Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 352 – 361
DOI : 10.15672/hujms.552216

Research Article

On covers of acts over monoids with Condition (P')

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Abstract

In this paper we consider two different definitions of cover, one of them is Enochs' notion of a cover and the other is the one that initiated by Mahmoudi and Renshaw which concerned with the coessential epimorphisms. We show that these definitions are not equivalent in our case and restrict our attention to (P')-covers (coessential-covers that satisfy Condition (P')). We give a necessary and sufficient condition for a cyclic act to have a (P')-cover and a sufficient condition for every act to have a $\mathcal{P}'$-cover (Enochs' $\mathcal{P}'$-cover where $\mathcal{P}'$ is the class of S-acts satisfying Condition (P')). We also obtain numerous classes of monoids over which indecomposable acts satisfying Condition (P') are locally cyclic.

Mathematics Subject Classification (2010). 20M30, 20M50

Keywords. monoids, acts, condition (P'), covers

1. Introduction and Preliminaries

For almost five decades, an active area of research in semigroup theory has been the classification of monoids S by so-called flatness properties of their associated S-acts. The properties in question, arranged in strictly decreasing order of strength, are as follows:

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{strongly flat} \Rightarrow \text{condition (P)} \Rightarrow \text{flat}$$
$$\Rightarrow \text{weakly flat} \Rightarrow \text{principally weakly flat} \Rightarrow \text{torsion-free.}$$

In [4] the authors introduced a generalization of Condition (P), called Condition (P'), and gave a characterization of monoids by this condition of their (Rees factor) acts. Note that if we know monoids over which Condition (P') of their acts imply Condition (P), then we know monoids over which torsion freeness of acts imply Condition (P).

On the other hand, over the past several decades, the covers of modules have been investigated by many authors and ample results have been obtained. Covers of acts over monoids are studied in [1–3, 6, 8, 10, 11]. In [10] the authors consider almost exclusively covers of cyclic acts and restrict their attention to strongly flat and Condition (P) covers. They provide a necessary and sufficient condition for the existence of such covers. In this

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Received: 01.07.2017; Accepted: 11.12.2018

paper we will focus on covers of acts over monoids with Condition (P') and shall obtain some results in this respect.

Throughout this paper S denotes a monoid, all acts will be right S-acts and all congruences right S-congruences. We refer the reader to [5] and [7] for basic definitions and terminology relating to semigroups and acts over monoids.

A monoid S is called *right reversible* if for every $s, t \in S$ there exist $p, q \in S$ such that $ps = qt$.

A right S-act A satisfies Condition (P) if for all $a, a' \in A$, $s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. A right S-act A satisfies Condition (P') if for all $a, a' \in A$, $s, s', z \in S$, $as = a's'$ and $sz = s'z$ imply that there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. A right S-act A satisfies Condition (E) if for all $a \in A$, $s, s' \in S$, $as = as'$, implies that there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us = us'$. A right S-act X is said to be *locally cyclic* if for all $x, y \in X$ there exist $z \in X$, $s, t \in S$ with $x = zs$, $y = zt$.

Let S be a monoid and A be an S-act. Let $\mathcal{X}$ be a class of S-acts which is closed under isomorphisms. By an $\mathcal{X}$-*precover* of A we mean an S-morphism $g : X \longrightarrow A$ for some $X \in \mathcal{X}$ such that for every S-morphism $g' : X' \longrightarrow A$, with $X' \in \mathcal{X}$, there exists an S-morphism $f : X' \longrightarrow X$ with $g' = gf$:

$$\begin{tikzcd} & X' \arrow[dl, dashed, "f"'] \arrow[d, "g'"] \\ X \arrow[r, "g"] & A \end{tikzcd}$$

If, in addition, the precover $g : X \longrightarrow A$ satisfies the condition that each S-morphism $f : X \longrightarrow X$ with $gf = g$ is an isomorphism, then we shall call it an $\mathcal{X}$-*cover*. We shall of course frequently identify the (pre)cover with its domain. Obviously an S-act A is an $\mathcal{X}$-cover of itself if and only if $A \in \mathcal{X}$. Note that this definition of cover is the Enochs' notion of cover. In Section 3 we replace the class $\mathcal{X}$ by the class $\mathcal{P}'$ of Condition (P') acts and consider $\mathcal{P}'$-covers.

Now we recall the concept of cover which has been used in [10] by Mahmoudi and Renshaw. Let S be a monoid and $f : C \longrightarrow A$ be an S-epimorphism. Then f is called *coessential* if for each S-act B and each S-morphism $g : B \longrightarrow C$, if fg is an epimorphism then g is an epimorphism. We shall say that an act C together with an S-epimorphism $f : C \longrightarrow A$ is a (P')-*cover* of A if C satisfies Condition (P') and f is coessential.

2. (P')-Covers

In this section we give a necessary and sufficient condition for a cyclic act to have a (P')-cover.

Recall [4] that a submonoid $R \subseteq S$ is said to be *weakly right reversible* if

$$(\forall s, s' \in R)(\forall z \in S)(sz = s'z \Rightarrow (\exists u, v \in R)(us = vs')).$$

Lemma 2.1. *Let ρ be a right congruence on S such that the right S-act S/ρ satisfies Condition (P') and let $R = [1]_\rho$. Then R is a weakly right reversible submonoid of S.*

Proof. Obviously, R is a submonoid of S. Let $s, s' \in R$ and $sz = s'z$ for $z \in S$. Then $(1s)\ \rho\ (1s')$ and by [4, Theorem 3.1] there exist $u, v \in S$ such that $us = vs'$, $u\ \rho\ 1$ and $v\ \rho\ 1$. It follows from $u\ \rho\ 1$ and $v\ \rho\ 1$ that $u, v \in R$. This means that R is weakly right reversible. $\square$

Theorem 2.2. *Let R be a weakly right reversible submonoid of S. Set $H = \{(p, q) \in R \times R \mid \exists z \in S\ ;\ pz = qz\} \cup \{(p, 1) \mid p \in R\}$ and let $\sigma = \sigma(H)$ be the right congruence on S generated by H. Then S/σ satisfies Condition (P').*

Proof. Set $C = \{(p,q) \in R \times R \mid \exists z \in S\ ;\ pz = qz\}$ and $D = \{(p,1) \mid p \in R\}$. Let $(xs)\,\sigma\,(ys')$ and $sz = s'z$, $x, y, s, s', z \in S$. Then there exist $p_1, \cdots, p_n, q_1, \cdots, q_n, w_1, \cdots, w_n \in S$, where for $i = 1, \cdots, n$, $(p_i, q_i) \in H$ or $(q_i, p_i) \in H$, such that

$$\begin{array}{llll} xs = p_1w_1, & q_2w_2 = p_3w_3, & \cdots & q_nw_n = ys'. \\ q_1w_1 = p_2w_2, & q_3w_3 = p_4w_4, & \cdots & \end{array} \tag{2.1}$$

Then there are two cases as follows:
Case 1. For all $i = 1, \cdots, n$, $(p_i, q_i) \in C$ or $(q_i, p_i) \in C$. Then there exists $z_1 \in S$ such that $p_1z_1 = q_1z_1$. Since R is weakly right reversible, then there exist $u_1, v_1 \in R$ such that $u_1p_1 = v_1q_1$. So $u_1xs = u_1p_1w_1 = v_1q_1w_1 = v_1p_2w_2$. Since $(p_2, q_2) \in C$ or $(q_2, p_2) \in C$, there exists $z_2 \in S$ such that $p_2z_2 = q_2z_2$ and $v_1p_2z_2 = v_1q_2z_2$ and so there exist $u_2, v_2 \in R$ such that $u_2(v_1p_2) = v_2(v_1q_2)$. Since $(p_3, q_3) \in C$ or $(q_3, p_3) \in C$, there exists $z_3 \in S$ such that $p_3z_3 = q_3z_3$ and $(v_2v_1)p_3z_3 = (v_2v_1)q_3z_3$ and so there exist $u_3, v_3 \in R$ such that $u_3(v_2v_1p_3) = v_3(v_2v_1q_3)$. Now we have

$$u_1(xs) = v_1p_2w_2,$$

$$u_2u_1(xs) = u_2(v_1p_2w_2) = (v_2v_1q_2)w_2 = v_2v_1(p_3w_3),$$

$$u_3u_2u_1(xs) = u_3(v_2v_1p_3w_3) = (v_3v_2v_1q_3)w_3 = v_3v_2v_1(p_4w_4).$$

Continuing in this way, we get $u_1, \cdots, u_n, v_1, \cdots, v_n \in R$ with

$$u_n \cdots u_1(xs) = v_n \cdots v_1(q_nw_n) = v_n \cdots v_1(ys').$$

Put $u = u_n \cdots u_1x$ and $v = v_n \cdots v_1y$. Then we have $u\sigma x$ and $v\sigma y$, since $u_n \cdots u_1, v_n \cdots v_1 \in R$ and

$$(u_n \cdots u_1, 1), (v_n \cdots v_1, 1) \in \sigma.$$

Also, $us = vs'$. Therefore by [4, Theorem 3.1], the result follows, that is S/σ satisfies Condition (P').
Case 2. There is at least one k, $1 \leq k \leq n$, such that $(p_k, q_k) \in D$ or $(q_k, p_k) \in D$ and the other pairs are in C. If two or more (p_i, q_i) or (q_i, p_i) are in D and the others are in C then we can get the result in a similar way to the case that there exists exactly one $1 \leq k \leq n$ such that $(q_k, 1) \in D$. So we may assume without loss of generality that there exists exactly one $1 \leq k \leq n$ such that $(q_k, 1) \in D$ and for $i \neq k$, $(p_i, q_i) \in C$ or $(q_i, p_i) \in C$. Then,

$$\begin{array}{rcl} xs & = & p_1w_1 \\ q_1w_1 & = & p_2w_2 \\ & \cdots & \\ q_{k-1}w_{k-1} & = & 1w_k \\ q_kw_k & = & p_{k+1}w_{k+1} \\ q_{k+1}w_{k+1} & = & p_{k+2}w_{k+2} \\ & \cdots & \\ q_nw_n & = & ys'. \end{array} \tag{2.2}$$

Since $(p_1, q_1) \in C$ or $(q_1, p_1) \in C$, there exists $z_1 \in S$ such that $p_1z_1 = q_1z_1$ and then $q_kp_1z_1 = q_kq_1z_1$, and so, by the assumption, there exist $u_1, v_1 \in R$ such that $u_1q_kp_1 = v_1q_kq_1$. Similarly, there exists $z_2 \in S$ such that $p_2z_2 = q_2z_2$ and then $(v_1q_k)p_2z_2 = (v_1q_k)q_2z_2$, and so there exist $u_2, v_2 \in R$ such that $u_2(v_1q_kp_2) = v_2(v_1q_kq_2)$. Continuing in this way, we get $u_{k-1}, v_{k-1} \in R$ such that

$$u_{k-1}(v_{k-2} \cdots v_1q_kp_{k-1}) = v_{k-1}(v_{k-2} \cdots v_1q_kq_{k-1}).$$

Hence,

$$\begin{aligned}
u_{n-1}\cdots u_k u_{k-1}\cdots u_1 q_k xs &= u_{n-1}\cdots u_k u_{k-1}\cdots u_1 q_k(p_1w_1)\\
&= u_{n-1}\cdots u_k u_{k-1}\cdots u_2(v_1q_kq_1)w_1\\
&= u_{n-1}\cdots u_k u_{k-1}\cdots u_2v_1q_k(p_2w_2)\\
&= u_{n-1}\cdots u_k u_{k-1}\cdots u_3(v_2v_1q_kq_2)w_2\\
&= u_{n-1}\cdots u_k u_{k-1}\cdots u_3v_2v_1q_k(p_3w_3)\\
&\ \ \vdots\\
&= u_{n-1}\cdots u_{k-1}v_{k-2}\cdots v_1q_k(p_{k-1}w_{k-1})\\
&= u_{n-1}\cdots (v_{k-1}v_{k-2}\cdots v_1q_kq_{k-1})w_{k-1}\\
&= u_{n-1}\cdots u_kv_{k-1}v_{k-2}\cdots v_1q_k(w_k)\\
&= u_{n-1}\cdots u_kv_{k-1}v_{k-2}\cdots v_1(p_{k+1}w_{k+1}).
\end{aligned}$$

Now, since $(p_{k+1}, q_{k+1}) \in C$ or $(q_{k+1}, p_{k+1}) \in C$, there exists $z_k \in S$ such that $p_{k+1}z_k = q_{k+1}z_k$ and then

$$(v_{k-1}\cdots v_1)p_{k+1}z_k = (v_{k-1}\cdots v_1)q_{k+1}z_k,$$

so there exist $u_k, v_k \in R$ such that

$$u_k(v_{k-1}.\cdots v_1p_{k+1}) = v_k(v_{k-1}\cdots v_1q_{k+1}).$$

Continuing in this way, we get

$$\begin{aligned}
u_{n-1}\cdots u_k\cdots u_1q_kxs &= u_{n-1}u_{n-2}\cdots u_ku_{k-1}\cdots_1 q_k(p_1w_1)\\
&= u_{n-1}\cdots u_ku_{k-1}\cdots u_2(v_1q_kq_1)w_1\\
&= u_{n-1}\cdots u_ku_{k-1}\cdots u_2v_1q_k(p_2w_2)\\
&\ \ \vdots\\
&= u_{n-1}\cdots u_ku_{k-1}v_{k-2}\cdots v_1q_k(p_{k-1}w_{k-1})\\
&= u_{n-1}\cdots u_k(v_{k-1}v_{k-2}\cdots v_1q_kq_{k-1})w_{k-1}\\
&= u_{n-1}\cdots u_kv_{k-1}v_{k-2}\cdots v_1q_k(w_k)\\
&= u_{n-1}\cdots u_{k+1}u_kv_{k-1}v_{k-2}\cdots v_1(p_{k+1}w_{k+1})\\
&= u_{n-1}\cdots u_{k+1}(v_kv_{k-1}\cdots v_1q_{k+1})w_{k+1}\\
&= u_{n-1}\cdots u_{k+1}v_kv_{k-1}\cdots v_1(p_{k+2}w_{k+2})\\
&\ \ \vdots\\
&= u_{n-1}v_{n-2}\cdots v_1(p_nw_n)\\
&= (v_{n-1}v_{n-2}\cdots v_1q_n)w_n\\
&= v_{n-1}\cdots v_1(ys').
\end{aligned}$$

Put $u = u_{n-1}u_{n-2}\cdots u_1q_kx$ and $v = v_{n-1}v_{n-2}\cdots v_1y$. Then $us = vs'$. Note that, since $q_k, u_{n-1}\cdots u_1, v_{n-1}\cdots v_1 \in R$,

$$(u_{n-1}u_{n-2}\cdots u_1q_k, 1), (v_{n-1}v_{n-2}\cdots v_1, 1) \in \sigma.$$

Consequently, $u\ \sigma\ x$ and $v\ \sigma\ y$. Hence S/σ satisfies Condition (P'), by [4, Theorem 3.1]. □

Theorem 2.3. *Let S be a monoid. Then the cyclic S-act S/ρ has a (P')-cover if and only if $[1]_\rho$ contains a weakly right reversible submonoid R such that for all $u \in [1]_\rho$, $uS\cap R \neq \emptyset$.*

Proof. First of all, it is easy to observe that any (P')-cover of a cyclic act is cyclic. Suppose that S/ρ has a (P')-cover S/σ. Then, by [10, Theorem 2.7], we can assume that $R = [1]_\sigma \subseteq [1]_\rho$ and for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$. Moreover, R is weakly right reversible, by Lemma 2.1.

Conversely, suppose that R is a weakly right reversible submonoid of $[1]_\rho$ such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$. Set

$$C = \{(p,q) \in R \times R \mid \exists z \in S\ ;\ pz = qz\}\ ,\ D = \{(p,1) \mid p \in R\}$$

and define a right congruence σ on S by $\sigma = \sigma(H)$ (the right congruence generated by H) where $H = C \cup D$. Then clearly $R \subseteq [1]_\sigma$. By a similar proof of [10, Theorem 2.8], we get that S/σ is a cover of S/ρ. Further, S/σ satisfies Condition (P'), by Theorem 2.2. □

Corollary 2.4. *The one-element S-act Θ_S has a (P')-cover if and only if there exists a weakly right reversible submonoid R of S such that for all $u \in S$, there exists $s \in S$ with $us \in R$.*

Recall, from [7], that a submonoid R of a monoid S is said to be *left unitary* if for every $r \in R$, $s \in S$ we have $rs \in R$ only if $s \in R$.

Proposition 2.5. *Let S be a monoid. Then every cyclic S-act has a (P')-cover if and only if every left unitary submonoid T of S contains a weakly right reversible submonoid R such that for all $u \in T$, $uS \cap R \neq \emptyset$.*

Proof. By [7, Corollary 1.4.39], every left unitary submonoid of S is a ρ-class containing 1_S, for some right congruence ρ on S. Hence the result follows. □

Since commutative monoids are necessarily weakly right reversible, we can deduce that

Theorem 2.6. *Let S be a commutative monoid. Then every cyclic S-act has a (P')-cover.*

Proof. It is clear from Theorem 2.3. □

Remark 2.7. Let S be a monoid.

(1) If S has a right zero then every S-act that satisfying Condition (P'), satisfies Condition (P).
(2) If S has a left zero then the only (P')-cover of Θ_S is Θ_S itself. In fact if $S/\sigma \longrightarrow \Theta_S$ is a (P')-cover of Θ_S, then $[1]_\sigma$ contains the left zero by Theorem 2.3, hence $\sigma = S \times S$.
(3) If S is right cancellative then all its submonoids are weakly right reversible and so every cyclic S-act has a (P')-cover.

Let X be a non-empty set, X^+ be the free semigroup generated by X, for each w in X^+, the *content* $C(w)$ is defined as the (necessarily finite) set of elements of X appearing in w. Let R be a subsemigroup of a free semigroup X^+. We recall the *content* $C(R)$ of R as

$$C(R) = \bigcup_{w \in R} C(w).$$

It was shown in [11, Lemma 2.2] that if X^+ is the free semigroup generated by a non-empty set X, and R is a subsemigroup of X^+ then R is right reversible if and only if $|C(R)| = 1$. If $X = \{x, y\}$ and $R = \langle xy \rangle$ is the subsemigroup of X^+ generated by xy, then R is right reversible but $|C(R)| \neq 1$. So their result is false and we have the following lemma.

Lemma 2.8. *Let X be a non-empty set, X^* the free monoid generated by X and R be a submonoid of X^*. Then R is right reversible if and only if R is generated by one element.*

Proof. *Necessity.* If R is generated by more than one element, say $R = \langle x, y \rangle$, then the elements x^2 and xy belong to R, and so by the property of the free monoid X^*, R is not right reversible which is a contradiction. Hence, R is generated by one element.

Sufficiency. Suppose $R = \langle x \rangle$, $x \in X^*$. For any $s, t \in R$, there exist $p, q \in \mathbb{N}$ such that $s = x^p$ and $t = x^q$. It is clear that $x^q \cdot x^p = x^p \cdot x^q$. Hence R is right reversible. $\square$

Lemma 2.9. *Let X be a set with at least two elements and $S = X^*$, the free monoid generated by X. Then, the one-element S-act Θ_S has no (P)-cover.*

Proof. Suppose R is a right reversible submonoid of $S = X^*$. By Lemma 2.8, R is generated by one element, suppose $R = \langle x \rangle$, $x \in S$. There exists $u \in S$ such that $uS \cap R = \emptyset$. By [10, Corollary 4.3], the result follows. $\square$

It is clear that every (P)-cover is also a (P')-cover and so all of the (P)-covers considered in [10] are also (P')-covers. The following example shows that the converse is not true.

Example 2.10. Let $X = \{x, y\}$ and $S = X^*$. Then S is a cancellative monoid and so it is weakly right reversible, but it is not right reversible. Hence, by Remark 2.7 (3), every cyclic S-act has a (P')-cover. However, the one-element act Θ_S has no (P)-cover, by Lemma 2.9. Therefore S is a monoid over which all cyclic right acts have (P')-cover and not all cyclic acts (for example Θ_S) have (P)-covers.

Theorem 2.11. *Let S be a monoid with a right zero. The cyclic S-act S/ρ has a (P)-cover if and only if S/ρ has a (P')-cover.*

Proof. *Necessity.* Obvious.

Sufficiency. This follows from Remark 2.7 (1). $\square$

Now we show that (P')-covers of cyclic S-acts need not be unique.

Remark 2.12. 1) If S is a group then, by [4, Theorem 2.5], all S-acts satisfy Condition (P') and so S/ρ is a (P')-cover of itself for any congruence ρ on S. However $[1]_\rho$ is a subgroup of S and so S is a (P')-cover of S/ρ, by [10, Theorem 2.10]. Furthermore, if S/ρ is a proper cyclic S-act and $S/\sigma \longrightarrow S/\rho$ is onto then S/σ is trivially a (P')-cover of S/ρ.
2) If S is a monoid and $S/\sigma \longrightarrow S/\rho$ is a (P')-cover then $S/\psi \longrightarrow S/\rho$, given by $[s]_\psi \mapsto [s]_\rho$, is a (P')-cover, where $\psi = \psi(H)$ is the right congruence on S generated by H,

$$H = \{(p, q) \in R \times R \mid \exists z \in S;\ pz = qz\} \cup \{(p, 1) \mid p \in R\}$$

and $R = [1]_\sigma$.

In the above remark we see that $\psi \subseteq \sigma$, but they need not to be equal.

Lemma 2.13. *If S is a monoid, K_S is a proper right ideal of S and ρ_{K_S} is the Rees congruence on S then for any $\sigma \subseteq \rho_{K_S}$, $S/\sigma \longrightarrow S/\rho_{K_S}$ is a coessential epimorphism.*

Proof. Since $\sigma \subseteq \rho_{K_S}$, then $S/\sigma \longrightarrow S/\rho_{K_S}$, given by $[s]_\sigma \mapsto [s]_{\rho_{K_S}}$, is a well-defined epimorphism. For any $u \in [1]_{\rho_{K_S}}$, $uS \cap [1]_\sigma \neq \emptyset$. Hence, by [10, Theorem 2.7], $S/\sigma \longrightarrow S/\rho_{K_S}$ is a coessential epimorphism. $\square$

By Theorem 3.7 of [12], each indecomposable act satisfying Condition (P) is locally cyclic. A question that could be brought up is whether this is valid for indecomposable acts satisfying Condition (P'). The following example answers this question negatively. We do not know what the structure of indecomposable acts satisfying Condition (P') is, and we leave it as an open problem.

Example 2.14. Let $S = (\mathbb{N}, \cdot)$, $A_S = \mathbb{N} \backslash \{1\}$ with multiplication by natural numbers as its action. It is easily seen that A_S dose not satisfy Condition (P), but it satisfies Condition (P'). Since there is no $z \in A_S$ such that $2\mathbb{N} \cup 3\mathbb{N} \subseteq z\mathbb{N}$, A_S is not locally cyclic. By the fact that the set of all prime numbers is a generating set for A_S, we conclude that A_S is indecomposable.

Theorem 2.15. *Let S be a monoid. Then every indecomposable S-act satisfying Condition (P') is locally cyclic if S satisfies any of the following:*

(1) *S has a right zero.*
(2) *S is a group.*
(3) *S is a right nil monoid.*
(4) *S is a commutative aperiodic monoid.*
(5) *S is a commutative idempotent monoid.*

Proof. (1) It follows from Remark 2.7 (1) and the discussion provided before Example 2.14.

(2) It follows from Lemma 2.3 in [9].

(3) Let A be an indecomposable S-act satisfying Condition (P') and S be a right nil monoid. Let $a, a' \in A$. Since A is indecomposable, there exists a set of equations

$$\begin{array}{rcl} a & = & a_1u_1 \\ a_1v_1 & = & a_2u_2 \\ & \cdots & \\ a_mv_m & = & a'. \end{array}$$

Since S is right nil, there exists $n_1 \in \mathbb{N}$ such that $u_1^{n_1}$ is a right zero of S, so $1u_1^{n_1} = u_1u_1^{n_1}$. From $a = a_1u_1$ and $u_1^{n_1} = u_1u_1^{n_1}$ we conclude that there exist $b_1 \in A_S$, $s_1, t_1 \in S$ with $a = b_1s_1$, $a_1 = b_1t_1$ and $s_1 = t_1u_1$, since A satisfies Condition (P'). Hence $(b_1t_1)v_1 = a_2u_2$ and, since u_2 is a right nilpotent element of S, there exists $n_2 \in \mathbb{N}$ such that $u_2^{n_2}$ is a right zero of S, so $(t_1v_1)u_2^{n_2} = u_2u_2^{n_2}$, and hence there exist $b_2 \in A_S$, $s_2, t_2 \in S$ with $b_1 = b_2s_2$, $a_2 = b_2t_2$ and $s_2t_1v_1 = t_2u_2$. Continuing in this way, we deduce that there exists $n_{m+1} \in \mathbb{N}$ such that $t_mv_mv_m^{n_{m+1}} = 1v_m^{n_{m+1}}$, and hence there exist $b_{m+1} \in A_S$, $s_{m+1}, t_{m+1} \in S$ with $b_m = b_{m+1}s_{m+1}$, $a' = b_{m+1}t_{m+1}$ and $s_{m+1}(t_mv_m) = t_{m+1}$. Consequently,

$$a = b_1s_1 = b_2s_2s_1 = \cdots = b_ms_m \cdots s_2s_1 = b_{m+1}s_{m+1}s_m \cdots s_2s_1$$

and $a' = b_{m+1}t_{m+1}$, as required.

(4) Let A be an indecomposable S-act satisfying Condition (P') and S be a commutative aperiodic monoid. Let $a, a' \in A$. Since A is indecomposable, there exists a set of equations

$$\begin{array}{rcl} a & = & a_1u_1 \\ a_1v_1 & = & a_2u_2 \\ & \cdots & \\ a_mv_m & = & a'. \end{array}$$

Since S is aperiodic, there exists $n_1 \in \mathbb{N}$ such that $u_1^{n_1} = u_1^{n_1+1}$, and so $1u_1^{n_1} = u_1u_1^{n_1}$. From $a = a_1u_1$ and $u_1^{n_1} = u_1u_1^{n_1}$, we conclude that there exist $b_1 \in A_S$, $s_1, t_1 \in S$ with $a = b_1s_1$, $a_1 = b_1t_1$ and $s_1 = t_1u_1$, since A satisfies Condition (P'). Hence $(b_1t_1)v_1 = a_2u_2$ and, since u_2 and t_1v_1 are aperiodic elements of S, there exist $n_2, n_3 \in \mathbb{N}$ such that $u_2^{n_2} = u_2^{n_2+1}$ and $(t_1v_1)^{n_3} = (t_1v_1)^{n_3+1}$, so $t_1v_1(u_2^{n_2}(t_1v_1)^{n_3}) = u_2(u_2^{n_2}(t_1v_1)^{n_3})$, and hence there exist $b_2 \in A_S$, $s_2, t_2 \in S$ with $b_1 = b_2s_2$, $a_2 = b_2t_2$ and $s_2t_1v_1 = t_2u_2$. Continuing in this way, we get $b_{m+1} \in A_S$, $s_{m+1}, t_{m+1} \in S$ with $b_m = b_{m+1}s_{m+1}$, $a' = b_{m+1}t_{m+1}$ and $s_{m+1}(t_mv_m) = t_{m+1}$. Consequently,

$$a = b_1s_1 = b_2s_2s_1 = \cdots = b_ms_m \cdots s_2s_1 = b_{m+1}s_{m+1}s_m \cdots s_2s_1$$

and $a' = b_{m+1}t_{m+1}$, as required.

(5) Since every idempotent monoid is aperiodic, then the proof is similar to (4). □

3. $\mathcal{P}'$-covers

In this section we study Enochs' notion of cover in the category of acts over monoids and focus on $\mathcal{P}'$-covers where $\mathcal{P}'$ is the class of S-acts satisfy Condition (P').

It is easy to show that $\mathcal{P}'$-covers, when they exist, are unique up to isomorphism, whereas this is not true, in general, for (P')-covers by Remark 2.12. Now it follows that not every $\mathcal{P}'$-cover is a (P')-cover.

The following theorem illustrates a close relationship between covers and precovers.

Theorem 3.1. *Let A be an S-act and $\mathcal{X}$ be a class of S-acts. Then the $\mathcal{X}$-cover of A, if it exists, is a retract of any $\mathcal{X}$-precover of A.*

Proof. Let $X \longrightarrow A$ be the $\mathcal{X}$-cover and $X' \longrightarrow A$ be an $\mathcal{X}$-precover. Since X and X' are precovers then there exist S-morphisms $g : X \to X'$ and $f : X' \to X$ such that the following diagram is commutative.

$$\begin{tikzcd}
X \arrow[d, dashed, "g"'] \arrow[dr] & \\
X' \arrow[r] \arrow[d, dashed, "f"'] & A \\
X \arrow[ur] &
\end{tikzcd}$$

But then $X \xrightarrow{g} X' \xrightarrow{f} X$ is an automorphism. So fg has a right inverse h, and so $f(gh) = \mathrm{id}_X$. We conclude that X is a retract of X'. □

Recall that $\mathcal{CP}$-covers are coessential-covers that satisfy Condition (P) and considered by Mahmoudi and Renshow in [10].

Lemma 3.2. *If A is a right S-act and $g : X \longrightarrow A$ is a $\mathcal{P}'$-cover, where X satisfies Condition (P), then X is a $\mathcal{CP}$-cover.*

Proof. If X' is an S-act satisfies Condition (P) and $f : X' \longrightarrow A$ is an S-morphism then X' satisfies Condition (P') and so there exists $h : X' \longrightarrow X$ with $gh = f$ and so $g : X \longrightarrow A$ is a $\mathcal{CP}$-cover. □

The concept of directed colimits in the category of S-acts, $\mathbf{Act} - S$, is identical to that in the category of R-modules, where R is a ring with identity. We refer the reader to [13] for more details.

Let I be a set with a preorder (that is, a reflexive and transitive relation). A *direct system* is a collection of S-acts $(X_i)_{i \in I}$ together with S-morphisms $\phi_{i,j} : X_i \longrightarrow X_j$ for all $i \leq j \in I$ such that

1. $\phi_{i,i} = 1_{X_i}$, for all $i \in I$; and
2. $\phi_{j,k}\phi_{i,j} = \phi_{i,k}$ whenever $i \leq j \leq k$.

The *colimit* of the system $(X_i, \phi_{i,j})$ is an S-act X together with S-morphisms $\alpha_i : X_i \longrightarrow X$ such that

1. $\alpha_j \phi_{i,j} = \alpha_i$, whenever $i \leq j$, and
2. If Y is an S-act and $\beta_i : X_i \longrightarrow Y$, $i \in I$, are S-morphisms such that $\beta_j \phi_{i,j} = \beta_i$ whenever $i \leq j$, then there exists a unique S-morphism $\psi : X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd}
X_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_i"'] & X \arrow[dl, dashed, "\psi"] \\
Y &
\end{tikzcd}$$

commutes for all $i \in I$.

If the indexing set I satisfies the property that for all $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$ then we say that I is *directed*. In this case we call the colimit a *directed colimit*.

In what follows, we shall show that the directed colimit and coproduct of acts that satisfy Condition (P'), satisfies Condition (P'), too. For this purpose we will use the following basic property of directed colimits. For more details we refer the reader to [13].

Theorem 3.3 ([1, Theorem 2.2]). *Let S be a monoid, let $(X_i, \phi_{i,j})$ be a direct system of S-acts with directed index set I and let X be an S-act and $\alpha_i : X_i \longrightarrow X$ be S-morphisms such that*

$$\begin{tikzcd} X_i \arrow[rr, "\phi_{i,j}"] \arrow[dr, "\alpha_i"'] & & X_j \arrow[dl, "\alpha_j"] \\ & X & \end{tikzcd}$$

commutes for all $i \leq j$ in I. Then (X, α_i) is the directed colimit of $(X_i, \phi_{i,j})$ if and only if

1. *for all $x \in X$ there exists $i \in I$ and $x_i \in X_i$ such that $x = \alpha_i(x_i)$,*
2. *for all $i, j \in I$, $\alpha_i(x_i) = \alpha_j(x_j)$ if and only if $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$ for some $k \geq i, j$.*

The proof of the following result is easy, because of Theorem 3.3.

Theorem 3.4. *Let S be a monoid. Every directed colimit of a direct system of acts that satisfy Condition (P'), satisfies Condition (P').*

Proof. Let $(X_i, \phi_{i,j})$ be a direct system of S-acts satisfying Condition (P') with directed index set and with directed colimit (X, α_i). Suppose that $xs = x't$ in X and $sz = tz$ for some $z \in S$. Then, there exists $x_i \in X_i$, $x_j \in X_j$ with $x = \alpha_i(x_i)$, $x' = \alpha_j(x_j)$. Then, by Theorem 3.3, there exists $k \geq i, j$ with $\phi_{i,k}(x_i)s = \phi_{j,k}(x_j)t$ in X_k. Consequently, there exist $y \in X_k$, $u, v \in S$ with $\phi_{i,k}(x_i) = yu$, $\phi_{j,k}(x_j) = yv$, $us = vt$. Hence,

$$x = \alpha_i(x_i) = \alpha_k(\phi_{i,k}(x_i)) = \alpha_k(yu) = \alpha_k(y)u.$$

In a similar way, $x' = \alpha_k(y)v$, and the result follows. $\square$

It is a direct consequence of the definition that the following lemma holds.

Lemma 3.5. *Let S be a monoid and let $X = \dot{\bigcup}_{i \in I} X_i$ be the coproduct of S-acts $(X_i)_{i \in I}$. Then X satisfies Condition (P') if and only if each X_i satisfies Condition (P').*

It is clear that a necessary condition for an S-act A to have an $\mathcal{X}$-precover is that there exists $X \in \mathcal{X}$ with $\hom_S(X, A) \neq \emptyset$. This condition is always satisfied in the category of modules over a ring (or indeed any category with a zero object), as every hom-set is always non-empty, but this is not always the case for S-acts.

Now, let $\mathcal{X}$ be the class $\mathcal{P}'$. We know, from [7, Theorem II.3.3], that every S-act A is a surjective image of a free S-act and every free S-act is isomorphic to $\dot{\bigcup}_{i \in I} S_i$, where $S_i \cong S_S$ for all $i \in I$, for some non-empty set I ([7, Theorem I.5.13]) and it is clear that $\dot{\bigcup}_{i \in I} S_i \in \mathcal{P}'$. Hence, by the above discussion and Lemma 3.5, conditions (1) and (2) of [1, Theorem 4.14] hold. It remains to provide situations under which condition (3) of [1, Corollary 4.14] holds.

Theorem 3.6. *Let S be a right cancellative monoid, Then, every S-act has a $\mathcal{P}'$-cover.*

Proof. By Theorem 3.4 and [1, Theorem 4.11], if an S-act has a $\mathcal{P}'$-precover then it has a $\mathcal{P}'$-cover. By the above discussion and [1, Corollary 4.14], we just prove that every indecomposable S-act which satisfies Condition (P') has a bound on its cardinality. Since every S-act which satisfies Condition (P') is torsion free, the result follows by [2, Theorem 5.4]. $\square$

Acknowledgment. The authors would like to thank the referees for their useful and valuable comments. Also, the authors gratefully acknowledges the many helpful suggestions of Professor M. Mehdi Ebrahimi during the preparation of this work. They would also like to thank the Department of Mathematics of Hakim Sabzevari University for their kind hospitality during their two weeks stay in summer there (2012).

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