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Research Article

Cartan-Eilenberg Ding projective complexes

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Abstract

In this article, Cartan-Eilenberg Ding projective complexes are introduced and investigated. It is shown that a complex C is Cartan-Eilenberg Ding projective if and only if C_n and $C_n/\mathrm{B}_n(C)$ are Ding projective in R-Mod for each $n \in \mathbb{Z}$ when R is a Ding-Chen ring. Some applications are also given.

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1. Introduction and Preliminaries

In classical homological algebra, the projective and injective modules play important and fundamental roles. In Chapter XVII of Homological Algebra, Cartan and Eilenberg [1] gave the definitions of projective and injective resolutions of a complex of modules. Subsequently, Verdier considered these resolutions and called them Cartan-Eilenberg projective and injective resolutions of a complex. Also, the definitions of Cartan-Eilenberg injective, projective and flat complexes were introduced [11].

Recently, Enochs studied Cartan-Eilenberg projective and injective complexes, Cartan-Eilenberg Gorenstein injective complexes are also introduced and studied [5]. We also considered Cartan-Eilenberg FP-injective complexes in [9]. In this paper, our main purpose is to introduce and investigate the concept of Cartan-Eilenberg Ding projective complexes.

It is an important question to establish relationships between a complex X and the modules $X_n, \mathrm{Z}_n(X), \mathrm{B}_n(X)$ and $\mathrm{H}_n(X)$, $n \in \mathbb{Z}$. As we know, a complex C is Gorenstein projective if and only if C_n is Gorenstein projective in R-Mod for $n \in \mathbb{Z}$ (see [12]) and a complex C is Ding projective if and only if C_n is Ding projective in R-Mod for $n \in \mathbb{Z}$ and $\mathrm{Hom}_R(C, F)$ is exact for all flat complexes F (see [13]). In [5], it was shown that a complex C is Cartan-Eilenberg Gorenstein injective if and only if $\mathrm{B}_n(X)$ and $\mathrm{H}_n(X)$ are Gorenstein injective in R-Mod for $n \in \mathbb{Z}$.

Recall from [10] that a left R-module E is called FP-*injective* if $\mathrm{Ext}^1_R(F, E) = 0$ for all finitely presented left R-modules F. More generally, the FP-*injective dimension* of a left R-module N is defined to be the least integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}_R(F, N) = 0$

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for all finitely presented left R-modules F. The FP-injective dimension of N is denoted FP-id(N) and equals ∞ if no such n above exists.

A ring R is called an *n-FC ring* if it is both left and right coherent and FP-id$({}_RR)$ and FP-id(R_R) are both less than or equal to n, see [2,3]. A ring R is called *Ding-Chen* (which is renamed by Gillespie [7]) if it is an n-FC ring for some non-negative integer n.

In the following, R will be a Ding-Chen ring. Our main result in this note can be stated as follows (cf. Theorem 2.13 and Proposition 2.15).

Theorem 1.1. *Let C be a complex.*
(1) *Then C is Cartan-Eilenberg Ding projective if and only if C_n and $C_n/\mathrm{B}_n(C)$ are Ding projective in R-*Mod *for each $n \in \mathbb{Z}$.*
(2) *If $\mathrm{Z}_i(C)$ and $\mathrm{B}_i(C)$ have finite flat dimension in R-*Mod *for each $i \in \mathbb{Z}$, then C is Cartan-Eilenberg Ding projective if and only if C is DG-Ding projective with $\mathrm{H}_i(C)$ Ding projective in R-*Mod *for each $i \in \mathbb{Z}$.*

As an application of Theorem 1.1, we get the following observation which establishes a relationship between Cartan-Eilenberg Ding projective complexes and Ding projective complexes (cf. Corollary 3.3).

Corollary 1.2. *Let G be a complex. Then the following statements are equivalent:*
(1) *G is Cartan-Eilenberg Ding projective.*
(2) *G is Ding projective and $G_i/\mathrm{B}_i(G)$ is Ding projective in R-*Mod *for each $i \in \mathbb{Z}$.*

A complex C is Ding projective if and only if $\mathrm{Z}_n(C)$ is Ding projective in R-Mod for $n \in \mathbb{Z}$ whenever C is an exact complex such that this complex remains exact when $\mathrm{Hom}_R(-, F)$ is applied to it for any flat R-module F, see [13]. As a direct consequence of Theorem 1.1, we have the following result (cf. Corollary 3.7).

Corollary 1.3. *Let G be an exact complex with $\mathrm{Z}_i(G)$ finite flat dimension in R-*Mod *for each $i \in \mathbb{Z}$. Then the following statements are equivalent*:
(1) *G is Ding projective.*
(2) *$\mathrm{Z}_n(G)$ is Ding projective in R-*Mod *for each $n \in \mathbb{Z}$.*

For the rest of the paper we will use the abbreviation C-E for Cartan-Eilenberg.

Throughout this paper, R denotes a ring with unity. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of R-modules will be denoted by (C, δ) or C.

We will use superscripts to distinguish complexes. So if $\{C^i\}_{i \in I}$ is a family of complexes, C^i will be

$$\cdots \xrightarrow{\delta_2} C^i_1 \xrightarrow{\delta_1} C^i_0 \xrightarrow{\delta_0} C^i_{-1} \xrightarrow{\delta_{-1}} \cdots .$$

Given a left R-module M, we use the notation $D^m(M)$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the mth and $(m-1)$th positions. We also use the notation $S^m(M)$ to denote the complex with M in the mth place and 0 in the other places.

Given a complex C and an integer l, the lth homology module of C is the module $\mathrm{H}_l(C) = \mathrm{Z}_l(C)/\mathrm{B}_l(C)$ where $\mathrm{Z}_l(C) = \mathrm{Ker}(\delta^C_l)$ and $\mathrm{B}_l(C) = \mathrm{Im}(\delta^C_{l+1})$.

Let C and D be complexes of left R-modules. We will denote by $\mathrm{Hom}_R(C, D)$ the complex of abelian groups with $\mathrm{Hom}_R(C, D)_n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}(C_t, D_{n+t})$ and such that if $f \in \mathrm{Hom}_R(C, D)_n$ then $(\delta_n(f))_m = \delta^D_{m+n} f_m - (-1)^n f_{m+1} \delta^D_m$. f is called a *chain map* of degree n if $\delta_n(f) = 0$. A chain map of degree 0 is called a *morphism*. We will use $\mathrm{Hom}(C, D)$ to denote the abelian group of morphisms from C to D and Ext^i for $i \geq 0$ will denote the groups we get from the right derived functor of Hom.

General background materials can be found in [8].

For a ring R, R-Mod denotes the category of left R-modules, $\mathcal{C}(R)$ denotes the abelian category of complexes of left R-modules.

Definition 1.4 ([5, Definition 3.1]). A complex P is said to be C-E projective if $P, \mathrm{Z}(P), \mathrm{B}(P)$ and $\mathrm{H}(P)$ are complexes consisting of projective modules.

A complex I is said to be C-E injective if $I, \mathrm{Z}(I), \mathrm{B}(I)$ and $\mathrm{H}(I)$ are complexes consisting of injective modules.

A complex F is said to be C-E flat if $F, \mathrm{Z}(F), \mathrm{B}(F)$ and $\mathrm{H}(F)$ are complexes consisting of flat modules.

Definition 1.5 ([5, Definition 5.3]). A sequence of complexes

$$\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$$

is said to be C-E exact if
(1) $\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots,$
(2) $\cdots \to \mathrm{Z}(C^{-1}) \to \mathrm{Z}(C^0) \to \mathrm{Z}(C^1) \to \cdots,$
(3) $\cdots \to \mathrm{B}(C^{-1}) \to \mathrm{B}(C^0) \to \mathrm{B}(C^1) \to \cdots,$
(4) $\cdots \to C^{-1}/\mathrm{Z}(C^{-1}) \to C^0/\mathrm{Z}(C^0) \to C^1/\mathrm{Z}(C^1) \to \cdots,$
(5) $\cdots \to C^{-1}/\mathrm{B}(C^{-1}) \to C^0/\mathrm{B}(C^0) \to C^1/\mathrm{B}(C^1) \to \cdots,$
(6) $\cdots \to \mathrm{H}(C^{-1}) \to \mathrm{H}(C^0) \to \mathrm{H}(C^1) \to \cdots$
are all exact.

By [5, Proposition 6.3], we can compute derived functors of $\mathrm{Hom}(-,-)$ using C-E projective resolutions or C-E injective resolutions. For given C and D we will denote these derived functors applied to (C, D) as $\overline{\mathrm{Ext}}^n(C, D)$. It is obvious that $\overline{\mathrm{Ext}}^n(C, D) \subseteq \mathrm{Ext}^n(C, D)$.

Definition 1.6 ([4, Definition 2.1]). A left R-module M is called Ding projective (or strongly Gorenstein flat), if there exists an exact sequence of projective left R-modules

$$\mathbb{P} : \cdots \to P_{-1} \to P_0 \to P_1 \to P_2 \to \cdots$$

with $M \cong \mathrm{Im}(P_0 \to P_1)$ and such that the functor $\mathrm{Hom}_R(-, F)$ leaves $\mathbb{P}$ exact whenever F is flat. In this case, we say that $\mathbb{P}$ is a strongly complete projective resolution of M.

2. Cartan-Eilenberg Ding projective complexes

We start with the following definition.

Definition 2.1. A complex G is said to be C-E Ding projective if there is a C-E exact sequence of C-E projective complexes

$$\mathbb{P} : \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$$

with $G \cong \mathrm{Ker}(P^0 \to P^1)$ and such that the functor $\mathrm{Hom}(-, F)$ leaves $\mathbb{P}$ exact whenever F is C-E flat.

Proposition 2.2. *If G is a* C-E *Ding projective complex, then $G_n/\mathrm{Z}_n(G)$ and $\mathrm{H}_n(G)$ are Ding projective in R*-Mod *for all $n \in \mathbb{Z}$.*

Proof. It is similar to the proof of [5, Theorem 8.5]. □

Remark 2.3. (1) According to Proposition 2.2 and [14, Theorem 2.6], if G is a C-E Ding projective complex, then $G_n, \mathrm{Z}_n(G), \mathrm{B}_n(G), \mathrm{H}_n(G), G_n/\mathrm{Z}_n(G)$ and $G_n/\mathrm{B}_n(G)$ are Ding projective in R-Mod for all $n \in \mathbb{Z}$.
(2) If $\mathbb{P} =: \cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$ is a $\mathrm{Hom}(-, F)$ exact C-E exact sequence of C-E projective complexes for any C-E flat complex F, then by symmetry, all the images, the kernels and the cokernels of $\mathbb{P}$ are C-E Ding projective.

Let C be a complex. Then a C-E flat resolution of C we mean a complex of complexes

$$\cdots \to F^2 \to F^1 \to F^0 \to C \to 0,$$

where each F^n is a C-E flat complex, and $F^0 \to C$, $F^1 \to \mathrm{Ker}(F^0 \to C)$ and $F^n \to \mathrm{Ker}(F^{n-1} \to F^{n-2})$ for $n \geq 2$ are C-E flat precovers.

A complex of left R-modules C is said to have C-E flat dimension at most n (denoted $\mathrm{CEfd}C \leq n$) if there is a C-E flat resolution of the form

$$0 \to F^n \to F^{n-1} \to \cdots \to F^1 \to F^0 \to C \to 0$$

of C. If n is the least, then we set $\mathrm{CEfd}C = n$ and if there is no such n, we set $\mathrm{CEfd}C = \infty$.

Lemma 2.4 ([5, Proposition 10.1 and its dual version])**.** (1) *For a complex C the equality $\overline{\mathrm{Ext}}^1(C, -) = \mathrm{Ext}^1(C, -)$ holds if and only if C is exact;*
(2)*For a complex C the equality $\overline{\mathrm{Ext}}^1(-, C) = \mathrm{Ext}^1(-, C)$ holds if and only if C is exact.*

A routine proof gives the following lemma using Lemma 2.4.

Lemma 2.5. *If F is a complex of finite* C-E *flat dimension and G is* C-E *Ding projective, then $\overline{\mathrm{Ext}}^n(G, F) = 0$ for each $n \geq 1$. Moreover,* $\mathrm{Ext}^1(G, F) = 0$ *whenever G is exact.*

Lemma 2.6. *Let G be a complex with G_i a Ding projective module for each $i \in \mathbb{Z}$. Then* $\mathrm{Hom}_R(G, F)$ *is exact for any* C-E *flat complex F if and only if* $\mathrm{Ext}^1(G, F) = 0$ *for any* C-E *flat complex F.*

Proof. It follows from [6, Lemma 2.1]. □

Definition 2.7 ([6, Definition 3.3])**.** Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod and X a complex.
(1) X is called an $\mathcal{A}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{A}$ for $n \in \mathbb{Z}$.
(2) X is called a $\mathcal{B}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{B}$ for $n \in \mathbb{Z}$.
(3) X is called a DG-$\mathcal{A}$ complex if $X_n \in \mathcal{A}$ for $n \in \mathbb{Z}$, and $\mathrm{Hom}_R(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.
(4) X is called a DG-$\mathcal{B}$ complex if $X_n \in \mathcal{B}$ for $n \in \mathbb{Z}$, and $\mathrm{Hom}_R(A, X)$ is exact whenever A is a $\mathcal{A}$ complex.

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of DG-$\mathcal{A}$ complexes by $DG\widetilde{\mathcal{A}}$. Similarly, the class of $\mathcal{B}$ complexes is denoted by $\widetilde{\mathcal{B}}$ and the class of DG-$\mathcal{B}$ complexes is denoted by $DG\widetilde{\mathcal{B}}$.

Lemma 2.8 ([6, Proposition 3.6])**.** *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R*-Mod*. Then $(\widetilde{\mathcal{A}}, DG\widetilde{\mathcal{B}})$ and $(DG\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are cotorsion pairs in $\mathcal{C}(R)$.*

$(\mathcal{DP}, \mathcal{W})$ is a complete hereditary cotorsion pair in R-Mod over Ding-Chen rings, where $\mathcal{DP}$ and $\mathcal{W}$ denote the class of Ding projective left R-modules and the class of left R-modules of finite flat dimension, respectively [7]. Taking $\mathcal{A} = \mathcal{DP}$ in Definition 2.7, we get DG-Ding projective complexes.

We define the flat dimension of a complex C to be the least integer $n \geq 0$ such that $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to C \to 0$ is exact. The flat dimension of C is denoted $\mathrm{fd}(C)$ and equals ∞ if no such n above exists. Then we get $\mathrm{fd}(C) \leq n$ if and only if C is exact and $\mathrm{fd}_R(\mathrm{Z}_i(C)) \leq n$, where $\mathrm{fd}_R(\mathrm{Z}_i(C))$ denotes the flat dimension of R-modules $\mathrm{Z}_i(C)$.

In the following R will be a Ding-Chen ring.

Lemma 2.9. *Let X be a complex. Then X_n and $X_n/\mathrm{B}_n(X)$ are Ding projective in R*-Mod *if and only if $\overline{\mathrm{Ext}}^1(X, Y) = 0$ for every complex Y with Y_n and $Y_n/\mathrm{B}_n(Y)$ finite flat dimension.*

Proof. It follows by [5, Theorem 9.4]. □

As a direct consequence of Lemma 2.8, we get the following result.

Lemma 2.10. *A complex X is DG-Ding projective if and only if* $\mathrm{Ext}^1(X,Y)=0$ *every complex Y with finite flat dimension.*

Lemma 2.11. *Let X be a complex. If X_n and $X_n/\mathrm{B}_n(X)$ are Ding projective in R-*Mod*, then X is DG-Ding projective.*

Proof. Since $\overline{\mathrm{Ext}}^1(X,Y)=0$ for every complex Y with Y_n and $Y_n/\mathrm{B}_n(Y)$ finite flat dimension by Lemma 2.9, $\overline{\mathrm{Ext}}^1(X,Y)=0$ for every exact complex Y with $\mathrm{Z}_n(Y)$ finite flat dimension. And so $\mathrm{Ext}^1(X,Y)=0$ for every exact complex Y with $\mathrm{Z}_n(Y)$ finite flat dimension by Lemma 2.4. Thus X is DG-Ding projective. □

Lemma 2.12. *Let F be a* C-E *flat complex. Then $F=K\oplus L$, where K is a flat complex and L is a graded module with L_i flat for each $i\in\mathbb{Z}$.*

Proof. By [5, Proposition 3.4 and Theorem 7.2], $F=\varinjlim(P\oplus Q)^i=\varinjlim P^i\oplus\varinjlim Q^i$, where P^i is a projective complex for each $i\in\mathbb{Z}$, Q^i is a graded module with Q^i_t projective for each $i,t\in\mathbb{Z}$. Take $K=\varinjlim P^i$ and $L=\varinjlim Q^i$. Then K is a flat complex, L is a graded module with L_i flat for each $i\in\mathbb{Z}$. □

Theorem 2.13. *Let C be a complex. Then the following conditions are equivalent*:
(1) *C is* C-E *Ding projective.*
(2) *C is DG-Ding projective and $C_n/\mathrm{B}_n(C)$ is Ding projective in R-*Mod *for each $n\in\mathbb{Z}$.*
(3) *C_n and $C_n/\mathrm{B}_n(C)$ are Ding projective in R-*Mod *for each $n\in\mathbb{Z}$.*

Proof. (2) ⇔ (3) is obvious by Lemma 2.11.

(1) ⇒ (3) follows by Remark 2.3.

(3) ⇒ (1). Note that $C_n/\mathrm{Z}_n(C)$ and $\mathrm{H}_n(C)$ are Ding projective for all $n\in\mathbb{Z}$ by [14, Theorem 2.6]. Then $C_n/\mathrm{Z}_n(C)$ and $\mathrm{H}_n(C)$ have strongly complete projective resolutions. Thus C_n has a strongly complete projective resolution since $0\to\mathrm{H}_n(C)\to C_n/\mathrm{B}_n(C)\to C_n/\mathrm{Z}_n(C)\to 0$ and $0\to\mathrm{B}_n(C)\to C_n\to C_n/\mathrm{B}_n(C)\to 0$ are exact.

Suppose $P^{C_n/\mathrm{Z}_n(C)}$ and $P^{\mathrm{H}_n(C)}$ are strongly complete projective resolutions of $C_n/\mathrm{Z}_n(C)$ and $\mathrm{H}_n(C)$, respectively. By the Horseshoe Lemma, we can construct a strongly complete projective resolution of $C_n/\mathrm{B}_n(C)$: $P^{C_n/\mathrm{B}_n(C)}=P^{C_n/\mathrm{Z}_n(C)}\oplus P^{\mathrm{H}_n(C)}$. Similarly, consider the exact sequence of modules

$$0\to\mathrm{B}_n(C)\to C_n\to C_n/\mathrm{B}_n(C)\to 0,$$

and we can construct a strongly complete projective resolution of C_n: $P^{C_n}=P^{\mathrm{B}_n(C)}\oplus P^{C_n/\mathrm{B}_n(C)}=P^{\mathrm{B}_n(C)}\oplus P^{C_n/\mathrm{Z}_n(C)}\oplus P^{\mathrm{H}_n(C)}$. We notice that $C_n/\mathrm{Z}_n(C)\cong\mathrm{B}_{n-1}(C)$. Then $P^{C_n}=P^{\mathrm{B}_n(C)}\oplus P^{\mathrm{B}_{n-1}(C)}\oplus P^{\mathrm{H}_n(C)}$. Set $P^i_n=P_i^{\mathrm{B}_n(C)}\oplus P_i^{\mathrm{B}_{n-1}(C)}\oplus P_i^{\mathrm{H}_n(C)}$ and $d_n^{P^i}:P^i_n\to P^i_{n-1}$ which maps (x,y,z) to $(y,0,0)$ for all $i,n\in\mathbb{Z}$.

By construction, it is easily seen that P^i is a C-E projective complex for all $i\in\mathbb{Z}$ and $C=\mathrm{Ker}\,(P^{-1}\to P^0)$. For any $n\in\mathbb{Z}$,

$$\cdots\to P^{-1}_n/\mathrm{B}_n(P^{-1})\to P^0_n/\mathrm{B}_n(P^0)\to P^1_n/\mathrm{B}_n(P^1)\to\cdots$$

is a strongly complete projective resolution of $C_n/\mathrm{B}_n(C)$ with

$$C_n/\mathrm{B}_n(C)=\mathrm{Ker}(P^0_n/\mathrm{B}_n(P^0)\to P^1_n/\mathrm{B}_n(P^1)),$$

and

$$\cdots\to P^{-1}_n\to P^0_n\to P^1_n\to\cdots$$

is a strongly complete projective resolution of C_n with $C_n=\mathrm{Ker}(P^0_n\to P^1_n)$, so they both are exact. Hence, we can get that

$$\mathbb{P}=\cdots\longrightarrow P^{-1}\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots \tag{1}$$

is C-E exact. Using Lemma 2.9, $\mathrm{Hom}(-, F)$ leaves the C-E exact sequence (1) exact when F is C-E flat. Therefore, C is a C-E Ding projective complex. □

Example 2.14. (1) If P is a C-E projective complex, then P is C-E Ding projective by Definition 2.1.

(2) Consider the quasi-Frobenius local ring $R = k[X]/(X^2)$ where k is a field. Then

$$P =: \cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to \cdots$$

is a strongly complete projective resolution in R-Mod and $\mathrm{Z}_i(P)$ is not projective in R-Mod. So P is a C-E Ding projective complex by Theorem 2.13 and P is not a C-E projective complex.

Proposition 2.15. *Let C be a complex with $\mathrm{Z}_i(C)$ and $\mathrm{B}_i(C)$ finite flat dimension in R-*Mod *for each $i \in \mathbb{Z}$. Then C is* C-E *Ding projective if and only if C is DG-Ding projective with $\mathrm{H}_i(C)$ Ding projective in R-*Mod *for each $i \in \mathbb{Z}$.*

Proof. ($\Rightarrow$) It follows by Remark 2.3 and Theorem 2.13.

($\Leftarrow$) Let C be a *DG*-Ding projective complex with $\mathrm{H}_i(C)$ Ding projective in R-Mod for each $i \in \mathbb{Z}$. Then we get exact sequences

$$0 \to K_n \to P_n \to \mathrm{B}_n(C) \to 0$$

and

$$0 \to 0 \to \mathrm{H}_n(C) \to \mathrm{H}_n(C) \to 0$$

with $P_n \to \mathrm{B}_n(C)$ Ding projective precovers of $\mathrm{B}_n(C)$, K_n finite flat dimension in R-Mod for each $n \in \mathbb{Z}$.

Notice that $\mathrm{B}_i(C)$ has finite flat dimension in R-Mod for each $i \in \mathbb{Z}$ and $0 \to \mathrm{B}_n(C) \to \mathrm{Z}_n(C) \to \mathrm{H}_n(C) \to 0$ is an exact sequence for each $n \in \mathbb{Z}$. Then $0 \to \mathrm{Hom}_R(A, \mathrm{B}_n(C)) \to \mathrm{Hom}_R(A, \mathrm{Z}_n(C)) \to \mathrm{Hom}_R(A, \mathrm{H}_n(C)) \to 0$ is exact for any Ding projective left R-module A and each $n \in \mathbb{Z}$. Using the horseshoe lemma and snake lemma, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & K_n & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_n & \longrightarrow & P_n \oplus \mathrm{H}_n(C) & \longrightarrow & \mathrm{H}_n(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{B}_n(C) & \longrightarrow & \mathrm{Z}_n(C) & \longrightarrow & \mathrm{H}_n(C) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We also get the following commutative diagram with exact rows and columns by an argument analogous to the above.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & L_n & \longrightarrow & K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_n \oplus \mathrm{H}_n(C) & \longrightarrow & P_n \oplus \mathrm{H}_n(C) \oplus P_{n-1} & \longrightarrow & P_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Z}_n(C) & \longrightarrow & C_n & \longrightarrow & \mathrm{B}_{n-1}(C) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Set $D_n = P_n \oplus \mathrm{H}_n(C) \oplus P_{n-1}$ for each n. Then the map $d_n : D_n \to D_{n-1}$ is the composition $D_n \to P_{n-1} \to P_{n-1} \oplus \mathrm{H}_{n-1}(C) \to D_{n-1}$. So we get the C-E exact sequence $0 \to L \to D \to C \to 0$, where $D = (D_n)_{n\in\mathbb{Z}}$ with d given as above and the kernel of $D \to C$ is the complex L with $L = (L_n)_{n\in\mathbb{Z}}$.

Note that $0 \to K_n \to L_n \to K_{n-1} \to 0$ is exact and K_n has finite flat dimension in R-Mod for each $n \in \mathbb{Z}$. Then the complex L is a complex of finite flat dimension. This yields the exact sequence

$$\mathrm{Hom}(C, D) \to \mathrm{Hom}(C, C) \to 0,$$

which means $0 \to L \to D \to C \to 0$ is split. Then C is a C-E Ding projective complex since D is C-E Ding projective. □

3. Applications

In this section, R is a Ding-Chen ring. We will give some interesting results using Theorem 2.13 and Proposition 2.15.

Notice that $(\mathcal{DP}, \mathcal{W})$ is a hereditary cotorsion pair in R-Mod. As an immediate consequence of Lemma 2.10, Theorem 2.13 and [6, Corollary 3.13], we have the following result.

Corollary 3.1. *Let G be a complex. Then the following statements are equivalent*:
(1) *G is* C-E *Ding projective.*
(2) $\mathrm{Ext}^1(G, F) = 0$ *for every complex F of finite flat dimension and $G_i/\mathrm{B}_i(G)$ is Ding projective in R-*Mod *for each $i \in \mathbb{Z}$.*
(3) $\mathrm{Ext}^n(G, F) = 0$ *for every complex F of finite flat dimension and $n \geq 0$, and $G_i/\mathrm{B}_i(G)$ is Ding projective in R-*Mod *for each $i \in \mathbb{Z}$.*

Lemma 3.2. *A complex G is Ding projective if and only if* $\mathrm{Ext}^n(G, F) = 0$ *for every complex F of finite flat dimension and $n \geq 1$.*

Proof. It is clear by the definition of Ding projective complexes. □

A characterization of C-E Ding projective complexes is given, as a direct consequence of Corollary 3.1, Lemma 3.2 and [13, Theorem 3.7].

Corollary 3.3. *Let G be a complex. Then the following statements are equivalent:*
(1) *G is* C-E *Ding projective.*
(2) *G is Ding projective and $G_i/\mathrm{B}_i(G)$ is Ding projective in R-*Mod *for each $i \in \mathbb{Z}$.*
(3) *G_i, $G_i/\mathrm{B}_i(G)$ are Ding projective in R-*Mod *for each $i \in \mathbb{Z}$ and* $\mathrm{Hom}_R(G, F)$ *is exact for any flat complex F.*

As a direct corollary to Theorem 2.13, the following result also can be obtained.

Corollary 3.4. *Let G be an exact complex. Then G is* C-E *Ding projective if and only if* $\mathrm{Z}_i(G)$ *is Ding projective in* R-Mod *for all* $i \in \mathbb{Z}$.

Similarly, we get the following result using Proposition 2.15.

Corollary 3.5. *Let G be a complex with* $\mathrm{Z}_i(G)$ *and* $\mathrm{B}_i(G)$ *finite flat dimension in* R-Mod *for each* $i \in \mathbb{Z}$. *Then the following statements are equivalent*:
(1) *G is* C-E *Ding projective.*
(2) $\mathrm{Ext}^1(G, F) = 0$ *for every complex F of finite flat dimension and* $\mathrm{H}_i(G)$ *is Ding projective in* R-Mod *for each* $i \in \mathbb{Z}$.
(3) $\mathrm{Ext}^n(G, F) = 0$ *for every complex F of finite flat dimension and* $n \geq 0$, *and* $\mathrm{H}_i(G)$ *is Ding projective in* R-Mod *for each* $i \in \mathbb{Z}$.

Corollary 3.6. *Let G be a complex with* $\mathrm{Z}_i(G)$ *and* $\mathrm{B}_i(G)$ *finite flat dimension in* R-Mod *for each* $i \in \mathbb{Z}$. *Then the following statements are equivalent*:
(1) *G is* C-E *Ding projective.*
(2) *G is Ding projective and* $\mathrm{H}_i(G)$ *is Ding projective in* R-Mod *for each* $i \in \mathbb{Z}$.

Using Corollaries 3.4 and 3.6, the following corollary is obtained.

Corollary 3.7. *Let G be an exact complex with* $\mathrm{Z}_i(G)$ *finite flat dimension in* R-Mod *for each* $i \in \mathbb{Z}$. *Then the following statements are equivalent*:
(1) *G is Ding projective.*
(2) $\mathrm{Z}_i(G)$ *is Ding projective in* R-Mod *for each* $i \in \mathbb{Z}$.

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