithstanding all this, however, Al-Khwârazmî's figure looms large against the horizons of the history of science in several major issues.

Al-Khwârazmî's work in the field of practical arithmetic has its controversial points. Al-Khwârazmî's book on arithmetic in its Arabic text has not come down to our time, but its Latin versian or translation is known to have played an important part in the spread, in Western Europe, of the decimal place-value system of numerals and the methods of computation with that system. This is witnessed by the fact that in Europe this system of calculation was called algorism, or algorithm, a word derived from the very name of Al-Khwârazmî. The fact that the decimal positional system of numerals was called the "Arabic numerals" in Western Europe corroborates the paramount importance of this transmission of knowledge. The term "Arabic numerals" was first used in the twelfth century by Adelard of Bath.

The earliest example of the positional system goes back to the Sumerians who lived in Mesopotamia up to four thousand years ago. This was a sexagesimal system for whole numbers as well as for fractions. But the concept of zero was explicitly integrated into the system only gradually, and even in Assyrian and Seleucid times this concept did not develop, in a formal sense, to the point of being used fully consistently. Nevertheless,

[177] Zeki Velidi Togan, "Bîrûnî", *Encyclopedia of Islam* (Turkish), vol. 2, p. 638, column 1.

the shortcomings of this sexagesimal system from the standpoint of full consistency may be said to have been small indeed when viewed within the perspective of such examples in history in its full extent.[178]

The Greeks chose the Babylonian sexagesimal system to express their fractions and patched this upon their alphabetic numeral system, and this usage was taken over by the astronomers of Islam. But for whole numbers the mathematicians of Islam used a positional decimal system and continued, together with it, the Greek Hellenistic tradition of positional sexagesimal fractions expressed in alphabetical numerals.

It is the positional decimal system, and the method of computation based upon this system, to the spread of which inside the Islamic World and in Western Europe Al-Khwârazmî is known to have largely contributed. And Al-Khwârazmî himself, as the name of his book clearly indicates, mentions India specifically as the origin of the decimal place-value system and the method of computation based upon it, which were introduced into the World of Islam by Al-Khwârazmî in particular. Very little is known concerning the exact mode of the evolution of this Indian system which involved a full use of the concept and symbol of zero and the principle of place-value numeration system.[179]

The question of the birth of the place-value system as connected with decimal numeration is far from having been completely clarified. Neugebauer believes it to have been the result of the diffusion of the Greek version of the positional sexagesimal fractions into India. That is, the Greeks adopted the Babylonian sexagesimal fractions and expressed them with their own alphabetical numerals. This influenced then the Indians. Neugebauer says, "It seems to me rather plausible to explain the decimal place-value notation as a modification of the sexagesimal place-value nota-

[178] See, George Sarton, "Decimal Systems Early and Late", *Osiris,* vol. 9, 1950, pp. 581-601; O. Neugebauer, *The Exact Sciences in Antiquity,* 1957, pp. 13, 16-17, 20, 22, 33-34.

[179] See, Solomon Gandz, "Review on Datta and Singh, History of Hindu Mathematics", *Isis,* vol. 25, 1936, pp. 478-488. See also, G.R. Kaye, "Notes on Indian Mathematics-Arithmetical Notation", *Journal of the Asiatic Society of Bengal,* vol. 3, number 7, July 1907, pp. 475-508; G.R. Kaye, "The Use of Abacus in Ancient India", *Journal of the Asiatic Society of Bengal,* vol. 4, number 6, June 1908, pp. 293-297; G.R. Kaye, "References to Indian Mathematics in Certain Medieval Works", *Journal of the Asiatic Society of Bengal,* vol. 7, number 11, December 1911, pp. 801-816; D.M. Bose, *A Concise History of Science in India,* Indian National Science Academy, 1971, pp. 172-183.

tion with which the Hindus became familiar through Hellenistic astronomy."[180]

In Egypt and Syria the Arab conquerors found a tradition of Byzantine administration of state revenues and financial matters. At first they left the established tradition more or less intact, but during the reigns of the caliphs ʿAbd al-Malik (685-705) and Walîd (705-715) the language of these public registers were changed from Greek into Arabic. The tradition of computational work and techniques seem, however, to have continued to be performed with the old Greek alphabetic numerals. Such numerals are seen to have lived for many centuries in Morocco where they were called *al-qalam al-Fâsî.* But how did they spread into the Maghrib? According to Georges S. Colin, the Greek alphabetic numerals were used extensively, and they were carried by the Arabs from Syria and Egypt into Morocco through Spain.

Colin supplies evidence to show that, in the thirteenth century, Spain was familiar with the Greek alphabetic numerals. He also points out that in the fourteenth century Ibn Khaldûn refers to the use of Greek alphabetic numerals in North Africa.[181]

Europe adopted the *ghubâr* numarals, i.e., the "Hindu-Arabic" numeral signs as they were used in Spain. But Colin says that the system of calculation based on the decimal place-value system seems to have been used in Spain only in connection with scientific work wherein complicated calculations were involved which could not be performed on the abacus or just mentally.[182]

According to Colin, very likely, the use of the abacus with columns of numeral signs led to reducing the twenty seven signs of the abacus to the nine signs of the first column, i.e., the column of ones of the abacus, and, as a consequence, these nine signs acquired a positional value. This thesis is not original with Colin. He shares it with others who advanced such a theory previously.

[180] O. Neugebauer, *The Exact Sciences in Antiquity,* Brown University Press, 1957, p. 189.

[181] See, Georges S. Colin, "De l'Origine Grecque des 'Chiffres de Fez' et de Nos Chiffres Arabes", *Journal Asiatique,* Avril-Juin 1933, pp. 193-198.

[182] *Op. cit.,* p. 209.

The nine signs of the Greek alphabet, according to this thesis, infiltrated in an early date to India, and there simplified methods of calculation with them were invented. These numerals and methods of calculation were diffused into Islam, and later on infiltrated from the Islamic realm into Western Europe. Within this process of infiltration Al-Khwârazmî seems to have played a major part in the passage of influence from Islam into Europe, although Europe adopted the *ghubâr* numeral signs of Spain which were not those used by Al-Khwârazmî.[183]

Saidan writes as follows:

"Some of the texts used in this study do not use, and some do not even seem to know, the Hindu-Arabic numerals. They express numbers in words, and for fractions they resort to the scale of sixty or other scales derived from local metrologies. Their manipulational schemes are mental and rely upon finger-reckoning. The system they expose is commonly called *ḥisâb al-yadd,* i.e., hand arithmetic; Al-Uqlîdisî calls it as well *ḥisâb al-Rûm wa al-ʿArab,* the arithmetic of the Byzantines and the Arabs. It did involve the so-called *jummal* notation, which uses the Arabic alphabet, in the Aramaic *jummal* order, to denote numbers. The notation seems to help and coexist with finger-reckoning but belongs to the scale of sixty. Manipulations of this scale are usually called: *ḥisâb al-daraj wa al-daqâiq* (the arithmetic of degrees and minutes), *ḥisâb al-zîj* (the arithmetic of astronomical tables) or *ṭarîq al-munajjimîn* (the way of astronomers). This was a complete and independent system, standing side by side with *ḥisâb al-yadd,* relying to a lesser extent on finger reckoning, and having its own multiplication tables expressed in the *jummal* notation.

"These systems expressed the arithmetical tradition obtaining in the civilized world before Islam, in service of government, everyday life and astronomical, as well as astrological, calculations. The foundation was mainly Greco-Babylonian. It was inherited by the Muslims and served their purposes before and after the advent of Hindu arithmetic. ... To pursue the mutual influence of one system upon the other is a tempting task not easy to carry out satisfactorily. Hindu arithmetic had a perfect notation and well-defined techniques that required little mental reckoning. But we shall find more concepts in common between the three systems

[183] Colin, *op. cit.*, pp. 214-215.

than we may at first expect. The task of tracing the influence of one system upon the other is made particularly difficult by the Arabic authors themselves, who laboured hard to secure a unified system better than all. Thus Al-Uqlîdisî gives us Hindî arithmetic enriched with Rûmî and Arabic devices expressed by Hindî numerals. Abû al-Wafâ and Al-Karajî present finger-reckoning combined with the scale of sixty, but even in their attempt to turn their back on Hindî devices, they prove to have borrowed from them. Kushyâr gives the scale of sixty expressed in Hindî numerals. A text called Hindî (arithmetic) extracted from Al-Kâfi attempts to present finger-reckoning expressed by Hindî numerals. ..." [184]

Neugebauer says:

"Only the purely mathematical (cuneiform) texts, which we find well represented about 1500 years after the beginning of writing, have fully utilized the great advantage of a consistent sexagesimal place-value notation. Again, 1000 years later, this method became the essential tool in the development of mathematical astronomy, whence it spread to the Greeks and then to the Hindus, who contributed the final step, namely, the use of the place-value notation also for the smaller decimal units." [185]

Again, the same author writes:

"The advantage of the Babylonian place-value system over the Egyptian additive computation with unit fractions are so obvious that the sexagesimal system was adopted for all astronomical computations not only by the Greek astronomers but also by their followers in India and by the Islamic and European astronomers. Nevertheless the sexagesimal notation is rarely applied with the strictness with which it appears in the cuneiform texts of the Seleucid period in Mesopotamia. Ptolemy, for example, uses the sexagesimal place-value system exclusively for fractions but not for integers." [186]

Gandz writes:

"The Hindu and the Ghubâr Numerals. — The modern numerals with place-value and zero are commonly known as the *Arabic* numerals,

[184] A.S. Saidan, *The Arithmetic of Al-Uqlîdisî*, D. Reidel Publishing Company, 1978, pp. 7-8.

[185] O. Neugebauer, *op. cit.*, p. 20.

[186] *Ibid.*, p. 22.

as distinguished from the Roman numerals. ... The Arabs too distinguished two different types of numerals and characterized them by two names, the *Hindu* and the *ghubâr* numerals. The Hindu numerals were common among the Eastern Arabs and are, at present, still usual in the Arabic World. The *ghubâr* numerals were found in Spain among the Western Arabs. ... It will be seen that these *ghubâr* numerals resemble our modern numerals much more closely than the Hindu numerals do, and are almost identical with the forms of the abacus numerals given in the Boethius geometry.

"The name Hindu numerals is quite clear. It simply indicates the origin and source, it acknowledges the well-established fact that the Arabs learned them from the Hindus. Much less clear, however, is the meaning of the term *ghubâr* and the origin of the *ghubâr* numerals.

"... That in last line they are to be traced back to India, like the so-called Hindu numerals ... is the common opinion. But who brought them from India to Muslim Spain, and at which time were they introduced? On this question there are two general theories. 'The first is that they were carried by the Moors to Spain in the eighth or ninth century, and thence were transmitted to Christian Europe. The second advanced by Woepcke is that they were not brought to Spain by the Moors, but that they were already in Spain when the Arabs arrived there, having reached the West through the Neo-Pythagoreans.' The facts that support Woepcke's theory are: the *ghubâr* numerals differed materially from the Hindu numerals and resembled the abacus numerals. It was customary with the Arabs to adopt the numerical system of the countries they conquered. They adopted the Greek numerals in use in Damascus and Syria, and the Coptic in Egypt, and so on entering Spain it was only natural for them to adopt the abacus numerals in use there. Whether these *ghubâr* numerals belonged to the Hindu system and reached Spain through the Neo-Pythagoreans of Alexandria as early as c. 450 A.D., as Woepcke thinks, or whether, as Bubnov's theory holds, they were derived from the ancient Roman-Greek symbols used on the abacus, it is not our purpose to discuss, or to decide." [187]

[187] Solomon Gandz, "The Origin of the Ghubâr Numerals, or the Arabian Abacus and the Articuli", *Isis*, vol. 16, 1931, pp. 393-395.

Gandz also writes:

"This again goes to corroborate the theory of Woepcke claiming that the ***ghubâr*** numerals were learned by the Arabs in Spain from the Roman abacus. As we today speak of Roman and Arabic numerals, simply indicating the origin and source, so the Arabs speak of the Hindu and ***ghubâr*** numerals, both terms only giving the origin of the numerals." [188]

Gandz writes also as follows:

"The earliest Arabic documents containing the ***ghubâr*** numerals are two manuscripts of 874 and 888 A.D. The oldest definitely dated European document known to contain these numerals is a Latin manuscript written in 976 A.D." Then, quoting Smith and Karpinsky, he adds, "That Gerbert (930-1003) and his pupils knew the ***ghubâr*** numerals is a fact no longer open to controversy. ... It is probable that Gerbert was the first to describe these numerals in any scientific way in Christian Europe, but without zero." [189]

Thus we may conclude that Western Europe apparently adopted the so-called ***ghubâr*** numerals, including a zero sign, from Moslem Spain, but, it learned the principle of the new reckoning especially from Al-Khwârazmî's book on arithmetic, since it not only had it translated into Latin but also gave the name of Al-Khwârazmî to the new method of reckoning. Of course, on the other hand, the question of the shape of the numerical signs is, essentially, of secondary importance in comparison with the principle of place-value system, supplied with a special sign for zero, and as compared to the diffusion of the new methods of the so-called Indian calculation. Moreover, very likely, the transmission of these into Spain are to be associated, to a large extent, with influences exerted by Al-Khwârazmî through his book on arithmetic.

The question in its entirety has very complex facets especially in some of its aspects pertaining to detail. For one thing, a) the question of Spain's part in the transmission of knowledge to Europe looms large in certain other ways even if Al-Khwârazmî, from Eastern Islam, was the major carrier of influence in the process involved in this special case of

[188] *Ibid.*, p. 399.
[189] *Ibid.*, p. 394.

diffusion of knowledge. b) A second major question is the exact nature and scope of the knowledge Al-Khwârazmî acquired from India, and c) A third comprehensive question concern the history and the origin of the *ghubâr* numerals as a specific theme.

Time does not as yet seem ripe to bring definitive answers to these questions. But I shall try to give a summary account of them at least in order to throw some additional light on the personality of Al-Khwârazmî and on his scientific achievement, partly in a direct manner and partly as a question of scientific perspective within which we have to appraise Al-Khwârazmî's work in the fields of arithmetic and algebra.

Let us begin with the first of these, namely Spain's part in the transmission of scientific knowledge from Islam to Western Europe. In the field of algebra the accomplishments of several mathematicians, some of whom were active in periods very close to the time of Al-Khwârazmî, were quite important and their contributions were quite weighty. One of these was Abû Kâmil Shujâᶜ ibn Aslam and another one was Al-Karajî (or Al-Karkhî, as he was called until recently). We have mentioned before ᶜAbd al-Ḥamîd ibn Turk. Yet it was mainly through the influence exerted by Al-Khwârazmî's book that the knowledge of algebra was transmitted to Europe and began to flourish there.

David Eugene Smith writes:

"Algebra at one time stood a fair chance of being called *Fakhrî*, since this was the name given to the work of Al-Karkhî (c. 1020), one of the greatest Arab mathematicians. Had this work been translated into Latin, as Al-Khwârazmî's was, the title might easily have caught the fancy of the European world." [190]

E.S. Kennedy writes:

"Bîrûnî notes the existence of a book by Al-Farghânî, a younger contemporary of Khwârazmî, criticizing the latter's zîj, and Bîrûnî himself demonstrates an error in Khwârazmî's planetary equation theory. It is curious to note that in spite of the simultaneous existence of tables based on more refined theories, this zîj was used in Spain three centuries after it had been written, and thence translated into Latin." He also says, con-

[190] D.E. Smith, *History of Mathematics*, vol. 2, The Athenaeum Press, Boston 1925, p. 388.

cerning this zîj, that "In the original Arabic the work is not extant, but Adelard of Bath's Latin translation of the revision of Maslama al-Majrîṭî (fl. 1000) has been published by Björnbo and Suter" and also that "The zîj of Muḥammad ibn Mûsâ al-Khwârazmî ... is one of the only two zîjes out of the entire lot which has been published." [191]

A. Saidan writes:

"In Western Islam, Indian mathematical thought had deeper influence. The arithmetic and astronomy of Al-Khwârazmî, with their Hindu elements were spread in Spain and North Africa, when better books in the East had already surpassed Hindu lore to the extent that Al-Bîrûnî (973-1048) found it expedient to write and translate for the Indians books on geometry and the astrolabe. It was the teaching of Western Muslims that reached Europe first and thus established the prestige of Al-Khwârazmî. ..." [192]

According to Colin, Spain served also as a region through which cultural innovations or influences in general and matters related to computational techniques in particular were transmitted into Morocco and other parts of the Maghrib. An interesting example he dwells upon on this occasion concerns Greek alphabetic numerals. He points out that Ibn Ṣabʿîn in thirteenth century Spain used to write his name in the form of Ibn O, i.e., "ibn" followed by an omikron sign. [193] As this letter standing for 70, i.e., sabʿîn, in the Greek alphabetic numerals was adopted to represent zero in the sexagesimal system used by the astronomers, this example may serve to explain how it came about that while in the decimal place-value system of Eastern Islam zero was represented by a dot, in the *ghubâr* numerals zero had the form of a circle.

Saidan speaks of Sarton's reference to late-medieval European terms *abacist* and *algorist* and writes:

"He assumes that the abacists avoided Hindu arithmetic and that the algorists, like Al-Khwârazmî, adhered to it. He thus finds that the two names were used promiscuously, as Leonardo's Hindu arithmetic was

[191] E.S. Kennedy, "Islamic Astronomical Tables", 1956, p. 128. See also, above, p. 5 and footnote 13.

[192] Saidan, *The Arithmetic of Al-Uqlîdisî*, p. 7.

[193] Georges S. Colin, "De l'Origine Grecque des 'Chiffres de Fez' et de Nos Chiffres Arabes", *Journal Asiatique*, Avril-Juin 1933, pp. 204-205.

called *Liber Abaci* while that of Beldonandi, which contains an outspoken denunciation of the Hindu pattern, was called *algorismus.* Sarton concludes that 'minds were still befogged with regard to the main issue.' We can now state that minds were not befogged, but informed; the abacists were those who used the Hindu type of arithmetic, while algorists avoided it."[194]

Saidan quotes Al-Uqlîdisî's statement, e.g., to the effect that calculators disliked being seen with the dustboard in their hands, making their hands dirty, and wished to avoid being identified with, or mistaken for, the people who earned their living by doing astrological prognostications on the streets. Strangely enough, this and certain other items of information gleaned by Saidan seem to confirm in a general way Sarton's above-quoted statement to the effect that people were not clear in distinguishing the major issues involved in the place-value system from secondary matters not pertaining to its essential virtues or characteristics. And another point is that Sarton is speaking of the late medieval times in Western Europe while Saidan's authorities and items of evidence concern the earlier Islamic Middle Ages.

Speaking of the diffusion of Hindu numerals in Western Christiandom, in the twelfth century, Sarton says:

"The use of these numerals extended gradually but very slowly. They were forbidden in Florence and Padua, and this implies that some people at least were trying to make use of them."[195]

Again, on the same subject the same author writes:

"The Hindu numerals continued their diffusion in the second half of the thirteenth century, steadily, but slowly. As we might expect, it was in Italy that they were first put to practical purposes. We know indirectly that they were already used by business people before the end of the cen-

[194] Saidan, "The Earliest Extant Arabic Arithmetic, Kitâb al-Fuṣûl fi al Ḥisab al-Hindî of Abû al-Ḥasan Aḥmad ibn Ibrahîm al-Uqlîdisî", *Isis,* vol. 57, 1966, p. 480. Saidan is seen to have later on changed his verdict on this matter. See his reference indicated below, p. 83 and note 203.

[195] George Sarton, *Introduction to the History of Science,* vol. 2, part 2, 1931, p. 747. See also, George Sarton, "The First Explanation of Decimal Fractions and Measures (1585). Together with a History of the Decimal Idea", *Isis,* vol. 23, 1935, pp. 164-166.

tury, because the bankers were forbidden in 1299 to do so. Besides, the statutes, of the University of Padua, ordered that the stationer keep a list of books for sale with the prices marked 'non per cifras sed per literas claras.'"[196]

Now, there should be practically no doubt that this new kind of arithmetic was called algorism in Europe.

On another occasion Saidan refers to the *Liber Algorismi de Numero Indorum* (The Book of Al-Khwârazmî on Indian Number), which is supposed to be a translation, by Adelard of Bath (c. 1120), of Al-Khwârazmî's book on the Indian method of calculation, lost now in its Arabic original. Saidan also speaks of *Dixit Algorismi* (So Speaks Al-Khwârazmî), of 1143, allegedly quoting the Indian arithmetic of Al-Khwârazmî.[197] There is also the *Liber Algorismi* of John of Seville, again from the first half of the twelfth century, which deals with Al-Khwârazmî's Indian method of calculation.[198]

In all these examples, the Indian method of calculation is represented by the word *algorism*, by referring to Al-Khwârazmî in person. In fact, it was suggested and shown in about the middle of the nineteenth century that this word was merely a corruption of the word Al-Khwârazmî. This is, moreover, in line with a statement of Sacrobosco, of the thirteenth century, to the effect that the word algorism was derived from the name of a scholar, and it is strongly confirmed by the above-mentioned book names such as *Dixit Algorismi* and *Liber Algorismi de Numero Indorum*.

It is to be concluded that the origin of the word was forgetten soon after the twelfth century and, in fact, many of the early Latin writers suggested various fanciful etymologies for it. D.E. Smith too refers to the loose and inconsistent manner in which this word was used, giving several examples to illustrate it.[199]

[196] *Op. cit.*, vol. 2, part 2, p. 985.

[197] See, Saidan, *The Arithmetic of Al-Uqlîdisî*, Reidel Publishing Company, 1978, p. 22.

[198] See, M.F. Woepcke, "Mémoire sur la Propagation des Chiffres Indiens", *Journal Asiatique*, series 6, vol. 1, May-June 1863, p. 519.

[199] D.E. Smith, *History of Mathematics*, vol. 2, pp. 8-11. See also, Kurt Vogel, *Die Practica des Algorismus Ratisbonensis*, S.H. Becksche Verlagsbuchhandlung, München 1954, pp. 1-9, especially 1-3.

David Eugene Smith writes:

"The Hindu Forms (of the numerals) described by Al-Khwârazmî were not used by the Arabs, however. The Baghdad scholars evidently derived their forms from some other source, possibly from Kabul in Afghanistan, where they may have been modified in transit from India." [200]

We have already spoken of the two sets of numeral forms which were used in the Islamic World, one in the East and one in the West. The one used in the East was perhaps the same as that used by Al-Khwârazmî. The Central Asian or Kabul form referred to by D.E. Smith may have been the one adopted by Al-Khwârazmî, since he was a native of that region.

It is of great interest that the numerals adopted by Europe, which are those still used today, were the same as the *ghubâr* numerals, and these numerals seem to have a very complex history which was probably quite independent from Al-Khwârazmî, although D.E. Smith's statement quoted above seems to imply the assumption that Al-Khwârazmî used numerals close in shape to that of the *ghubâr* numerals.

It is so much the more interesting therefore that the passage to Europe of methods of reckoning based on the decimal place-value system owed much to Al-Khwârazmî, as the word *algorism* testifies. Europe's adoption of the *ghubâr* numerals of Spain too obviously had a great part to play in the passage of the computation methods based on the decimal place-value system from the World of Islam to the Western Christian World.

Saidan says:

"In Western Islam, Indian mathematical thought had deeper influence. The arithmetic and astronomy of Al-Khwârazmî, with their Hindu elements, were spread in Spain and North Africa, when better books in the East had already surpassed Hindu lore. ... It was the teaching of Western Muslims that reached Europe first and thus established the prestige of Al-Khwârazmî. ..." [201]

[200] *Op. cit.*, vol. 2, p. 72. David Eugene Smith does not give his source for this statement.

[201] *The Arithmetic of Al-Uqlîdisî*, p. 7.

This generalization should not be fully correct as far as astronomy is concerned, and its veracity for arithmetic is in need of further research.

Again, Saidan says:

"... Al-Khwârazmî wrote the first Arabic work on Indian arithmetic. This is lost to us, but we have a collection of Latin texts alleged to be partial translations of it or derived from it. From these it seems that neither the numeral forms nor the manipulational schemes given by Al-Khwârazmî agree with that spread later on in Islam under the name of Indian arithmetic." [202]

Saidan also writes:

"According to this assumption, two arithmetics must be attributed to Al-Khwârazmî, the Latin texts must be presentations, or translations, of his *Kitâb al-Hisâb al-Hindî*.

"This assumption justifies the two names given to mathematicians in Europe, viz., abacists and algorists; see Sarton's (89) section 78. Both seem to have drawn from Al-Khwârazmî; the former from his Hindî arithmetic, and the letter from his *Al-Jam' wa al-Tafrîq*. "[203]

This assertion of Saidan to the effect that Al-Khwârazmî's *Al-jam' wa't-Tafrîq,* lost in its Arabic original, influenced Europe is very interesting, but in need of proof.

The question of the origin of the *ghubâr* numerals has been the subject of quite profound investigations by Woepcke, Nicholas Bubnov, and Solomon Gandz, in particular. [204]

[202] See, *op. cit.,* p. 12.

[203] See, *op. cit.,* p. 23.

[204] See, M.F. Woepcke, "Mémoire sur la Propagation des Chiffres Indiens", *Journal Asiatique,* series 6, vol. I, 1863, pp. 27-291, 442-529; for Bubnov, see, Harriet Pratt Latin, "The Origin of our Present System of Notation According to the Theories of Nicholas Bubnov", *Isis,* vol. 19, 1933, pp. 181-194; David Eugene Smith and Louis Charles Karpinski, *The Hindu-Arabic Numerals,* Ginn and Co., Boston 1911; Solomon Gandz, "The Knot in Hebrew Literature, or From the Knot to the Alphabet", *Isis,* vol. 14, 1930, pp. 189-214; S. Gandz, "The Origin of the Ghubâr Numerals, or the Arabian Abacus and the Articuli", *Isis,* vol. 16, 1931, pp. 393-424; S. Gandz, "Review on Datta and Singh: History of Hindu Mathematics", *Isis,* vol. 25, 1936, pp. 478-488; Salih Zeki, *Âthâr-i Bâqiya* (in Turkish), vol. 2, Istanbul 1329 (1913), pp. 10-102.

What is the origin of the ghubâr numerals? These numerals are the same as the apex signs, i.e., the signs marked on the abacus blocks or apices, and they are found in the *Ars Geometrica* of Boethius (480-524 A.D.), Roman encyclopaedic scholar. They have each a particular name ranging from 1 to 9 inclusive. These names are igin (1), andras (2), ormis (3), arbas (4), quimas (5), kaltis (6), zenis (7), temenias (8), selentis (9). Moreover, these names incorporate also the idea of place-value. For while they represent these values on the first column of the abacus, on the second column they represent 10, 20, 30, 40, 50, 60, 70, 80, and 90, and on the third column they represent the hundreds. The system has no zero. But zero is represented by the absence of apices on the corresponding column.

Therefore, the apex signs go beyond the idea of utilizing a separate sign for each item of the ones, tens, and the hundreds, as so on, as in the alphabetical numerals. With these signs, in accord with the decimal place-value system, merely nine signs can be utilized on the abacus to represent any number within the range of the thousands and beyond.

If, therefore, this stage of development of the idea of representing numbers had already been attained by the time of Boethius, this would be earlier than Severus Sebokt and Al-Khwârazmî. Concerning this question D.E. Smith writes:

"In certain manuscripts of Boethius there appear similar forms (similar to the *ghubâr* numerals), but these manuscripts are not earlier than the tenth century and were written at a time when it was not considered improper to modernize a text. They do not appear in the arithmetic of Boethius where we might expect to find them, if at all, but in his geometry, and their introduction breaks the continuity of the text. It therefore seems very doubtful that they were part of the original work of Boethius."[205]

Another interesting side of these apex signs, regardless of the more or less exact chronology of their origin, is that they seem to contain Ural-Altaic, Finno-Ugrian and Semitic sounding elements.

Concerning the *ghubâr* numerals Harriet Pratt Lattin writes as follows: "On etymological grounds also Bubnov denies the Hindu-Arabic origin of our numerals. In manuscripts of the eleventh century and possibly of the

[205] D.E. Smith, *History of Mathematics*, vol. 2, pp. 73-74.

end of the tenth century are found strange names for the symbols used on the abacus, i.e., *igin, andras, ormis, arbas, quimas, caltis, zenis, zemenias,* or *temenias,* words unknown to the Hindus, and meaning 1, 2, 3, 4, 5, 6, 7, 8, 9. The words for 1, 2, 3, 6, 7 and 9 belong to the languages of the peoples of Ural-Altaic origin; thus *igin* is related to Hungarian ik, ekky, and to an Ugro-Finnish dialect of Siberia, *ögy, egid;* ormis, to the Hungarian *horom, harom; kaltis* to the Turkish *alti; zenis* to the Turkish *sekiz* or *senkis* without the *"k"*, celentis (pronounced *kelentis),* to the Hungarian *kilenez.* Only the names for 4, 5 and 8 are of Semitic origin. ... Such a mixture could have occured in Mesopotamia before the Christian era, if one accepts the fact that the people there were subjected to Semitic (Babylonian) domination. If our numerals had originated in India, the names would result from a mixture of Indo-European word roots and Semitic (Arabic). Our numerals and these strange names originated in Central Asia and from there spread both to India and to Western Asia where the Greeks became acquainted with them and through the Greeks they found a place on the abacus."[206]

According to Bubnov, place-value was a feature of the abacus and was constantly employed on the abacus, but not independently of the abacus "until the thirteenth century, due to the failure of the abacist to understand the theory of the zero which they actually used in practise." He also believed that the fundamental elements going into the making of the positional system of numerals were developed by a slow process, lasting hundreds if not thousands of years, and took place among different peoples and different cultures so that special individuals cannot lay claim to their origin. Again, according to Bubnov, Boethius may have known the symbols, i.e., the apex signs, "and according to Bubnov's theory there is no reason why he should not have, but he was *not* the author of any surviving geometries circulating under his name so that conclusions as to his part in the transmission of the numerals based on their contents are worthless."[207]

Apparently Budnov did not deal with the place-value system of numerals in Islam, and nor does he deal with Al-Khwârazmî's contributions

[206] Harriet Pratt Lattin, "The Origin of our Present System of Notation According to the Theories of Nicholaus Bubnov", *Isis,* vol. 19, 1933, pp. 185-186.

[207] *Ibid.*, pp. 183, 189, 190.

to the dissemination of this numeral system in Western Europe as a result of the Twelfth Century Renaissance of Europe.

In short, however, the origin of the *ghubâr* numerals seems therefore to involve, according to Bubnov, influences coming from Ural-Altaic, Finno-Ughrian, and Semitic languages. In his opinion these names must have originated from Central Asia where such interminglings could occur. Hence, Bubnov denies a Hindu-Arabic origin for the decimal place-value system of numeration which with the passage of time came to be adopted by Western Europe. He believes the system to have originated with the Greeks and to have resulted from a transfer of the instrumental arithmetic of the abacus to writing. As to the names of the apex signs, Bubnov believed, on etymological grounds, that they originated in Central Asia, and thus we come once more face to face with Central Asia which seems of great interest with respect to intellectual developments of medieval Islam.

Gerbert (930-1004 A.D.) knew the *ghubâr* numerals, abstraction, of course, being made of the zero sign. Gandz brings the words *ʿuqûd* and *articuli* into correspondence with each other and concludes that the origin of the use of this word in the sense of series of numerals goes back to Rome, in agreement with Woepcke. Gandz concludes that Persius (34-62 A.D.), Boethius, and Alcuin (735-804) knew the *ghubâr* numeral signs with the exception of zero and that the sign of zero was added to this system as a result of Indian influence transmitted through the World of Islam.[208]

Salih Zeki[209] speculates that the *ghubâr* numerals passed from the World of Islam to Europe as a result of the contact between Hârûn al-Rashîd (786-809 A.D.) and Charlemagne and their exchange of gifts. Gandz has the following to say concerning hypotheses of this nature:

"It is true that at the time of Alcuin and his royal friend Charlemagne there were some merchants, travellers and emissaries passing back and forth between the East and West, and with such ambassadors must have gone the adventurous scholar, inspired, as Alcuin says of Archbishop Albert of York (766-780), to seek the learning of other lands. There is also a cruciform brooch in the British Museum inlaid with a piece of paste on which is the Mohammedan inscription in Kufic characters "There is

[208] Gandz, "The Origin of the Ghubâr Numerals, or the Arabian Abacus and the Articuli", p. 411.

[209] *Op. cit.*, p. 62-63.

no god but God." How did such a brooch find its way, perhaps in the time of Alcuin, to England? And if these Kufic characters reached there, why not the numeral forms as well? So ask Smith and Karpinski. Similarly, Ruska thinks only of two possibilities: either Alcuin invented the term articulus, or he learned it from the Moors. ... In the writer's opinion, however, there would be more probability for the assumption that some of these emissaries, pilgrims and scholars came in touch with the Nestorian priests of Syria, who, like Severus Sebokht, were familiar with the Hindu numerals as early as 662. ..."[210]

The question seems rather complex, and there may be truth in more than one of the several theories advanced. One thing may also be said to emerge out of this complicated situation, and this is that there was apparently much inertia to change in this matter so closely tied up with established practices. But is it possible to conclude that Al-Khwârazmî appears to emerge out of this puzzling situation as a person of outstanding foresight in appreciating the essential advantages of a decimal place-value system of numeration and as a figure of far-reaching influence not only in Islam but also in Europe in the dissemination of that system and the method of calculation based upon it?

I have already quoted Saidan saying that some of the texts studied by him do not use and some do not even seem to know the Hindu-Arabic numerals. Reproducing a gist of his statements, we have, "... To pursue the mutual influence of one system upon the other is a tempting task not easy to carry out satisfactorily. ... But we shall find more concepts in common between the three systems than we may at first expect. The task of tracing the influence of one system upon the other is made particularly difficult by the Arabic authors themselves, who laboured hard to secure a unified system better than all. ..." We also have Saidan's thesis to the effect that Al-Khwârazmî's arithmetic as a representative of Indian mathematical thought had a greater influence in Spain than in Eastern Islam. To reproduce another statement of his, we have: "... It seems that neither the numeral forms nor the manipulational systems given by Al-Khwârazmî agree with that spread later on in Islam under the name of Indian arithmetic. ..."[211]

[210] Gandz, "The Origin of the Ghubâr Numerals...", pp. 410-411.

[211] See above, pp. 74-55, note 184, pp. 79-80, note 194, p. 81, note 198, pp. 82-83, notes 201, 202. See also, Saidan, *The Arithmetic of Al-Uqlîdisî*, pp. 7-8, 12.

Saidan ignores the extra-Islamic or pre-Islamic influences upon Spain in the matter of the *ghubâr* numerals as a specific group of symbols and as a type of calculation presumably deriving from an act of making abstraction of the columns of the abacus with the exception of the first column. This manner of conceiving the *ghubâr* numerals in their past history as the tools of a certain type of calculation akin to that of Al-Khwârazmî but deprived as yet of a zero sign serves to bridge the gap between Al-Khwârazmî as a representative of Eastern Islam and the *ghubâr* numerals as distinctive of Spain.

There seems to lurk behind all this the possibility of gaining more knowledge of detail without increasing our grasp of a question as a whole, of having difficulty in seeing the wood for the trees. The manuscripts that have come down to us may possibly not represent a balanced and realistic distribution of the different tendencies and preferences. The antidote to such a situation would be to consult and assess the views of others who were in a better situation from the standpoint of gaining a well-rounded perspective of the real circumstances.

Relevant views seem to be gleanable from, e.g., Ibn al-Qiftî and Abû'l-Qâsim Ṣâᶜid al-Andalusî. Ibn al-Qiftî speaks of Al-Khwârazmî as the person who materially helped spread the Indian arithmetic, declaring that the method of calculation disseminated by Al-Khwârazmî was clearly superior and preferable to all other methods available, and, naturally, he does not distinguish Eastern and Western Islam from one another as the scenes of diffusion of this influence exerted by Al-Khwârazmî.[212]

It is of interest also that Abû'l-Qâsim Ṣâᶜid al-Andalusî, speaking of the arithmetic of the Indians, refers to it as the "ghubâr calculation" *(hisâb al-ghubâr)* and says that it was through Abû Jaᶜfar Muḥammad ibn Mûsâ al-Khwârazmî that its use became more extensive.[213] Here the reference is to the method of calculation rather than to the type of numerals. Yet, Ṣâᶜid al-Andalusî thus associates indirectly the *ghubâr* numerals also with Al-Khwârazmî, or seems to do so. This may possibly be explained by the fact that he was from Spain. This is by no means clear. But the idea that

[212] Ibn al-Qiftî, *Ta'rîkh al-Ḥukamâ,* ed. Julius Lippert, Leipzig 1903, pp. 266-267.

[213] Abû'l-Qâsim Ṣâᶜid ibn Aḥmad al-Andalusî, *Kitâb Tabaqât al-Umam,* ed. P. Louis Cheikho, Beyrut 1912, p. 14, French translation by Régis Blachere (Livre des Catégories des Nations), Paris 1935, pp. 47-48.

emerges from his statement clearly is that Ṣâᶜid al-Andulusî did not contrast the *ghubâr* numerals of Spain with the "Indian" system of calculation of Eastern Islam.

Richard Lemay writes, "In Muslim Spain, on the other hand, as G. Menéndez Pidal has pointed out, the Indian system (of arithmetic) became known as early as the ninth century. It seems to have prospered more immediately there, although in a significantly different cultural context marked by the opposition of the Spanish ᶜUmayyads to the Abbasid culture of Baghdad. Starting at least with the tenth century under the first caliph of Cordoba, Abder Rahman III. an indigenous scientific and cultural tradition flourished in al-Andalus where astronomy, astrology and mathematics in particular were intensely cultivated. In view of its potential impact upon Western Europe, as shown by the example of Gerbert in the late tenth century, al-Andalus thus becomes a more natural focus of attention for the transmission of the "Hindu" numerals to Western Europe in the Middle Ages."[214]

In the Eastern parts of Islam too the Abbasid Caliphate, the Buwayhids, Samanids, Qarakhanids, and Ghaznawids, as well as the rulers of smaller kingdoms under the jurisdiction of sovereigns such as Qabûs and the rulers of Eastern and Western Khwârazm regions, were all good patrons of science, and they encouraged scientists and scholars in their intellectual pursuits both in the fields of the secular or intellectual sciences, i.e., *al-ᶜulûm al-ᶜaqliyya* or the *awâil* sciences, and the Arabic and religions sciences, i.e., *al-ᶜulûm al-ᶜArabiyya* and *al-ᶜulûm al-naqliyya.*

Naturally, this patronage did not distinguish between different approaches to specific scientific subjects or problems, and did not distinguish between detailed epistemological concerns either. It seems necessary therefore to consider our question dealing with numerals and methods of calculation in the narrower context related to this specific topic or theme.

For example, Spain was in favor of Al-Khwârazmî's "Indian" arithmetic, and this was quite plausible and well suited to the question dealt with. But this fame of Al-Khwârazmî seems to have perhaps led to the choice of his *zîj* for the publication of a revised version, whereas there

[214] Richard Lemay, "The Hispanic Origin of our Present Numeral Terms", *Viator* (Medieval and Renaissance Studies), vol. 8, 1977, University of California Press, p. 444.

were several other *zîjs* such as that of Al-Battânî that could or should have been preferred for such a purpose.[215]

The question is well posed, however. Spain played a prominent part in the acceptance by Western Europe of the decimal positional system of numeration. For the *ghubâr* type of numeral signs belonging to Spain were adopted by Western Europe. But Al-Khwârazmî too was outstanding in this passage of influence as unmistakebly seen in the coining of the term algorithm. We are, therefore, naturally interested in the answer to the question as to why did Spain constitute a favorable environment for the passage of this influence.

The question naturally divides itself into two parts. One is the ease with which Arabic Spain adopted the "Hindu" system of numerals. The second part, or phase, concerns the passage of this system of numeration from Spain to Western Europe. In this second phase one automatically thinks of geographical proximity as a manifest reason for the passage of influence from Spain to Western Europe. But the more relevant reason would exclude the factor of geographical proximity. For in the first phase concerning Arab Spain at any rate, i.e., concerning the question as to why did Arab Spain adopt the "Hindu" numeral system of Al-Khwârazmî more readily, the factor of geographical proximity does not come into play at all. In the second phase, i.e., the adoption of these numerals by Western Europe such a factor may have come in to play a part.

In short, therefore, we are essentially interested in the answer to the question as to why did Spain constitute a favorable environment for the passage of influence from Al-Khwârazmî in the field of the place-value numeral system and the Indian type of calculation. This question is much more specific in comparison with the patronage and encouragement of scientific work and intellectual pursuits, and it can be dealt with or taken up with greater clarity of purpose. For it concerns more directly the nature of conditions prevailing in a particular place with regard to the question studied.

Such specific conditions prevailing in Spain were that arithmetical calculations in Spain depended on the abacus operated with the help of

[215] See above, pp. 4, 6 and notes 11, 12, 14, 15 and also, p. 82, note 201, pp. 78-79, note 191. As to the degree to which scientific publications of the Eastern Islamic World were available in Arab Spain, see, M.S. Khan, "Qâdî Ṣâʿid al-Andalusî's Ṭabaqât al-Umam: The First World History of Science", *Islamic Studies*, vol. 30: 4, 1991, pp. 518, 520, 524.

the nine apex signs — in the absence of a sign for zero. The Arabs of Spain must have adopted this system locally, and as a matter of fact they did, as they did in many regions of the vast Islamic realm. But why did they take the next step, i.e., why did they easily adopt Al-Khwârazmî's number system and arithmetic with much relative ease? Very relevant to this circumstance is the following quotation Gandz gives from Alcuin of York (735-804), a scholar contemporary with Charlemagne:

"We see also that the progression of numbers through the articles, being so to say, certain units, grows up to infinity by a limited number of certain forms. For the first progression of numbers is from 1 to 10, the second from ten to a hundred, and the third from a hundred to thousand. ...

"Thus even as the number six is in the order of the units, ... so also must be the number sixty ... in the order of the tens. ..." Alcuin observes here that through the repetition of these three series or forms the numbers continue to grow in an unlimited progression.[216] Gandz concludes therefrom that Alcuin shows himself to be familiar with the Hindu system.[217]

Bernelinus describes Gerbert's abacus as divided into thirty columns "of which three were reserved for fractions, while the remaining 27 were divided into groups with three columns in each. In every group the columns were marked respectively by the letters C (centum), D (decem), and S (singularis) or M (monas). Bernelinus gives the nine numerals used, which are the apices of Boethius, and then remarks that the Greek letters may be used in their place. By the use of these columns any number can be written without introducing a zero, and all operations in arithmetic can be performed in the same way as we execute ours without the columns but with the symbol for zero."[218]

With Al-Khwârazmî and the passage of the decimal positional system of numeration to Western Europe we are dealing mainly with integers to the exclusion of decimal positional fractions. It is so much the more interesting therefore that the abacus of Gerbert as described by Bernelinus is seen to be designed so as to be equipped with the potentiality of applying

216 Gandz, "The Origin of the Ghubâr Numerals", *Isis*, vol. 16, 1931, p. 408.

217 Gandz, *ibid.*, p. 409.

218 This quotation is from Florian Cajori, *A History of Mathematics*, 1931, p. 116.

the place-value principle to fractions, as well as to integers, although deprived of a zero sign.

We have just seen that, from the words quoted from Alcuin, Gandz believed one must conclude that Alcuin was familiar with the so-called Hindu-Arabic numeral system, and that is a system including a sign for zero. It is clear, however, from the passage just quoted from Cajori that the words of Alcuin quoted above from Gandz need not refer exclusively to the Hindu-Arabic numerals, including zero. They might as well refer to the *ghubâr* numeral signs used on the abacus.

It is thus seen, therefore, that Spain was in a very favorable position to appreciate and adopt Al-Khwârazmî's "Hindu" system of number. This should be of considerable importance in trying to explain why, in the words of Richard Lemay, the Indian system of numeration seems to have prospered more immediately in Spain as compared to other parts of the Muslim World. For, as we have pointed out with some detail, the positional decimal system was for a considerably long time not sufficiently appreciated and easily adopted, neither in medieval Islam and nor in Western Europe of the late Middle Ages. According to Richard Lemay, Al-Beyrûnî states that among the Indians too "the system of nine figures and their use in positional value was far from being universally practiced since it had to compete within Indian tradition with two rival systems, the sexagesimal and the letter numerals."[219]

Otto Neugebauer writes: "Only in one point is the Greek (Hellenistic) notation less consistent than the Babylonian method. In the latter all numbers were written strictly sexagesimally, regardless of whether they are integers or fractions. In Greek astronomy, however, only the fractions were written sexagesimally, whereas for integer degrees or hours the ordinary alphabetic notation remained in use for numbers from 60 onwards. In other words, the Greeks already introduced the inconsistency which is still visible in modern astronomy, where one also would write $130^{\circ}\ 17'20''$. The other inconsistency of modern astronomical notation, namely to continue beyond the seconds with decimal fractions, is a recent innovation. It is interesting to see that it took about two thousand years of migration of astronomical knowledge from Mesopotamia via Greeks, Hindus, and Arabs to arrive at a truly absurd numerical system."[220]

[219] Richard Lemay, *op. cit.*, p. 443.

[220] O. Neugebauer, *The Exact Sciences in Antiquity,* Brown University Press, 1957, p. 16-17.

It is of much interest that with the same critical approach and appraisal as that of Neugebauer, we may describe the *ghubâr* numerals "as a system in which there were nine signs which in conjunction with the abacus could express numbers in a place-value system and in which one could perform arithmetical operations consistently with any integers as well as fractions expressed on a decimal scale." But because this system did not have a sign for zero, in the absence of the abacus, these numbers could not be written down, e.g., on paper. They could only be expressed with the help of the abacus.

This reminds us of the cuneiform sexagesimal place-value system of Mesopotamia in its earlier phases when it did not have a sign for zero. The introduction of a zero sign came as a gradual development in the Mesopotamian sexagesimal place-value number system. We may set up in our minds a parallelism between this process and the case of the *ghubâr* numerals, therefore, from such a standpoint also. By such a comparison it would seem reasonable to speculate that through contact with Al-Khwârazmî's "Indian" numeral system the *ghubâr* numerals should with relative ease remedy its disadvantage resulting from the absence of a sign for zero and should without much difficulty adopt the zero sign.

We have seen in our quotation from Shigeru Nakayama that the Chinese were not alien to the decimal fractions either; or, rather, that their use of the positional decimal fractions increased as a result of their adoption of the Futian calendar, i.e., as a result of contact with Central Asia.[221]

With Al-Khwârazmî and the passage of the decimal positional system to Western Europe, we are dealing mainly with integers to the exclusion of decimal fractions. It is so much the more interesting therefore, as pointed out above, that the abacus of Gerbert as described by Bernelinus is seen to be designed so as to be equipped with the possibility of applying the place-value principle to fractions as well as to integers. We learn from A.S. Saidan that Al-Uqlîdisî (fl. ca. 952) was familiar with decimal fractions, and Al-Uqlîdisî is the author of the earliest book of medieval Islam on arithmetic, the Arabic text of which has come down to our day. It is possible therefore that decimal fractions were not entirely unknown to Al-Khwârazmî.

221 See above, p. 50 and note 129. See also, Ronan, pp. 37-38.

A.S. Saidan, relying on Joseph Needham, says that the Chinese mathematicians of the third century A.D. may be considered the inventors of decimal fractions and adds that it can be safely said that the first mathematician "so far known" to have used decimal fractions in the Middle East is Al-Uqlîdisî of the tenth century.[222] The life times of Al-Khwârazmî and Al-Uqlîdisî were separated by about five generations, assuming that generations are renewed every twenty-five years, so that Al-Uqlîdisî's father could have known Al-Khwârazmî in person.

According to Zeki Velidi Togan, a truly outstanding scholar not only in the fields of Turkish medieval Islam and Central Asia but also a foremost contributor to our knowledge of Al-Beyrûnî, Al-Beyrûnî considered the civilized world to be composed of two major parts, the East and the West. The Chinese, the Turks, and the people of India made up the East in his classification, and the World of Islam was a continuation of the Western civilization which was based on the classical Greek civilization. According to Zeki Velidi Togan, Al-Beyrûnî believed that the acceptance of the Moslem religion by the Turks caused a considerable expansion of the Western civilization, and that this constituted a great gain for humanity as a whole and especially for the cause of science.[223]

As we have seen, such examples as Jâbir's in chemistry, the propagation of the art of making rag paper, and the algebra of second degree equations corroborate Al-Beyrûnî's assertion that generally the Chinese and Turkish cultures and civilizations were somewhat tied up and related to each other. A similar situation may therefore have existed in number theory and arithmetic. As we have seen, moreover, Central Asia, and more particularly some Turkish elements of its population seem to have given somekind of impetus to China in the use of decimal fractions. Now, as the abacus used with the *ghubâr* numerals may be considered as having offered access to the use of decimal fractions, this may be interpreted as constituting a clue or an item of evidence in favor of Bubnov's contention, or suggestion, that the *ghubâr* numerals must have originated in Central Asia.

The question of the use of decimal fractions in China may possibly have an explanation connected with China's cultural relations with India

[222] A. S. Saidan, *The Arithmetic of Al-Uqlīdisī,* pp. 485, 486.

[223] Zeki Velidi Togan, "Bîrûnî", *Encyclopedia of Islam* (Turkish), vol. 2, 1949, p. 638.

directly or through Central Asia. Central Asia too may possibly come somewhat into the foreground in this regard. I have on an earlier occasion referred to a statement of D.E. Smith to the effect that numeral signs used by Baghdad scholars, and Arabs in general, were not the same as the signs described by Al-Khwârazmî and that they were probably derived from those used in pre-Islamic Afghanistan.[224] This is a rather vague statement. It may, nevertheless, by association of ideas, bring to our mind Bubnov's contention that the *ghubâr* numeral signs must have originated in Central Asia.

All this may also possibly tend to lead to some suggestions as to the nature of the "Indian" origin of Al-Khwârazmî's arithmetic, partly affecting our picture of the influence brought by Manka or Hanka of "India" to Baghdad during the reign of the Abbasid caliph Al-Manṣûr (754-775), or, at an earlier date (c. 650 A.D.) via the Nestorian Severus Sebokt.[225]

Neither Khwârazm, the home of Al-Khwârazmî, nor Khuttal and Gilân, or Jîlân, one of which must have been the birth place of ʿAbdu'l-Ḥamîd ibn Turk, is in North India, or, in the southern extention of Central Asia. They are both in Central Asia, more properly speaking. On the other hand, our sources tell us that ʿAbdu'l-Ḥamîd ibn Turk too, like Al-Khwârazmî, was the author of books on arithmetic. And they both belong, presumably at least, to the initial phases of the dissemination of the "Indian arithmetic" in the World of Islam.

ʿAbd al-Ḥamîd ibn Turk is said to have been the author of books on arithmetic,[226] three of them mentioned by name, and Al-Khwârazmî was the author of one, or, perhaps, of two books in this field.[227] Our source on the information concerning Ibn Turk's publications in the field of arithmetic leaves the impression that he was the earlier writer, as compared to Al-Khwârazmî and it is likely that his arithmetic also was of the Indian type.

[224] See, above, p. 82 and note 200.

[225] The words India and Indian are written in most of these passages within quotation marks in order to remind the reader that these words as used in the sources may be referring to Northern India and that "Northern India" may be taken to mean, more specifically, the southern extention of Central Asia.

[226] See, above, p. 17 and notes 51, 52, 53.

[227] See, above, p. 83 and note 203.

Hanka or Manka may therefore not be sufficient to bring to light the sources of ʿAbdu'l-Ḥamîd ibn Turk and Al-Khwârazmî in their knowledge of arithmetic; i.e., he may have not constituted the sole source of the knowledge of these two mathematicians in the field of arithmetic. Just as in the field of algebra, in the field of arithmetic too, ʿAbd al-Ḥamîd ibn Turk and Al-Khwârazmî may have possibly been indebted for at least part of their knowledge of arithmetic to their homeland in Central Asia.

I have spoken of decimal fractions as a topic which may constitute an item of evidence in favor of Bubnov's thesis to the effect that Central Asia may have been the source and origin of the *ghubâr* numeral signs. This contention of Bubnov's which rests on etymological considerations cannot be changed by replacing the term Central Asia by the word China. And the subject of decimal fractions is not very clearly known. Thus, a claim that the subject of decimal fractions helps increase the possibility of the veracity of Bubnov's thesis is not very convincing, and Bubnov's thesis stands in need of much more concrete verification.

Moreover, we should not exaggerate the importance of decimal fractions as an indirect evidence in support of the etymologically reasonable Bubnov thesis. For one thing the use of decimal fractions is not a sufficiently well-attested feature of the *ghubâr* numeral signs used in conjunction with the abacus either.

Joseph Needham says: "Place-value could and did exist without any symbol for zero, as in China from the late Chou (i.e., before the third century B.C.) onwards. But the zero symbol, as part of the numeral system, never existed, and could not have come into being, without place-value. It seems to be established that place-value was known to, and used by, the authors of the *Paulisa Siddhanta* in the early years of the +5th century, and certainly by the time of Aryabhata and Vraha-Mihira (c.+500). And this was the decimal place-value of earlier China, not the sexagesimal place-value of earlier Babylonia. It may be very significant that the older literary Indian references simply use the word *sûnya,* "emptiness", just as if they were describing the empty spaces in Chinese counting-boards."[228]

[228] Joseph Needham, *Science and Civilization in China,* vol. 3, Cambridge University Press, 1959, pp. 10-11 (note k). The Chou Dynasty period referred to above extends between -10th and the -3rd centuries. See, Joseph Needham, *ibid.,* p. 5.

Again, Joseph Needham writes: "In general therefore, it will be seen that the Shang numeral system was more advanced and scientific than the contemporary scripts of Old Babylonia and Egypt. ... All three systems agreed in that a new cycle of signs began at 10 and each of its powers. With one exception already noted, the Chinese repeated all the original nine numerals with the addition of a place-value component, which *was not itself a numeral.* The Old Babylonian system, however, was mainly addivite or cumulative, below 200, like the later Roman; and both employed subtractive devices; ... . Only in the sexagesimal notation of the astronomers, where the principle of place-value applied, was there better consistency, though even then special signs were used for such numbers as 3600, and the subtractive element was not excluded. Moreover, numbers less than 60 were expressed by 'pile-up' signs. The ancient Egyptians followed a cumulative system, with some multiplicative usages. It seems therefore that the Shang Chinese were the first to be able to express any desired number, however large, with no more than nine numerals. The subtractive principle of forming numerals was never used by them."[229]

I have dwelt at some lenght on the Chinese numerals in order to explore or examine the possibility of the Central Asian origin of the nine *ghubâr* numerals on the hypothesis of influence received by Central Asia from China especially because of Joseph Needham's statement just quoted to the effect that the Shang Chinese were able to express any decimal number, however large, with no more than nine numerals, and likewise Colin A. Ronan's assertion that "only the Shang Chinese were able to express any number, however large, using no more than nine numerals and a counting board."[230] These two statements can be applied to the *ghubâr* numerals without changing the wording, with this exception that in Colin A. Ronan's sentence the term "counting board" will have to be replaced by "calculating board", or the word "abacus", with some reservations with regard to technical detail. For, in connection with the *ghubâr* numerals for the sake of clarity we may specify the abacus as the abacus as described by Bernelinus.

I have dwelt on the Chinese numerals, as I have just said, because of the statements of Joseph Needham and Colin A. Ronan, in particular.

[229] Joseph Needham, *ibid.*, vol. 3, pp. 13-15.

[230] Colin A. Ronan, *The Shorter Science and Civilization in China:* 2, p. 5.

But I have decided that these statements are somewhat misleading perhaps because of an exaggerated importance attributed to the idea of "piled-up signs" and to the idea of "place-value components", neither of which concern the inherent characteristics essential to the concept of the place-value numeral system.

For the sake of brevity and simplicity we may have recourse to a mathematical definition of the place-value notation based only on the essential aspects or features of the system. The system may be decimal or sexagesimal, or based on some other convenient number. If decimal, then it is in need of ten signs, if sexagesimal in need of sixty signs, including zero in each case. The number signs, or symbols, may be plain, or simple, as in our present day decimal system, or based on a piling-up process of constituent elemental parts as in the old Mesopotamian sexagesimal system. In a sexagesimal system sixty independent elemental signs would make the system a bit unwieldy, so that the "piling-up" process could help making the system less cumbersome.

In the "mathematical" definition of the place-value system, a basic number sign such as three in a decimal place-value system such as ours has the place-value $3=3\times10^n$ where $n= \dots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$, and in the Mesopotamian sexagesimal system, a basic numeral sign such as eleven has the place-value  $=11\times60^n$ where $n= \dots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$, n representing the rank or order a special integer, or numeral sign, occupies.

Now, Al-Khwârazmî's decimal system had a zero sign, but so far as the value of n of our formula is concerned, it did not run through negative values. The *ghubâr* numerals used on the abacus did not have a zero sign, and as we have conjectured, it may have been used for values $n<0$, but it had only nine signs and was in need of a zero sign in order to be properly classified as a place-value system. This shows very clearly how close it was to the status of being a place-value system properly speaking. But the Old Egyptian or the Roman, Ionian, and the Phoenician numeral systems, e.g., and so far as I understand it, the Chinese numeral systems, cannot be fitted into our mathematical definition of a place-value system, or, at least not into a pattern closely similar to the *ghubâr* numerals used with an abacus resembling that of Gerbert.

The hypothetical Central Asian numeral system that constituted the origin of the *ghubâr* numbers without a zero sign does not thus seem to be confirmable or supportable on the basis of influences traceable to Chinese number systems. And the same may be said concerning its possible relations with the numeral systems of India.

This brief survey based on, or centering upon, our "mathematical" definition of the place-value number system should be of help to us by once more indicating clearly what a great advantage Spain had for transforming its numeral system into a place-value system. Indeed, this was to be done in the presence of a ready model, and the only change to be brought about was the adoption of its scheme of using a special additional zero sign.

We have tried to see if any features similar to the *ghubâr* numerals can be discovered in Chinese numerals, thinking that this may be construed as confirming the existence of an affinity or kinship between Central Asian numerals and the *ghubâr* number system. And we have failed to discover such similarities. But this does not of course mean that Central Asia cannot constitute the origin of the *ghubâr* numerals at all. For not every cultural trait of Central Asia has to be akin to that of China. For instance, the Turkish runic alphabet and the Chinese script were basically different from each other. So, we cannot infer that such a number system did not exist in Central Asia.

The problem remains, therefore, that it is difficult not to take Bubnov's theory of Central Asian origin for the *ghubâr* numerals seriously. For, with the sole exception of the country of the Khazars, i.e., Caucasia, it is virtually impossible to think of any region, or country, which could have given rise to the names of "Boethius' apexes", and the Khazars may be considered to have much in common with the authoctonous peoples of Central Asia.

One other possible candidate for the country of origin for the names of the apex signs used in a vague manner, is, it may be conjectured, Mesopotamia, as mentioned by Bubnov himself.[231] This requires, however, a chronology which is much too early for the *ghubâr* numerals, and with such early dates the etymological basis of the argument would loose much of its force.

[231] See, Harriet Pratt Lattin, *op. cit.*, pp. 185-186, 189, 190.

The "Central Asia" of Bubnov should conform, moreover, to a Central Asia either peripheral to "Islamic Central Asia" or it should refer to a Central Asia where the Arabic language was not the sole dominent cultural tongue. This geography would of course show some variation depending on chronology.