



Almost-s-Menger ditopological texture spaces

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Abstract

The aim of this article is to introduce and characterize almost-s-Menger and almost-co-s-Menger selection properties in ditopological texture spaces. We prove that every s-Menger ditopological texture space is almost-s-Menger and every s-compact ditopological texture space is almost-s-Menger, and give an example which shows that the converse is not true in general.

Mathematics Subject Classification (2010). 54D20, 54A05, 54C08, 54H12

Keywords. almost-s-Menger, selection principles, ditopological texture space

1. Introduction

Recently many mathematicians have worked on weaker forms of the selection principles in topological spaces. In 1999 Kočinac in [14] introduced almost-Menger and almost-Rothberger spaces. Kocev after systematical studying of this property in [13] give the characterization in term of regular open sets and almost continuous mappings. In 2016 Sabah et. al in [23] defined and studied almost-s-Menger and almost-s-Rothberger spaces. Also from different points of view some weaker forms of these properties in topological spaces were investigated in [1, 2, 10, 15–17, 22, 24].

Ditopological texture spaces were introduced by L. M. Brown as a natural extension of representation of lattice-valued topologies by bitopologies. The concept of ditopology is more general than topology, bitopology and fuzzy topology. An adequate introduction to the theory of texture spaces and ditopological texture spaces may be obtained from [4–9]. For a study of selection principles in ditopological texture spaces see [18, 19].

We will define and extend the idea of almost-s-Menger and almost-s-Rothberger spaces in the setting of ditopological texture spaces.

2. Preliminaries

We now recall some important and basic definitions used in sequel as follow:

Texture space: [4] If S is a set, a texturing $\mathfrak{S} \subseteq P(S)$ is complete, point separating, completely distributive lattice containing S and \emptyset , and, for which finite join \bigvee coincides with union \bigcup and arbitrary meet \bigwedge coincides with intersection \bigcap . Then the pair (S, \mathfrak{S}) is called texture space.

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Received: 14.05.2018; Accepted: 15.11.2018

A mapping $\sigma : \mathfrak{S} \rightarrow \mathfrak{S}$ satisfying $\sigma^2(A) = A$, for each $A \in \mathfrak{S}$ and $A \subseteq B$ implies $\sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \mathfrak{S}$ is called a complementation on (S, \mathfrak{S}) and $(S, \mathfrak{S}, \sigma)$ is then said to be a complemented texture [4]. The sets $P_s = \bigcap \{A \in \mathfrak{S} \mid s \in A\}$ and $Q_s = \bigvee \{P_t \mid t \in S, s \notin P_t\}$ defines conveniently most of the properties of the texture space and are known as *p-sets* and *q-sets* respectively.

For $A \in \mathfrak{S}$ the core A^b of A is defined by $A^b = \{s \in S \mid A \not\subseteq Q_s\}$. The set A^b does not necessarily belong to \mathfrak{S} .

If $(S, P(S)), (\mathcal{L}, \mathfrak{S}_2)$ are textures, then the product texture of $(S, P(S))$ and $(\mathcal{L}, \mathfrak{S}_2)$ is $P(S) \otimes \mathfrak{S}_2$ for which $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ denotes the *p-sets* and *q-sets* respectively. For $s \in S, t \in \mathcal{L}$ we have p-sets and q-sets in the product space as following:

$$\overline{P}_{(s,t)} = \{s\} \times P_t$$

$$\overline{Q}_{(s,t)} = (S \setminus \{s\} \times T) \cup (S \times Q_t).$$

Definition 2.1 ([6]). Let $(S, \mathfrak{S}_1), (\mathcal{L}, \mathfrak{S}_2)$ be textures. Then for $r \in \mathcal{P}(S) \otimes \mathfrak{S}_2$ satisfying:

$$(R_1) \ r \not\subseteq \overline{Q}_{(s,t)} \text{ and } P_s \not\subseteq Q_s \text{ implies } r \not\subseteq \overline{Q}_{(s,t)},$$

$$(R_2) \ r \not\subseteq \overline{Q}_{(s,t)} \text{ then there is } s \in S \text{ such that } P_s \not\subseteq Q_s \text{ and } r \not\subseteq \overline{Q}_{(s,t)},$$

is called *relation* and for $R \in \mathcal{P}(S) \otimes \mathfrak{S}_2$ such that

$$(CR_1) \ \overline{P}_{(s,t)} \not\subseteq R \text{ and } P_s \not\subseteq Q_s \text{ implies } \overline{P}_{(s,t)} \not\subseteq R,$$

$$(CR_2) \ \text{If } \overline{P}_{(s,t)} \not\subseteq R \text{ then there exists } s \in S \text{ such that } P_s \not\subseteq Q_s \text{ and } \overline{P}_{(s,t)} \not\subseteq R,$$

is called a *corelation* from $(S, P(S))$ to $(\mathcal{L}, \mathfrak{S}_2)$. The pair (r, R) together is a *direlation* from (S, \mathfrak{S}_1) to $(\mathcal{L}, \mathfrak{S}_2)$.

Lemma 2.2 ([6]). Let (r, R) be a *direlation* from (S, \mathfrak{S}_1) to $(\mathcal{L}, \mathfrak{S}_2)$, J be an index set, $A_j \in \mathfrak{S}_1, \forall j \in J$ and $B_j \in \mathfrak{S}_2, \forall j \in J$. Then:

$$(1) \ r \leftarrow \left(\bigcap_{j \in J} B_j \right) = \bigcap_{j \in J} r \leftarrow B_j \text{ and } R \rightarrow \left(\bigcap_{j \in J} A_j \right) = \bigcap_{j \in J} R \rightarrow A_j,$$

$$(2) \ r \rightarrow \left(\bigvee_{j \in J} A_j \right) = \bigvee_{j \in J} r \rightarrow A_j \text{ and } R \leftarrow \left(\bigvee_{j \in J} B_j \right) = \bigvee_{j \in J} R \leftarrow B_j.$$

Definition 2.3. Let (f, F) be a *direlation* from (S, \mathfrak{S}_1) to $(\mathcal{L}, \mathfrak{S}_2)$. Then $(f, F) : (S, \mathfrak{S}_1) \rightarrow (\mathcal{L}, \mathfrak{S}_2)$ is a *difunction* if it satisfies the following two conditions:

$$(DF1) \ \text{For } s, s' \in S, P_s \not\subseteq Q_{s'} \implies t \in \mathcal{L} \text{ with } f \not\subseteq \overline{Q}_{(s,t)} \text{ and } \overline{P}_{(s',t)} \not\subseteq F.$$

$$(DF2) \ \text{For } t, t' \in \mathcal{L} \text{ and } s \in S, f \not\subseteq \overline{Q}_{(s,t)} \text{ and } \overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t.$$

Definition 2.4 ([6]). Let $(f, F) : (S, \mathfrak{S}_1) \rightarrow (\mathcal{L}, \mathfrak{S}_2)$ be a *difunction*. For $A \in \mathfrak{S}_1$, the image $f \rightarrow (A)$ and coimage $F \leftarrow (A)$ are defined as:

$$f \rightarrow (A) = \bigcap \{Q_t : \forall s, f \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\},$$

$$F \leftarrow (A) = \bigvee \{P_t : \forall s, \overline{P}_{(s,t)} \not\subseteq F \implies P_s \subseteq A\},$$

and for $B \in \mathfrak{S}_2$, the inverse image $f \leftarrow (B)$ and inverse coimage $F \leftarrow (B)$ are defined as:

$$f \leftarrow (B) = \bigvee \{P_s : \forall t, f \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\},$$

$$F \leftarrow (B) = \bigcap \{Q_s : \forall t, \overline{P}_{(s,t)} \not\subseteq F \implies B \subseteq Q_t\}.$$

For a *difunction*, the inverse image and the inverse co-image are equal, but the image and co-image are usually not.

Lemma 2.5 ([6]). For a direlation (f, F) from (S, \mathfrak{S}_1) to (T, \mathfrak{S}_2) the following are equivalent:

- (1) (f, F) is a difunction.
- (2) The following inclusion holds
 - (a) $f \leftarrow (F \rightarrow(A)) \subseteq A \subseteq F \leftarrow (f \rightarrow(A)); \forall A \in \mathfrak{S}_1$, and
 - (b) $f \rightarrow (F \leftarrow(B)) \subseteq B \subseteq F \rightarrow (f \leftarrow(B)); \forall B \in \mathfrak{S}_2$
- (3) $f \leftarrow(B) = F \leftarrow(B); \forall B \in \mathfrak{S}_2$.

Definition 2.6 ([6]). Let $(f, F) : (S, \mathfrak{S}_1) \rightarrow (\mathcal{L}, \mathfrak{S}_2)$ be a difunction. Then (f, F) is called surjective if it satisfies the condition:

(SUR) For $t, t' \in \mathcal{L}, P_t \not\subseteq Q_{t'} \implies \exists s \in S, f \not\subseteq Q_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Similarly, (f, F) is called injective if it satisfies the condition

(INJ) For $s, \acute{s} \in S$, and $t \in \mathcal{L}$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(\acute{s},t)} \not\subseteq F \implies P_s \not\subseteq Q_{\acute{s}}$.

We now recall the notion of ditopology on texture spaces.

Definition 2.7 ([4]). A pair (τ, κ) of subsets of \mathfrak{S} is said to be a ditopology on a texture space (S, \mathfrak{S}) , if $\tau \subseteq \mathfrak{S}$ satisfies:

- (1) $S, \emptyset \in \tau$.
- (2) $G_1, G_2 \in \tau$ implies $G_1 \cap G_2 \in \tau$ and
- (3) $G_\alpha \in \tau, \alpha \in I$ implies $\bigvee_\alpha G_\alpha \in \tau$,

and $\kappa \subseteq \mathfrak{S}$ satisfies

- (1) $S, \emptyset \in \kappa$.
- (2) $F_1, F_2 \in \kappa$ implies $F_1 \cup F_2 \in \kappa$ and
- (3) $F_\alpha \in \kappa, \alpha \in I$ implies $\bigcap F_\alpha \in \kappa$.

Where the members of τ are called open sets and members of κ are closed sets. Also τ is called topology, κ is called cotopology and (τ, κ) is called ditopology. If (τ, κ) is a ditopology on (S, \mathfrak{S}) then $(S, \mathfrak{S}, \tau, \kappa)$ is called ditopological texture space.

The idea of semi-open sets in topological spaces was first introduced by Norman Levine in 1963 in [21]. Ş Dost extended this concept of semi-open sets from topological spaces to ditopological texture spaces in 2012 in [11]. Let $] [$ denote the interior and let $] F [$ denote the closure operator.

It is known in [11] that in a ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$:

- (1) $A \in \mathfrak{S}$ is semi-open if and only if there exists a set $G \in O(S)$ such that $G \subseteq A \subseteq [G]$.
- (2) $B \in \mathfrak{S}$ is semi-closed if and only if there exists a set $F \in C(S)$ such that $]F[\subseteq B \subseteq F$.
- (3) $O(S) \subseteq SO(S)$ and $C(S) \subseteq SC(S)$. The collection of all semi-open (resp. semi-closed) sets in \mathfrak{S} is denoted by $SO(S, \mathfrak{S}, \tau, \kappa)$ or simply $SO(S)$ (resp. $SC(S, \mathfrak{S}, \tau, \kappa)$ or simply $SC(S)$). $SR(S)$ is the collection of all the semi-regular sets in S . A set A is semi-regular if A is semi-open as well as semi-closed in S .
- (4) Arbitrary join of semi-open sets is semi-open.
- (5) Arbitrary intersection of semi-closed sets is semi-closed.

If A is semi-open in ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ then its complement may not be semi-closed. Every open set is semi-open, whereas a semi-open set may not be open. The intersection of two semi-open sets may not be semi-open, but intersection of an open set and a semi-open set is always semi-open.

In general there is no connection between the semi-open and semi-closed sets, but in case of complemented ditopological texture space $(S, \mathfrak{S}, \sigma, \tau, \kappa)$, $A \in \mathfrak{S}$ is semi-open if and only if $\sigma(A)$ is semi-closed. If $()_\circ$ denotes the semi-interior and $\underline{()}$ denotes the semi-closure, then we have the following definition.

Definition 2.8 ([20]). Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathfrak{S}$. We define:

- (i) The semi-closure \underline{A} of A under (τ, κ) by

$$\underline{A} = \bigcap \{B : B \in SC(S), \text{ and } A \subseteq B\}$$

- (ii) The semi-interior $(A)_\circ$ of A under (τ, κ) by

$$(A)_\circ = \bigvee \{B : B \in SO(S), \text{ and } B \subseteq A\}.$$

Definition 2.9 ([20]). Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space. A set $A \in \mathfrak{S}$ is called

- (a) semi-open if and only if $A \subseteq]A[$
- (b) semi-closed if and only if $]\!]A] \subseteq A$.

Lemma 2.10 ([20]). Let $(S, \mathfrak{S}, \sigma, \tau, \kappa)$ be a complemented ditopological texture space and $A \in \mathfrak{S}$. Then

- (1) $\sigma(\underline{A}) = (\sigma(A))_\circ$.
- (2) $\sigma(A)_\circ = \underline{\sigma(A)}$.
- (3) $(A)_\circ = \sigma(\underline{\sigma(A)})$.

For terms not defined here, the reader is referred to see [6, 9, 11, 16].

Definition 2.11. A difunction $(f, F) : (S, \mathfrak{S}_1, \tau_S, \kappa_S) \rightarrow (T, \mathfrak{S}_2, \tau_T, \kappa_T)$ is:

- (i) continuous [6]; if $F^{\leftarrow}(G) \in \tau_S$ where $G \in \tau_T$;
- (ii) cocontinuous [6]; if $f^{\leftarrow}(K) \in \kappa_S$ where $K \in \kappa_T$;
- (iii) bicontinuous [6]; if it is continuous and cocontinuous.

Corollary 2.12 ([3]). Let (f, F) be a difunction on (S_1, \mathfrak{S}_1) to (S_2, \mathfrak{S}_2) :

(1) If (f, F) is surjective then $F(f^{\leftarrow}(B)) = B = f(F^{\leftarrow}(B))$ for all $B \in \mathfrak{S}_2$, in particular

- (i) $F(A) \subseteq f(A), \forall A \in \mathfrak{S}_1$, and
- (ii) $\forall B_1, B_2 \in \mathfrak{S}_2, f^{\leftarrow}(B_1) \subseteq F^{\leftarrow}(B_2) \implies B_1 \subseteq B_2$.

(2) If (f, F) is surjective then $F^{\leftarrow}(f(A)) = A = f^{\leftarrow}(F(A))$ for all $A \in \mathfrak{S}_1$, in particular

- (i) $f(A) \subseteq F(A), \forall A \in \mathfrak{S}_1$, and
- (ii) $\forall A_1, A_2 \in \mathfrak{S}_1, F(A_1) \subseteq f(A_2) \implies A_1 \subseteq A_2$.

Lemma 2.13 ([12]). Let $(f, F) : (S_1, \mathfrak{S}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be a difunction between the ditopological texture spaces. Then the following are equivalent.

- (i) (f, F) is cocontinuous.
- (ii) $A \in \mathfrak{S}_1 \implies f^{\rightarrow}[A] \subseteq [f^{\rightarrow}(A)]$.
- (iii) $B \in \mathfrak{S}_2 \implies [f^{\leftarrow}(B)] \subseteq f^{\leftarrow}([B])$.

Lemma 2.14 ([12]). Let $(f, F) : (S_1, \mathfrak{S}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be a difunction between the ditopological texture spaces. Then the following are equivalent.

- (i) (f, F) is continuous.
- (ii) $A \in \mathfrak{S}_1 \implies]F^{\rightarrow}(A)[\subseteq F^{\rightarrow}(]A[)$.
- (iii) $B \in \mathfrak{S}_2 \implies F^{\leftarrow}(]B]) \subseteq]F^{\leftarrow}(B)[$.

Definition 2.15. Let (τ, κ) be a ditopology on texture space (S, \mathfrak{S}) and $A \in \mathfrak{S}$. The family $\{G_i : i \in I\}$ is said to be semi-open cover of A if $G_i \in SO(S), i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$.

Similarly a semi-closed cocover of A is a family $\{\mathcal{F}_i : i \in I\}$ with $\mathcal{F}_i \in SC(S)$ satisfying $\bigcap_{i \in I} \mathcal{F}_i \subseteq A$.

Definition 2.16 ([25]). Let $(S_i, \mathfrak{S}_i, \tau_i, \kappa_i), i = 1, 2$ be ditopological texture spaces. A difunction $(f, F) : (S_1, \mathfrak{S}_1) \rightarrow (S_2, \mathfrak{S}_2)$ is said to be

- (i) semi-continuous (resp. MS-continuous) if $F^{\leftarrow}(A) \in SO(S_1) \forall A \in O(S_2)$ (resp. $\forall A \in SO(S_2)$).
- (iii) semi-cocontinuous (resp. MS-cocontinuous) if $f^{\leftarrow}(A) \in SC(S_1) \forall A \in C(S_2)$ (resp. $\forall A \in SC(S_2)$).
- (iii) semi-bicontinuous (resp. MS-bicontinuous) if it semi-continuous and semi-cocontinuous (resp. MS-continuous and MS-cocontinuous).

Lemma 2.17 ([11]). Let $(f, F) : (S_1, \mathfrak{S}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be a difunction.

1. The following are equivalent.
 - (a) (f, F) is semi-irresolute.
 - (b) $(F^{\rightarrow}(A))_{\circ} \subseteq F^{\rightarrow}(A)_{\circ} \forall A \in \mathfrak{S}_1$.
 - (c) $f^{\leftarrow}(B)_{\circ} \subseteq (f^{\leftarrow}(B))_{\circ} \forall B \in \mathfrak{S}_2$.
2. The following are equivalent.
 - (a) (f, F) is semi-co-irresolute.
 - (b) $f^{\rightarrow}(\underline{A}) \subseteq (f^{\rightarrow}(A)) \forall A \in \mathfrak{S}_1$.
 - (c) $(F^{\leftarrow}(B)) \subseteq \overline{F^{\leftarrow}(B)} \forall B \in \mathfrak{S}_2$.

Lemma 2.18 ([11]). Let $(f, F) : (S_1, \mathfrak{S}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be a difunction.

1. The following are equivalent.
 - (a) (f, F) is semi-continuous.
 - (b) $]F^{\rightarrow}(A)[\subseteq F^{\rightarrow}(A)_{\circ} \forall A \in \mathfrak{S}_1$.
 - (c) $f^{\leftarrow}(]B[) \subseteq (f^{\leftarrow}(B))_{\circ} \forall B \in \mathfrak{S}_2$.
2. The following are equivalent.
 - (a) (f, F) is semi-cocontinuous.
 - (b) $f^{\rightarrow}(\underline{A}) \subseteq [f^{\rightarrow}(A)] \forall A \in \mathfrak{S}_1$.
 - (c) $(F^{\leftarrow}(B)) \subseteq F^{\leftarrow}[B] \forall B \in \mathfrak{S}_2$.

Definition 2.19. Let $(S_i, \mathfrak{S}_i, \tau_i, \kappa_i), i = 1, 2$ be ditopological texture spaces. A difunction $(f, F) : (S_1, \mathfrak{S}_1) \rightarrow (S_2, \mathfrak{S}_2)$ is said to be:

- (i) s-continuous; if for each semi-open set $B \in \mathfrak{S}_2, F^{\leftarrow}(B) \in \mathfrak{S}_1$ is a open set.
- (ii) s-cocontinuous; if for each semi-closed set $B \in \mathfrak{S}_2, f^{\leftarrow}(B) \in \mathfrak{S}_1$ is a closed set.
- (iii) s-bicontinuous; if it both s-continuous and s-cocontinuous.

3. Almost-s-Menger ditopological texture space

In 1999 Kočinac in [14] defined almost-Menger property in topological space and Sabah et. al. in [23] defined almost-s-Menger property in topological spaces. In 2015, Kočinac and Özçağ [18] introduced the notion of Menger and co-Menger property in ditopological texture space and Ullah and Khan in [26] defined almost-Menger and almost-co-Menger in ditopological texture spaces.

Here the notion of almost-s-Menger and almost-co-s-Menger selection properties of ditopological texture spaces are defined and studied in this section.

Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space. We use the following notation:

- \mathcal{O} -the collection of all open covers of S ;
- $s\mathcal{O}$ -the collection of all semi-open covers of S ;
- $s\mathcal{C}$ -the collection of all semi-closed cocovers of S ;
- $s\mathcal{O}$ -the collection of all families \mathcal{U} of semi-open subsets of S such that $\{\underline{U} : U \in \mathcal{U}\}$ is a cover of S ;
- $(s\mathcal{C})_{\circ}$ -the collection of all families \mathcal{F} of semi-closed subsets of S such that $\{(F)_{\circ} : F \in \mathcal{F}\}$ is a cocover of S .

Definition 3.1. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and \mathcal{A} be a member of \mathfrak{S} .

- (1) \mathcal{A} is said to have the *almost-s-Menger* (resp. almost-s-Rothberger) property for the ditopological texture space if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of \mathcal{A} there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ (resp. a sequence $(V_n : n \in \mathbb{N})$) such that for each $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n (resp. $V_n \in \mathcal{U}_n$) and $\mathcal{A} \subseteq \bigvee_{n \in \mathbb{N}} \bigcup \mathcal{V}_n$ (resp. $\mathcal{A} \subseteq \bigvee_{n \in \mathbb{N}} V_n$), where $\underline{\mathcal{V}}_n = \{\underline{V} : V \in \mathcal{V}_n\}$ (resp. $\underline{V}_n = \{\underline{V} : V \in V_n\}$). We say that $(S, \mathfrak{S}, \tau, \kappa)$ is almost-s-Menger (resp. almost-s-Rothberger) if the set S is almost-s-Menger (resp. almost-s-Rothberger).
- (2) \mathcal{A} is said to have the *almost-co-s-Menger* (resp. almost-co-s-Rothberger) property for the ditopological texture space if for each sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of semi-closed cocovers of \mathcal{A} there is a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ (resp. a sequence $(K_n : n \in \mathbb{N})$) such that for each $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of \mathcal{F}_n (resp. $K_n \in \mathcal{F}_n$) and $\bigcap_{n \in \mathbb{N}} \bigcap (\mathcal{K}_n)_\circ \subseteq \mathcal{A}$ (resp. $\bigcap_{n \in \mathbb{N}} (K_n)_\circ \subseteq \mathcal{A}$), where $(\mathcal{K}_n)_\circ = \{(K)_\circ : K \in \mathcal{K}_n\}$ (resp. $(K_n)_\circ = \{(K)_\circ : K \in K_n\}$). We say that $(S, \mathfrak{S}, \tau, \kappa)$ is almost-co-s-Menger (resp. almost-co-s-Rothberger) if the set \emptyset is almost-co-s-Menger (resp. almost-co-s-Rothberger).

Note that a ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is almost-s-Menger (resp. almost-s-Rothberger) if it satisfies the selection hypothesis $\mathcal{S}_{fin}(s\mathcal{O}, s\mathcal{O})$ (resp. $\mathcal{S}_1(s\mathcal{O}, s\mathcal{O})$) and almost-co-s-Menger (resp. almost-co-s-Rothberger) if it satisfies the selection hypothesis $\mathcal{S}_{cfin}(s\mathcal{C}, (s\mathcal{C})_\circ)$ (resp. $\mathcal{S}_{c1}(s\mathcal{C}, (s\mathcal{C})_\circ)$).

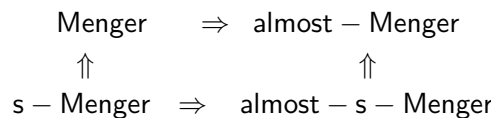
Every Menger space is almost Menger and every s-Menger space is almost-s-Menger. Here we give an example of s-Menger space which is almost-s-Menger in setting of ditopological texture spaces.

Example 3.2. There is a ditopological texture space which is s-Menger and hence almost-s-Menger.

Let $(\mathbb{R}, \mathfrak{S}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ be the real line with the texture $\mathfrak{R} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$, topology $\tau_{\mathbb{R}} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and cotopology $\kappa_{\mathbb{R}} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. But $(\mathbb{R}, \mathfrak{S}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is s-Menger and co-s-Menger. Let us prove that this space is s-Menger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of \mathbb{R} . We note that semi-open sets can be of the form $(-\infty, r)$ and $(-\infty, r]$. Write $\mathbb{R} = \bigcup \{(-\infty, n] : n \in \mathbb{N}\}$. For each n , \mathcal{U}_n is a semi-open cover of \mathbb{R} , hence there is some $r_n \in \mathbb{R}$ such that $(-\infty, n] \subseteq (-\infty, r_n] \in \mathcal{U}_n$. Then the collection $\mathcal{V}_n = \{(-\infty, r_n]\}$, $n \in \mathbb{N}$, such that $\mathcal{V}_n \subseteq \underline{\mathcal{V}}_n$ shows that $(\mathbb{R}, \mathfrak{S}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is almost-s-Menger.

Problem 3.3. Can we find an almost-s-Menger ditopological texture space which is not s-Menger?

Evidently we have the following diagram:



Remark 3.4. In this section we study almost-s-Menger ditopological texture spaces, but all properties of this structure can be investigated for almost-s-Rothberger ditopological texture spaces applying quite similar techniques for their proofs.

Definition 3.5 ([25]). Let (τ, κ) be a ditopology on the texture space (S, \mathfrak{S}) and take $A \in \mathfrak{S}$.

(1) A is said to be *s-compact* if whenever $\{G_\alpha : \alpha \in \nabla\}$ is semi-open cover of A , there is a finite subset ∇_0 of ∇ , with $A \subseteq \bigvee_{\alpha \in \nabla_0} G_\alpha$.

The ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is *s-compact* if S is *s-compact*. Every s-compact space in the ditopological texture space is compact but not conversely.

(2) A is said to be *co-s-compact* if whenever $\{\mathcal{F}_\alpha : \alpha \in \nabla\}$ is a semi-closed cocover of A , there is a finite subset ∇_0 of ∇ , with $\bigcap_{\alpha \in \nabla_0} \mathcal{F}_\alpha \subseteq A$. In particular the ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is *co-s-compact* if \emptyset is *co-s-compact*.

In general s-compactness and co-s-compactness are independent.

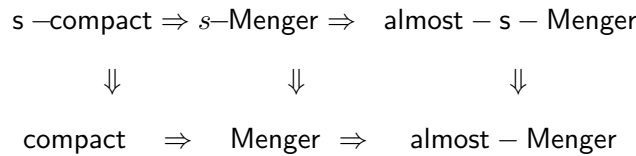
Example 3.6. There is a ditopological texture space which is s-compact and hence almost-s-Menger.

Let $L = (0, 1]$ with texturing $\mathfrak{L} = \{(0, r] \mid r \in [0, 1]\}$ and ditopology (τ, κ) such that $\tau = \{\emptyset, L\}$ and $\kappa = \{\mathfrak{L}\}$. Then it is s-compact and hence almost-s-Menger.

Example 3.7. There is a ditopological texture space which is almost-s-Menger, but not s-compact.

The above Example 3.2 is not compact by Example 3.4 [18] and hence not s-compact because the (open and hence) semi-open cover does not contain a finite subcover, but it is s-Menger and hence almost-s-Menger.

Evidently we have the following diagram:



Definition 3.8. Call a ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ σ -s-compact (resp. co- σ -s-compact) if there is a sequence $(A_n : n \in \mathbb{N})$ of s-compact (co-s-compact) subsets of S such that $\bigvee_{n \in \mathbb{N}} A_n = S$ (resp. $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$).

Theorem 3.9. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space.

- (1) If $(S, \mathfrak{S}, \tau, \kappa)$ is σ -s-compact, then $(S, \mathfrak{S}, \tau, \kappa)$ has the s-Menger property (and thus the almost-s-Menger property).
- (2) If $(S, \mathfrak{S}, \tau, \kappa)$ is co- σ -s-compact, then $(S, \mathfrak{S}, \tau, \kappa)$ has the co-s-Menger property (and thus the almost-co-s-Menger property).

Proof. (1). Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a sequence of semi-open covers of S . Since S is σ -s-compact therefore it can be represented in the form $S = \bigvee_{i \in \mathbb{N}} A_i$, where each A_i is s-compact $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose a finite set $\mathcal{V}_i \subseteq \mathcal{U}_i$ such that $A_i \subseteq \bigvee \mathcal{V}_i = \bigcup \mathcal{V}_i$. This shows that S is s-Menger.

(2). Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence of semi-closed cocovers of \emptyset . We have $\emptyset = \bigcap_{i \in \mathbb{N}} A_i$, where each A_i is co-s-compact and $A_i \supseteq A_{i+1}$, $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose a finite subset $\mathcal{K}_i \subseteq \mathcal{F}_i$ such that $\bigcap \mathcal{K}_i \subseteq A_i$. Then $\bigcap_{i \in \mathbb{N}} \bigcap \mathcal{K}_i \subseteq \bigcap_{i \in \mathbb{N}} A_i = \emptyset$, which means that S is co-s-Menger. \square

For complemented ditopological texture spaces we have:

Theorem 3.10. Let $(S, \mathfrak{S}, \sigma)$ be a texture with the complementation σ and let (τ, κ) be a complemented ditopology on $(S, \mathfrak{S}, \sigma)$. Then $S \in \mathfrak{S}_{fin}(s\mathcal{O}, \underline{s}\mathcal{O})$ if and only if $\emptyset \in \mathfrak{S}_{cfin}(s\mathbb{C}, (s\mathbb{C})_o)$.

Proof. Let $S \in \mathcal{S}_{fin}(s\mathcal{O}, \underline{s\mathcal{O}})$ and let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence of semi-closed cocovers of \emptyset . Then $(\sigma(\mathcal{F}_n) = \{\sigma(F) : F \in \mathcal{F}_n\})$ and $(\sigma(\mathcal{F}_n)_{n \in \mathbb{N}})$ is a sequence of semi-open covers of S . Since $S \in \mathcal{S}_{fin}(s\mathcal{O}, \underline{s\mathcal{O}})$, there is a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ of finite sets such that for each n , $\mathcal{V}_n \subseteq \sigma(\mathcal{F}_n)$ and $\bigvee_{n \in \mathbb{N}} \underline{\mathcal{V}_n} = S$. We have $(\sigma(\mathcal{V}_n) : n \in \mathbb{N})$ is a sequence of finite sets, and also

$$\emptyset = \sigma(S) = \sigma\left(\bigvee_{n \in \mathbb{N}} \underline{\mathcal{V}_n}\right) = \bigcap_{n \in \mathbb{N}} \bigcap \sigma(\mathcal{V}_n) = \bigcap_{n \in \mathbb{N}} \bigcap (\sigma(\mathcal{V}_n))_{\circ}$$

Hence $\emptyset \in \mathcal{S}_{cfm}(s\mathcal{C}, (s\mathcal{C})_{\circ})$.

Conversely let $\emptyset \in \mathcal{S}_{cfm}(s\mathcal{C}, (s\mathcal{C})_{\circ})$ and $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be sequence of semi-open covers of S . Then $(\sigma(\mathcal{U}_n) = \{\sigma(U) : U \in \mathcal{U}_n\})$ and $(\sigma(\mathcal{U}_n)_{n \in \mathbb{N}})$ is a sequence of semi-closed cocovers of \emptyset . Since $\emptyset \in \mathcal{S}_{cfm}(s\mathcal{C}, (s\mathcal{C})_{\circ})$, there is a sequence $(\kappa_n)_{n \in \mathbb{N}}$ of finite sets such that for each n , $\kappa_n \subseteq \sigma(\mathcal{U}_n)$ and $\bigcap_{n \in \mathbb{N}} \bigcap (\kappa_n)_{\circ} = \emptyset$. We have $(\sigma(\kappa_n)_{n \in \mathbb{N}})$ is a sequence of finite sets, such that

$$S = \sigma(\emptyset) = \sigma\left(\bigcap_{n \in \mathbb{N}} \bigcap (\kappa_n)_{\circ}\right) = \bigvee_{n \in \mathbb{N}} \bigvee (\sigma(\kappa_n)_{\circ})$$

Hence $S \in \mathcal{S}_{fin}(s\mathcal{O}, \underline{s\mathcal{O}})$. □

Theorem 3.11. *Let $(S, \mathfrak{S}, \sigma)$ be a texture space with complementation σ and let (τ, κ) be a complemented ditopology on $(S, \mathfrak{S}, \sigma)$. Then for $K \in \kappa$ with $K \neq S$, K is almost-s-Menger if and only if $G = \sigma(K)$ is almost-co-s-Menger for some $G \in \tau$ and $G \neq \emptyset$.*

Proof. (\Rightarrow) Let $G \in \tau$ with $G \neq \emptyset$. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence of semi-closed cocovers of G . Set $K = \sigma(G)$ and we obtain $K \in \kappa$ with $K \neq S$. Since K is almost-s-Menger, for $(\sigma(\mathcal{F}_n))_{n \in \mathbb{N}}$ the sequence of semi-open covers of K there is a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ such that for each n , \mathcal{V}_n is a finite subset of $\sigma(\mathcal{F}_n)$ and $\bigvee_{n \in \mathbb{N}} \underline{\mathcal{V}_n}$ is semi-open cover of K . Thus $(\sigma(\mathcal{V}_n))_{n \in \mathbb{N}}$ is a sequence such that for each n , $\sigma(\mathcal{V}_n)$ is a finite subset of \mathcal{F}_n and $\sigma\left(\bigvee_{n \in \mathbb{N}} \underline{\mathcal{V}_n}\right) = \bigcap_{n \in \mathbb{N}} \bigcap (\sigma(\mathcal{V}_n))_{\circ} \subseteq G$ which gives G is almost-co-s-Menger.

(\Leftarrow) $K \in \kappa$ with $K \neq S$. Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a sequence of semi-open covers of K . Since $K = \sigma(G)$ but G is almost-co-s-Menger, so for $(\sigma(\mathcal{U}_n))_{n \in \mathbb{N}}$ the sequence of semi-closed cocovers of G there is a sequence $(\mathcal{K}_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of $(\sigma(\mathcal{U}_n))$ and $\bigcap_{n \in \mathbb{N}} \bigcap (\mathcal{K}_n)_{\circ} \subseteq G$. Thus $\sigma(\mathcal{K}_n)$ is a sequence such that for each $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of \mathcal{U}_n and $\sigma\left(\bigcap_{n \in \mathbb{N}} \bigcap (\mathcal{K}_n)_{\circ}\right) = \bigvee_{n \in \mathbb{N}} \bigvee (\sigma(\mathcal{K}_n)) \supseteq K$ which gives K is almost-s-Menger. □

In 2009 Kocev in [13] proved that in the definition of almost-Menger we can use regular open sets instead of open sets. He proved that:

A topological space X is *almost-Menger* if and only if for each sequence $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of covers of X by *regular open sets*, there exists a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigcup \{\mathcal{V}'_n : n \in \mathbb{N}\}$ is a cover of X , where $\mathcal{V}'_n = \{cl(V) : V \in \mathcal{V}_n\}$.

Now we extend this idea to ditopological texture spaces and prove the following results.

Theorem 3.12. *A ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is almost-s-Menger if and only if for each sequence $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of covers of X by semi-regular sets, there exists a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigvee_{n \in \mathbb{N}} \bigcup \mathcal{V}_n$ is a cover of S .*

Proof. Let S be almost-s-Menger and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of S by semi-regular open sets. Since every semi-regular set is semi-open, so $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of semi-open covers. By assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigvee_{n \in \mathbb{N}} \bigcup \mathcal{V}_n$ covers S .

Conversely: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of S . Let $(\mathcal{U}'_n : n \in \mathbb{N})$ be a sequence defined by $\mathcal{U}'_n = \{\underline{U} : U \in \mathcal{U}_n\}$. Then each \mathcal{U}'_n is a cover of S by semi-regular sets. Thus there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}'_n and $\bigvee_{n \in \mathbb{N}} \bigcup \mathcal{V}_n$ is a cover of S . By construction, for each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $V = \underline{U}_V$. Hence, $\bigvee_{n \in \mathbb{N}} \{\bigcup \underline{U}_V : V \in \mathcal{V}_n\}$ covers S , so $(S, \mathfrak{S}, \tau, \kappa)$ is an almost-s-Menger space. \square

Theorem 3.13. *A ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is almost-co-s-Menger if and only if for each sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of cocovers of \emptyset by semi-regular sets, there exists a sequence $(\mathcal{K}_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of \mathcal{F}_n and $\bigcap_{n \in \mathbb{N}} \bigcap \mathcal{K}_n$ is a cocover of \emptyset .*

Proof. Let S be almost-co-s-Menger and let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of cocovers of \emptyset by semi regular sets. Since every semi regular set is semi-closed, so $(\mathcal{F}_n : n \in \mathbb{N})$ is a sequence of semi-closed cocovers. By assumption, there exists a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of \mathcal{F}_n and $\bigcap_{n \in \mathbb{N}} \bigcap \mathcal{K}_n$ is a cocover \emptyset .

Conversely: Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of semi-closed cocovers of \emptyset . Let $(\mathcal{F}'_n : n \in \mathbb{N})$ be a sequence defined by $\mathcal{F}'_n = \{(F)_\circ : F \in \mathcal{F}_n\}$. Then each \mathcal{F}'_n is a cocover of \emptyset by semi-regular sets.

Thus there exists a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of \mathcal{F}_n and $\bigcap_{n \in \mathbb{N}} \bigcap (\mathcal{K}_n)_\circ$ is a cocover of \emptyset . By construction, for each $n \in \mathbb{N}$ and $k \in \mathcal{K}_n$ there exists $F_K \in \mathcal{F}_n$ such that $K = (F_K)_\circ$. Hence, $\bigcap_{n \in \mathbb{N}} \{\bigcap (F_K)_\circ : k \in \mathcal{K}_n\}$ cocovers \emptyset , so $(S, \mathfrak{S}, \tau, \kappa)$ is an almost-co-s-Menger space. \square

Lemma 3.14 ([11]). *Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and $\mathcal{A} \in \mathfrak{S}$.*

- (1) $\underline{\mathcal{A}} = \mathcal{A} \cup \llbracket \mathcal{A} \rrbracket$.
- (2) $(\mathcal{A})_\circ = \mathcal{A} \cap \llbracket \mathcal{A} \rrbracket$.

Theorem 3.15. *If the ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is almost-s-Menger and $\llbracket \mathcal{A} \rrbracket$ is finite for any $\mathcal{A} \subset S$, then S is semi-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of S by semi-open sets. Since S is almost-s-Menger, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigvee_{n \in \mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\}$ covers S . Since for any $\mathcal{A} \subset S$, $\underline{\mathcal{A}} = \mathcal{A} \cup \llbracket \mathcal{A} \rrbracket$, by assumption that $\llbracket \mathcal{A} \rrbracket$ is finite, there exist finite sets $(\mathcal{F}_n : n \in \mathbb{N})$ subsets of S such that $S = \bigvee_{n \in \mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\} \cup \bigvee_{n \in \mathbb{N}} \mathcal{F}_n$. For each n let \mathcal{W}_n be a set of finitely many elements of \mathcal{U}_n which covers \mathcal{F}_n . Then the sequence $(\mathcal{V}_n \cup \mathcal{W}_n : n \in \mathbb{N})$ of finite sets witnesses that S is semi-Menger. \square

Theorem 3.16. *If the ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is almost-co-s-Menger and $\llbracket \mathcal{A} \rrbracket$ is finite for any $\mathcal{A} \subset S$, then \emptyset is co-s-Menger.*

Proof. Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of cocovers of \emptyset by semi-closed sets. Since \emptyset is almost-co-s-Menger, there exists a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, \mathcal{K}_n is a finite subset of \mathcal{F}_n and $\bigcap_{n \in \mathbb{N}} \bigcap \{(K)_\circ : K \in \mathcal{K}_n\}$ covers \emptyset . Since for any $\mathcal{A} \subset S$, $(\mathcal{A})_\circ = \mathcal{A} \cap \llbracket \mathcal{A} \rrbracket$, by assumption that $\llbracket \mathcal{A} \rrbracket$ is finite, there exist finite sets $(\mathcal{V}_n : n \in \mathbb{N})$ subsets of S such that $\bigcap_{n \in \mathbb{N}} \bigcap \{K : K \in \mathcal{K}_n\} \cap \bigcap_{n \in \mathbb{N}} \mathcal{V}_n = \emptyset$. For each n let \mathcal{W}_n be a set of finitely many elements of \mathcal{V}_n which covers \mathcal{V}_n . Then the sequence $(\mathcal{K}_n \cup \mathcal{W}_n : n \in \mathbb{N})$ of finite sets witnesses that \emptyset is co-s-Menger. \square

Theorem 3.17. *If a ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is a semi-regular space and S is an almost-s-Menger space, then S is also an s-Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of S . Since S is a semi-regular space, then by definition, there exists for each n a semi-open cover \mathcal{V}_n such that $\mathcal{V}'_n = \{\underline{V} : V \in \mathcal{V}_n\}$ forms a refinement of \mathcal{U}_n . By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n , \mathcal{W}_n is a finite subset of \mathcal{V}_n and $\bigcup\{\mathcal{W}'_n : n \in \mathbb{N}\}$ is a cover of S , where $\mathcal{W}'_n = \{\underline{W} : W \in \mathcal{W}_n\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}_n$ we can choose $U_W \in \mathcal{U}_n$ such that $\underline{W} \subset U_W$. Let $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$. We shall prove that $\bigvee_{n \in \mathbb{N}} \bigcup \mathcal{U}'_n$ is a semi-open cover of S . Let $x \in S$. There exists $n \in \mathbb{N}$ and $\underline{W} \in \mathcal{W}'_n$ such that $x \in \underline{W}$. By construction, there exists $U_W \in \mathcal{U}'_n$ such that $\underline{W} \subset U_W$. Hence $x \in U_W$. \square

Theorem 3.18. *If a ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is a semi-co-regular space and \emptyset is an almost-co-s-Menger space, then \emptyset is a co-s-Menger space.*

Proof. Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of semi-closed cocovers of \emptyset . Since S is a semi-co-regular space, then by definition, there exists for each n a semi-closed cover \mathcal{V}_n such that $\mathcal{V}'_n = \{(V)_\circ : V \in \mathcal{V}_n\}$ forms a refinement of \mathcal{F}_n . By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n , \mathcal{W}_n is a finite subset of \mathcal{V}_n and $\bigcap\{\mathcal{W}'_n : n \in \mathbb{N}\}$ is a cocover of \emptyset , where $\mathcal{W}'_n = \{(W)_\circ : W \in \mathcal{W}_n\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}_n$ we can choose $F_W \in \mathcal{F}_n$ such that $F_W \subset (W)_\circ$. Let $\mathcal{F}'_n = \{F_W : W \in \mathcal{W}_n\}$. We shall prove that $\bigcap_{n \in \mathbb{N}} \bigcap \mathcal{F}'_n$ is a semi-closed cocover of \emptyset . Let $x \in S$. There exists $n \in \mathbb{N}$ and $(W)_\circ \in \mathcal{W}'_n$ such that $x \in (W)_\circ$. By construction, there exists $F_W \in \mathcal{F}'_n$ such that $(W)_\circ \subset F_W$. Hence $x \in F_W$. \square

Theorem 3.19. *Every semi-regular subset of an almost-s-Menger ditopological texture space is almost-s-Menger.*

Proof. Let F be a semi-regular subset of the almost-s-Menger ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of F . Then $\mathcal{V}_n = \{\mathcal{U}_n\} \cup \{S \setminus F\}$ is a semi-open cover of S for every $n \in \mathbb{N}$. Since S is almost-s-Menger, there exists finite subsets \mathcal{V}'_n of \mathcal{V}_n for every $n \in \mathbb{N}$ so that $\bigvee_{n \in \mathbb{N}} \bigcup \{\underline{V} : V \in \mathcal{V}'_n\} = S$. But $S \setminus F$ is semi-regular, so $\underline{S \setminus F} = S \setminus F$ and $\bigvee_{n \in \mathbb{N}} \bigcup \{\underline{V} : V \in \mathcal{V}'_n, V \neq S \setminus F\}$ covers S . \square

Theorem 3.20. *Every semi-co-regular subset of an almost-co-s-Menger ditopological texture space is almost-co-s-Menger.*

Proof. Let F be a semi-co-regular subset of almost-co-s-Menger ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$, and let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of semi-closed cocovers of F . Then $\mathcal{K}_n = \{\mathcal{F}_n\} \cup \{S \setminus F\}$ is a semi-closed cocover of S for every $n \in \mathbb{N}$. Since S is almost-co-s-Menger, there exists finite subsets \mathcal{K}'_n of \mathcal{K}_n for every $n \in \mathbb{N}$ so that $\bigcap_{n \in \mathbb{N}} \bigcap \{(V)_\circ : V \in \mathcal{K}'_n\} = \emptyset$. But $S \setminus F$ is semi-co-regular, so $(S \setminus F)_\circ = S \setminus F$ and $\bigcap_{n \in \mathbb{N}} \bigcap \{(V)_\circ : V \in \mathcal{K}'_n, V \neq S \setminus F\}$ cocovers \emptyset . \square

In what follows we consider preservation of almost-s-Menger and almost-co-s-Menger properties by mappings. We follow some ideas from [17].

Theorem 3.21. *Let $(f, F) : (S_1, \mathfrak{S}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be a semi-irresolute difunction between S_1 and S_2 ditopological texture spaces. If $\mathcal{A} \in \mathfrak{S}_1$ is almost-s-Menger, then $f \rightarrow (\mathcal{A}) \in \mathfrak{S}_2$ is almost-s-Menger.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of τ_2 -semi-open covers of $f \rightarrow (\mathcal{A})$. Then by Lemma 2.5(2a) and Lemma 2.2(2) alongwith semi-irresoluteness of (f, F) , for each n we have

$$\mathcal{A} \subseteq F \leftarrow (f \rightarrow (\mathcal{A})) \subseteq F \leftarrow (\bigvee \mathcal{V}_n) = \bigvee F \leftarrow (\mathcal{V}_n)$$

so that each $F \leftarrow (\mathcal{V}_n)$ is a τ_1 -semi-open cover of \mathcal{A} . As \mathcal{A} is almost-s-Menger, for each n there exist finite subsets $\mathcal{U}_n \subseteq \mathcal{V}_n$ such that $\mathcal{A} \subseteq \bigvee_{n \in \mathbb{N}} (\bigcup F \leftarrow (\mathcal{U}_n))$. Again by Lemma 2.5(2b), Lemma 2.2(2) and Lemma 2.17 (2c) we have

$$\begin{aligned} f \rightarrow (\mathcal{A}) &\subseteq f \rightarrow (\bigvee_{n \in \mathbb{N}} (\bigcup F \leftarrow (\mathcal{U}_n))) \subseteq f \rightarrow (\bigvee_{n \in \mathbb{N}} \bigcup F \leftarrow (\mathcal{U}_n)) \subseteq \bigvee_{n \in \mathbb{N}} \bigcup f \rightarrow F \leftarrow (\mathcal{U}_n) \\ \implies f \rightarrow (\mathcal{A}) &\subseteq \bigvee_{n \in \mathbb{N}} \bigcup \mathcal{U}_n \end{aligned}$$

This proves that $f \rightarrow (\mathcal{A})$ is an almost-s-Menger space. □

Theorem 3.22. Let $(S_1, \mathfrak{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a semi-co-irresolute difunction between them. If $\mathcal{A} \in \mathfrak{S}_1$ is almost-co-s-Menger, then $F \rightarrow (\mathcal{A}) \in \mathfrak{S}_2$ is also almost-co-s-Menger.

Proof. Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of κ_2 -semi-closed cocovers of $F \rightarrow (\mathcal{A})$. Then by Lemma 2.5(2a) and Lemma 2.2(2) with semi-co-irresoluteness of (f, F) , for each n we have

$$\bigcap \mathcal{F}_n \subseteq F \rightarrow (\mathcal{A}) \implies f \leftarrow (\bigcap \mathcal{F}_n) \subseteq f \leftarrow (F \rightarrow (\mathcal{A})) \subseteq \mathcal{A} \implies \bigcap f \leftarrow (\mathcal{F}_n) \subseteq \mathcal{A}$$

This gives each $f \leftarrow (\mathcal{F}_n)$ is a κ_1 -semi-closed cocover of \mathcal{A} . As \mathcal{A} is almost-co-s-Menger, for each n there exist finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n \in \mathbb{N}} \bigcap (f \leftarrow (\mathcal{K}_n))_\circ \subseteq \mathcal{A}$. Again by Lemma 2.5(2b), Lemma 2.2(2) and Lemma 2.17 (1c) we have

$$\bigcap_{n \in \mathbb{N}} \bigcap (f \leftarrow (\mathcal{K}_n))_\circ \subseteq \mathcal{A} \implies F \rightarrow (\bigcap_{n \in \mathbb{N}} \bigcap (f \leftarrow (\mathcal{K}_n))_\circ) \subseteq F \rightarrow (\mathcal{A})$$

$$\text{But } f \leftarrow (\mathcal{K}_n)_\circ \subseteq (f \leftarrow \mathcal{K}_n)_\circ \implies F \rightarrow (\bigcap_{n \in \mathbb{N}} \bigcap f \leftarrow (\mathcal{K}_n)_\circ) \subseteq \bigcap_{n \in \mathbb{N}} \bigcap F \rightarrow f \leftarrow (\mathcal{K}_n)_\circ$$

$$\implies \bigcap_{n \in \mathbb{N}} \bigcap F \rightarrow f \leftarrow (\mathcal{K}_n)_\circ \subseteq F \rightarrow (\mathcal{A}) \implies \bigcap_{n \in \mathbb{N}} \bigcap (\mathcal{K}_n)_\circ \subseteq F \rightarrow (\mathcal{A})$$

This proves that $F \rightarrow (\mathcal{A})$ is almost-co-s-Menger. □

Theorem 3.23. Let $(f, F) : (S_1, \mathfrak{S}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be a semi-continuous difunction between S_1 and S_2 ditopological texture spaces. If $\mathcal{A} \in \mathfrak{S}_1$ is almost-s-Menger, then $f \rightarrow (\mathcal{A}) \in \mathfrak{S}_2$ is almost-Menger.

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of τ_2 -open covers of $f \rightarrow (\mathcal{A})$. Then by Lemma 2.5(2a) and Lemma 2.2(2) alongwith semi-continuity of (f, F) , for each n we have

$$\mathcal{A} \subseteq F \leftarrow (f \rightarrow (\mathcal{A})) \subseteq F \leftarrow (\bigvee \mathcal{V}_n) = \bigvee F \leftarrow (\mathcal{V}_n)$$

so that each $F \leftarrow (\mathcal{V}_n)$ is a τ_1 -semi-open cover of \mathcal{A} . As \mathcal{A} is almost-s-Menger, for each n there exist finite subsets $\mathcal{U}_n \subseteq \mathcal{V}_n$ such that $\mathcal{A} \subseteq \bigvee_{n \in \mathbb{N}} (\bigcup F \leftarrow (\mathcal{U}_n))$. Again by Lemma 2.5(2b), Lemma 2.2(2) and Lemma 2.18 (2c) we have

$$\begin{aligned} f \rightarrow (\mathcal{A}) &\subseteq f \rightarrow (\bigvee_{n \in \mathbb{N}} (\bigcup F \leftarrow (\mathcal{U}_n))) \subseteq f \rightarrow (\bigvee_{n \in \mathbb{N}} \bigcup F \leftarrow (\mathcal{U}_n)) \subseteq \bigvee_{n \in \mathbb{N}} \bigcup f \rightarrow F \leftarrow (\mathcal{U}_n) \\ \implies f \rightarrow (\mathcal{A}) &\subseteq \bigvee_{n \in \mathbb{N}} \bigcup \mathcal{U}_n \end{aligned}$$

This proves that $f \rightarrow (\mathcal{A})$ is an almost-Menger space. □

Theorem 3.24. Let $(S_1, \mathfrak{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a semi-cocontinuous difunction between them. If $\mathcal{A} \in \mathfrak{S}_1$ is almost-co-s-Menger, then $F \rightarrow(\mathcal{A}) \in \mathfrak{S}_2$ is also almost-co-Menger.

Proof. Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of κ_2 -closed cocovers of $F \rightarrow(\mathcal{A})$. Then by Lemma 2.5(2a) and Lemma 2.2(2) with semi-cocontinuity of (f, F) , for each n we have

$$\bigcap \mathcal{F}_n \subseteq F \rightarrow(\mathcal{A}) \Rightarrow f \leftarrow(\bigcap \mathcal{F}_n) \subseteq f \leftarrow(F \rightarrow(\mathcal{A})) \subseteq \mathcal{A} \Rightarrow \bigcap f \leftarrow(\mathcal{F}_n) \subseteq \mathcal{A}$$

This gives each $f \leftarrow(\mathcal{F}_n)$ is a κ_1 -semi-closed cocover of \mathcal{A} . As \mathcal{A} is almost-co-s-Menger, for each n there exist finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n \in \mathbb{N}} \bigcap (f \leftarrow(\mathcal{K}_n))_o \subseteq \mathcal{A}$. Again by Lemma 2.5(2b), Lemma 2.2(2) and Lemma 2.18 (1c) we have

$$\bigcap_{n \in \mathbb{N}} \bigcap (f \leftarrow(\mathcal{K}_n))_o \subseteq \mathcal{A} \Rightarrow F \rightarrow(\bigcap_{n \in \mathbb{N}} \bigcap (f \leftarrow(\mathcal{K}_n))_o) \subseteq F \rightarrow(\mathcal{A})$$

$$\begin{aligned} \text{But } f \leftarrow]\mathcal{K}_n[\subseteq (f \leftarrow(\mathcal{K}_n))_o &\implies F \rightarrow(\bigcap_{n \in \mathbb{N}} \bigcap f \leftarrow]\mathcal{K}_n[) \subseteq \bigcap_{n \in \mathbb{N}} \bigcap F \rightarrow f \leftarrow]\mathcal{K}_n[\subseteq F \rightarrow(\mathcal{A}) \\ &\implies \bigcap_{n \in \mathbb{N}} \bigcap]\mathcal{K}_n[\subseteq F \rightarrow(\mathcal{A}) \end{aligned}$$

This proves that $F \rightarrow(\mathcal{A})$ is almost-co-Menger. \square

Acknowledgment. The authors thank the referee for helpful comments and suggestions which have improved the presentation of the paper.

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