

ISTANBUL MANUSCRIPTS OF AL-KHWÂRIZMÎ'S TREATISES

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The existence of Istanbul manuscripts of two astronomical treatises of Al-Khwârizmî (≈ 783 – ≈ 850) was pointed out for the first time by F. Sezgin in the sixth volume of his "Geschichte des arabischen Schrifttums" [1, p. 143]; enumerating the manuscripts of Al-Khwârizmî's astronomical works F. Sezgin mentioned the manuscript of "Witty ideas in the actions of Muḥammad ibn Mûsâ al-Khwârizmî: the determination of the azimuth using the astrolabe" and "The construction of hours on the plane sundial by Muḥammad ibn Mûsâ al-Khwârizmî" in the codex of the Süleymaniye Library of Istanbul, registered under Ayasofya No. 4830, f. 198b-199b and f. 231b-235a. One of the co-authors of this article J. ad-Dabbâgh in 1983 recopied these two treatises, translated them into Russian and published them with commentaries in [2, p. 216-234].

§ 1. *Codex Ayasofya 4830*

The codex No. 4830 of Ayasofya Library was studied by M. Krause in [3] and Y. Dold-Samplonius in [4]. This codex contains numerous unique manuscripts: the Arabic translation of "Book on the section of a line in a given ratio" of Apollonius (f. 2a-52b), three mathematical treatises of the famous philosopher Al-Kindî "Book on the supreme art", "Treatise on the determination of thought of numbers" and a treatise on the determination of distances to inaccessible objects (see [5, vol. 2, p. 67]) (f. 53a-86b, 204b-210b), an anonymous treatise on geometric algebra (f. 86b-89a), "The Book of Assumptions" of Aqâṭun, published with English translation in [4] (f. 89b-102b), a part of the "Spherics" of Theodosius (f. 102b), an anonymous treatise on the theory of numbers (f. 103a-108b), the treatise of Thâbit ibn Qurra on amicable numbers, translated into Russian by G. P.

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Matvievskaya from another manuscript [6] (see [5, vol. 2, p. 86-87]) (f. 110a-121b), eight mathematical, astronomical and logical treatises of Ahmad ibn as-Surā (see [5, vol. 2, p. 332-334]) (f. 122b-160a), eight geometrical treatises of Wījan al-Kūhī (see [5, vol. 2, p. 189-193]) (f. 161a-182a), the treatises we are going to talk about in more detail (f. 183a-199b and 228b-235a), the treatise of Muḥammad aṣ-Ṣabbāh on the sundial (see [5, vol. 2, p. 59]) (f. 200a-204a), the above-mentioned treatise of Al-Kindī on the determination of distances to inaccessible objects and treatises of Abū Bakr ibn ‘Ābis, An-Nairīzī and Ar-Rayyī on the same question (see [5, vol. 2, p. 117, 257 and 296]) (f. 210b-227b). The manuscript was copied in 1229—1232, but the majority of treatises in it were written in the 9-10th centuries the latest—the treatises of Ibn as-Surā—in the first half of the twelfth century. Neither Krause nor Dold-Samplonius mentioned the treatises of Al-Khwārizmī; Krause considers the treatises in f. 228b-235a to be an anonymous treatise.

2. *Al-Khwārizmī's treatises in the Codex Ayasofya 4830*

The name of Al-Khwārizmī appears only on the two treatises mentioned above. The fact that all the group of treatises that are in f. 182a-199b and f. 228b-235a of this codex belongs to Al-Khwārizmī was established by D. A. King [7] on the basis of closeness of the language and the topics studied in these treatises, to which it should be added such a distinctive feature for the time of Al-Khwārizmī as the value of the “the complete sine” (the radius of the trigonometric circle) equal to 150 which was borrowed from Brahmagupta (a similar thing exists in the Berlin manuscript of Al-Khwārizmī's treatises MS No. 5793 of the Library Preussischer Kulturbesitz (West Berlin) which contains treatises or parts of treatises of Al-Khwārizmī on the construction of astrolabe, on the use of astrolabes, on the sundial, and on the sine quadrant, from which the name of Al-Khwārizmī was written only in the treatise on the use of astrolabes, published in German translation by J. Frank [8] and in Russian translation by G. P. Matvievskaya [9, p. 255-266]—see [7, p. 23-31] and [2, p. 151-154].

D. A. King established as well the fact that the manuscript of the Al-Bīrūnī Institute of Oriental Studies of the Academy of Science of the Uzbek SSR (Tashkent) No. 177/3 (f. 148a-151a)[7,

p. 12-13] also was written by Al-Khwârizmî. He found a textual coincidence between this anonymous treatise and the Istanbul manuscript of Al-Khwârizmî on the determination of the qibla. The Tashkent manuscript was copied about 1250.

The manuscript of the Ayasofya Library, as well as the Berlin manuscript, is divided into "chapters" of different lengths. Following J. Frank in this publication of the Berlin manuscript we give a number to each of these chapters.

The first 19 chapters concern the determination of the azimuth of the sunrise (rising amplitude) in a given day and the determination of the azimuth of the Sun at a given instant. As we mentioned above, the text of Al-Khwârizmî's treatises is placed on f. 183a-199b and on f. 228b-235a of codex No. 4830 of the Ayasofya Library. However, chapters 1 and 2 and the beginning of chapter 3 are placed on f. 228a, while the end of chapter 3 and the following chapters are on f. 183a and the following folios. Chapter 1 is entitled: "The determination of the azimuth of the sunrise at any city according to what Ptolemy had done for the diameter of the celestial sphere to be equal to 120 degrees." The following chapters are: (2) "The determination of azimuth from altitude for the southern zodiacal signs" (f. 228b), (3) "The determination of the azimuths for the northern zodiacal signs" (f. 228b, 183a), (4) "The determination of the operation with the azimuth, the shadow, and the altitude" (f. 183a-183b), (5) "The determination of altitude from the equal as well as the temporal hours" (f. 183b), (6) "The determination of the azimuth" (f. 183b), (7) "Also the determination of the azimuth" (f. 183b-184a), (8) "The determination of the azimuth for every hour and every city" (f. 184a), (9) "The determination of the azimuth of sunrise of any city" (f. 184a), (10) "Another chapter [on this topic]" (f. 184a), (11) "Another chapter [on this topic]" (f. 184a), (12) "Another chapter [on this topic]" (f. 184a), (13) "The determination of the azimuth for the altitude of given magnitude" (f. 184a-184b), (14) "The procedure of determining the azimuth by another method" (f. 184b), (15) "The procedure of determining the azimuth of the sunrise" (f. 184b), (16) "The description of determining the azimuth from the ascendent" (f. 184b), (17) "The determination of the azimuth of any hour you need for any degree you wish from all zodiacal signs" (f. 184b-185b), (18) "The construction of the hours by the azimuth" (f. 185b-186a), (19) "Also the construction of the altitude for the hours" (f. 196a).

The following six chapters concern, mainly, the determination of the azimuth of the *qibla*. These are: (20) "The determination of the direction of the *qibla* in any city you wish" (f. 186a-186b), (21) "The determination of the meridian line" (f. 186b), (22) "The determination of the *qibla* also for any latitude" (f. 187a), (23) "Another chapter on the determination of the *qibla*" (f. 187a-187b), (24) "One more short chapter on the determination of the *qibla*" (f. 187b), (25) "The determination of the azimuth of the *qibla* from the table" (f. 187b-188a). After the table of chapter 25 follows chapter 26 which contains "A table of shadows and inclinations" (f. 188b).

The chapters 27-31 are devoted to the description of different clock instructions: (27) "The construction of the *binkāns*, i.e., the troughs, or the tubs for the measurement of hours, equal as well as temporal" (f. 189a-190a), (28) "The construction of the *binkān* that shoots pebbles" (f. 190a-190b), (29) "The construction of the horary lifting wheel" (f. 191a-192a), (30) "The construction of the conical sundial for the [determination of] hours" (f. 192a), (31) "Operating with the [ritual] sundial [that is called] *miknasa*" (f. 193a).

Then follows chapter 32 "Geometrical construction of the azimuth of sunrise you wish for [any] zodiacal sign in the latitude you wish" (f. 193b). The following three chapters are devoted to the determination of latitudes and longitudes of cities: these are (33) "The determination of the latitude of a city" (f. 193b-194a), (34) "The geometrical determination of the latitude of a city", (f. 194a) (35) "Cities situated in all climates" (a table of longitudes and latitudes of 163 cities) (f. 194b-196b).

The next three chapters are again devoted to the description of astronomical and mathematical instruments: (36) "Operating with a horary quadrant" (f. 196b-197a), (37) "Pair of compasses for a camp" (f. 197a-197b), (38) "A board for determining the [lunar] eclipse without the appropriate instruments" (f. 198a). After that follows (39) "Witty ideas in the construction of Muḥammad ibn Mūsā al-Khwārizmī: the determination of the azimuth using the astrolabe" (f. 198b-199a) and (40) "The determination of the latitudes of climates" (f. 199a-199b).

The last 19 chapters (the beginning of the first of them is not there) are devoted to the measurement of time. These are: (41) The

determination of "times of prayers from the table" (f. 229a-229b), (42) The determination of "the midday altitude in any day for any latitude" (f. 229b), (43) The determination of "the surplus of the day" from the table (f. 229b), (44) Determining "when [the Sun] is in the zenith" (f. 229b), (45) Determining "when [the Sun] is for the second time in the zenith" (f. 229b), (46) The determination of "the shadow of the hour you wish with the aid of a gnomon" (f. 229b-230a), (47) "The determination of shadows for the times [of prayers]" (f. 230a), (48) "[The determination of the midday] altitude in any day and any latitude" (f. 230a), (49) "More about the altitude in any day and the time of *'aṣr* and *zuhr*" (f. 230b), (50) "The midday altitude for the latitude 34 degrees from the verified inclination" (a table) (f. 230b), (51) "The altitude for the latitude 33 degrees" (a table) (f. 231a), (52) "The construction of the hours in the plane of the sundial by Muḥammad ibn Mūsâ al-Khwârizmî" (f. 231b), (53) "This sundial—a sundial for Baghdad [in which the shadows and the azimuths appear] every one sixth [of every hour for the beginnings of all zodiacal signs]" (a table) (f. 231b-232b), (54) Tables of the first and the last azimuths of the time of *'aṣr* for the beginnings of Capricorn, Aries, and Cancer and the first and the last shadows of *'aṣr* (f. 232b), (55) "The sundial for the latitudes 15, 18, 21, 24, 27, 30, 33, 36, 38, 40" (a table of the altitudes, azimuths, and shadows for the hours 1, 2, 3, 4, 5, 6 for the signs Cancer and Capricorn) (f. 233a-234a), (56) "The sundial for Surra man Ra'â (Samarra) [the latitude] 34" (a table of the azimuths and shadows every half an hour for the same zodiacal signs) (f. 234a), (58) "The sundial for the place which has no latitude, i.e., for the equator" (f. 234a-234b), (59) "The construction of the sundial by geometric method for any latitude" (f. 234b-235a).

Chapters 39-40 and 52-59 were published in [2]. As we see the majority of chapters are devoted to two problems: The determination of the azimuth of some directions and the determination of time.

The Tashkent manuscript is entitled: "The determination of the direction of the azimuth of the *qibla* in any city that you wish" (*Ma'rifat taqwîm samt al-qibla fî ayyi balad shi'ta*). This manuscript consists of five chapters, the last four of which coincide with chapters 22-25 of the Istanbul manuscript.

§ 3. *Treatise on the determination of the qibla*

First of all let us consider some chapters of these treatises concerning the determination of azimuths. Azimuth (*samt*, literally “direction”) is the angle between the direction of a certain point on the horizontal plane and some initial direction (the origin of the word “azimuth” is the Arabic word *as-sumût* —the plural of *as-samt*). The azimuth of a celestial body or a point of the celestial sphere is the azimuth of the orthogonal projection of this point on the plane of the horizon, the azimuth of a celestial body with its altitude — the angle between the direction of this body and the plane of the horizon — form one of the spherical coordinates on the celestial sphere, in which the circle of the horizon stands for the equator and the zenith and nadir for the poles. The *qibla* is the direction of Mecca—the main holy place of the Muslims. The knowledge of the direction of the *qibla* was necessary in the building of mosques since Muslim worshippers should face Mecca when they pray; therefore the determination of the azimuth of the *qibla* was one of the important practical problems in Muslim countries. The azimuth of the *qibla* for a given city means the angle between the direction to Mecca and the direction of the meridian in this city. The azimuth of sunrise or rising (ortive amplitude, *si‘at al-mashriq*) means the arc of the horizon circle from the east point to the point of sunrise in the given day. In all cases, the problems of determining the azimuths can be solved by means of spherical trigonometry.

In chapter 20 of the Istanbul manuscript and chapter 1 of the Tashkent manuscript approximate methods of determining the azimuth of the *qibla* are given. At first two cases in which the latitudes or the longitudes of the given city X and Mecca M are equal are considered. In these cases Al-Khwârizmî takes as the direction of the *qibla* respectively the direction of east-west and north-south. The second of these rules is exact but the first one is rough and approximate. Then an approximate construction of the zenith of Mecca on the celestial sphere of the given city is suggested: find the difference $\lambda_X - \lambda_M$ of these cities and subtract it from half of the day arc for the day corresponding to 4° of Gemini, i.e., approximately to the day in which the sun in Mecca is in the zenith. Take the difference you get as the altitude by which you determine the azimuth according to what was described in the previous chapters. In the Tashkent manusc-

ription an other approximate representation of the city M is drawn by the latitudes and longitudes of the cities X and M in a circle with center at X where the direction of the line XM is taken as the direction of the *qibla*. We give here the trigonometric solution of this problem in chapter 22 of the Istanbul manuscript and in chapter 2 of the Tashkent manuscript. Here is the Arabic passage :

عمل القبلة أيضا في كل عرض

تأخذ ما بين عرض مكة وعرض البلد الذي تريد. فاجعله جيبا ثم اضربه في مثله ، ثم خذ فضل ما بين طول مكة وطول مدينتك ، فاجعله جيبا واضربه في جيب تمام عرض مكة ، واقسمه على قن أبدا ، فما خرج فهو الجيب المعدل. فاضربه في مثله وزده على المضروب في مثله الذي حفظته. فما بلغ فخذ جذره فاحفظه. ثم اضرب الجيب المعدل في قن واقسمه على الجذر. فما بلغ قفوسه واحفظه. ثم أدر في المسجد الذي تريد دائرة هندية ثم اعرف خط الزوال. فان كانت مدينتك غربية عن مكة فعد مثل تلك القوس عن خط الزوال الجنوبي الى ناحية المشرق. فحيث انتهت فهو موضع القبلة من موضع العود الذي في وسط الدائرة. وان كانت مدينتك شرقية عن مكة فعد من ناحية المغرب وافعل كما فعلت اولاً.

Its English translation is as follows :

(22) "The determination of the *qibla* also for any latitude. Take [the difference] between the latitude of Mecca and the latitude of the city in which you are, take its sine, multiply it by itself and keep it in mind. Then take the difference between the longitudes of Mecca and your city, take its sine multiply it by the sine of the complement of the latitude of Mecca and divide this by 150. The thing you get is 'the corrected sine.' Multiply it by itself and add it to that product by itself which you kept in mind. Extract the root of the sum and keep it in mind. After that multiply the 'corrected sine' by the greatest sine and divide it by the root. Take the arc of the quotient and keep it in mind. Then draw an Indian circle in the mosque that you want [to build] and determine the meridian line. If your city is to the west of Mecca, cut off what is equal to this arc to the south of the meridian line to the east direction. The place where the arc reaches will be the place of the *qibla* with respect to the gnomon in the middle of the circle. If your city is to the east of Mecca, cut off the arc in the west direction and act in the same way as in the beginning." (f. 187a).

If we denote Mecca, the given city, and the north pole of the earth, by M , X , and N , respectively; then the azimuth of the *qibla* is the angle NXM in the spherical triangle NXM (fig. 1). "The corrected sine" (*al-jaiib al-mu'addal*)—the result of the "correction" of the sine line, $\text{Sin} (\lambda_X - \lambda_M) = R \sin (\lambda_X - \lambda_M)$, i.e., its product by $\cos \varphi_M R$ is "the complete sine", the radius of the Earth which is taken in the treatise equal to 150; in the Tashkent manuscript the number 150 is replaced by the words *jumlat al-jaiib* — "complete sine". $\cos \varphi_M$ is $\text{Cos} \varphi_M / R$: this multiplier indicates the number that must be multiplied by the linear dimensions of the great circle of the terrestrial globe in order to get the corresponding linear dimensions of the parallel of Mecca. The method given here is based on plane trigonometry, i.e., it is assumed that the part of the terrestrial surface, with the cities M and X is plane. The arc MY of the parallel of Mecca is identified with "the corrected sine"

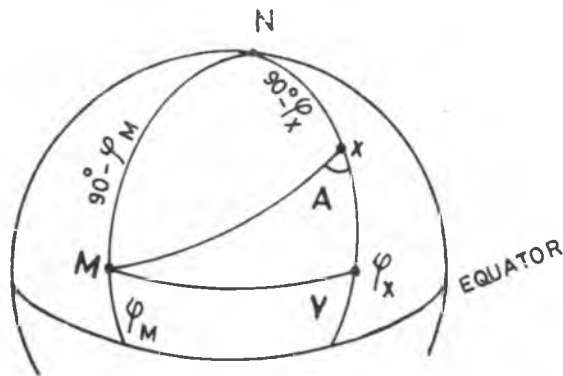


Fig. 1

$$\text{Sin} \eta = (\text{Sin} [\lambda_X - \lambda_M] \text{Cos} \varphi_M) / R$$

(in its computation the sphericity of the surface of the earth is taken into account). The arc XY is identified with sine line

$\text{Sin} (\varphi_X - \varphi_M)$, and the hypotenuse $\sqrt{\text{Sin}^2 (\varphi_X - \varphi_M) + \text{Sin}^2 \eta}$ of the right angled triangle with legs MY and XY is taken for the arc MX . Further, in the triangle MXY , taken as a right triangle with straight sides, the sine of the angle A is found by the formula

$\sin A = (MY/MX) \cdot R$. "The Indian circle" found by Indians for determining the meridian line ("midday line") is a circle drawn on the horizontal plane with vertical gnomon in the centre: the meridian line joins the centre of the circle with the midpoint of its arc joining the points of intersection with the line described by the end of the shadow of the gnomon.

In chapter 25 of the Istanbul manuscript and chapter 5 of the Tashkent manuscript there is a table with 20 columns and 20 rows on each of which the numbers 1 to 20 are marked the numbers 1, 2, ..., 20 above the columns denote the differences of latitudes $\varphi_X - \varphi_M$ (above these numbers the word "latitude" is written), the numbers 1, 2, ..., 20 on the right side of the rows denote the differences of longitudes $\lambda_X - \lambda_M$ (above these numbers from up to down the word "longitude" is written), in the squares at the intersection of columns and rows corresponding to these differences, the azimuth A of the *qibla* in these differences $\varphi_X - \varphi_M$ and $\lambda_X - \lambda_M$ is indicated. We note that a table very similar to this is given in the manuscript of the library Taymur in Cairo No. 103/2 (f. 101a-102a) under the title "Table of twenty (*al-jadwal al-'ashrîni*) of Abû Ja'far Muḥammad ibn Mûsâ al-Khwârizmî", this gives an additional argument for ascribing this treatise to Al-Khwârizmî. There is a similar table in a *zîj* in the MS No. 9116 of the British Library (London), written in Yemen in the 13th c. [7, p. 15-16].

The determination of the azimuth of the *qibla* by applying spherical trigonometry to the spherical triangle MXN is carried out in "The book on the use as astrolabe" of Abdu'r-Raḥmân aṣ-Ṣûfî (10th c.) [10, p. 297-298] and in "al-Qânûn al-Mas'ûdî" of Al-Bîrûnî [11, p. 425-427].

§ 4. *The first treatise on the determination of the rising amplitude*

Chapter 1 "The determination of the azimuth of the sunrise in any city" begins as follows. Here is the Arabic passage :

معرفة سعة المشرق في كل بلد...

فاذا أردت معرفة ذلك تأخذ من نصف القطر ربعه وهو $\frac{1}{4}$ فتضربه في مثله، وتضرب جميع درج الميل في مثله، ثم تجمعها وتأخذ جذرها، فما كان فهو سعة مشرق تلك المدينة.

Its English translation is as follows :

“If you want to determine this take one fourth of the diameter, this is 15 [degrees], and multiply this by itself. Multiply the degrees of declination taken as a whole by themselves. Then add these and extract the roots of this sum. The thing you get is the azimuth of the sunrise of this city.” (F. 228b). If we denote the celestial equator by EF , and the circle of the horizon by ES (fig. 2), then the point of the sunrise is the point E of intersection of these two circles. If the Sun on the given day is at the point S of the day circle which is at a spherical distance δ from the celestial equator (the arc δ is called the declination of the Sun), then the azimuth of the sunrise θ is the hypotenuse of the right angled spherical triangle ESF with right angle F ; the leg SF of this triangle is equal to the declination δ , and the angle E is equal to the complement of the latitude φ of the given city to 90° . Therefore, by virtue of the spherical sine theorem

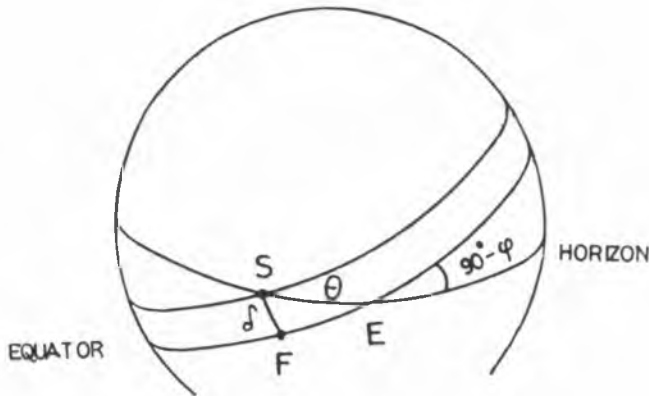


Fig. 2

$$\sin\theta/R = \sin\delta/\cos\varphi, \quad (1)$$

i.e.,

$$\sin\theta = R (\sin\delta/\cos\varphi). \quad (2)$$

In the title of the chapter Al-Khwārizmī refers to Ptolemy. The problem of determining the azimuth of the sunrise for the day of the solstice when the declination δ of the Sun is equal to the maximal declination, i.e., to the angle between the ecliptic and the celestial

equator was solved by Ptolemy in chapter 3 of Book II of the "Almagest" [12, p. 65), where he gives a rule equivalent to the formula (1). Al-Khwârizmî uses here the division of the radius into 60 parts that had been used by Ptolemy (he mentions this in the title of the treatise, since he usually uses the division of the radius into 150 parts): Al-Khwârizmî calls here the 60th parts of the radius "degrees". In the above mentioned problem Al-Khwârizmî, as well as Ptolemy, finds the azimuth of the sunrise for the day solstice, i.e. for $\delta = \epsilon$. In this case he considers the triangle ESF as a plane triangle, and the latitude of the city equal to the latitude of Baghdad $\varphi \approx 33^\circ$. Therefore, $90^\circ - \varphi \approx 57^\circ$, $EF \approx SF/\tan 57^\circ \approx 24/1,6 = 15$; that is the reason why Al-Khwârizmî considers this leg equal to 15 "degrees". Al-Khwârizmî defines the azimuth of the sunrise θ as the hypotenuse ES of the plane triangle ESF , i.e., by the formula $\theta = \sqrt{EF^2 + \delta^2}$.

After that Al-Khwârizmî solves the same problem for each zodiacal sign, i.e. for any month. In the case when the Sun is in the ecliptic with ecliptical longitude λ , it follows from the spherical sine theorem for the right angled spherical triangle γSF (fig. 3) that

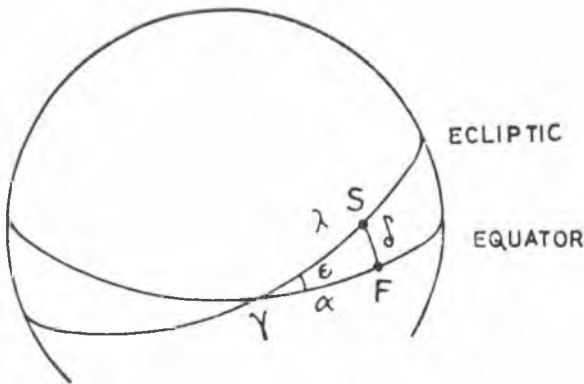


Fig. 3

$$\text{Sin}\lambda/R = \text{Sin}\delta/\text{Sin}\epsilon$$

Al-Khwârizmî approximates the ratio $\text{Sin}\delta/\text{Sin}\epsilon$ replacing it by the ratio δ/ϵ and considers $\text{Sin}\lambda = R(\delta/\epsilon)$. In this case he takes

the hypotenuse ES , of the triangle ES_1F_1 with a leg S_1F_1 equal to δ as the azimuth of the sunrise (fig. 4). Therefore

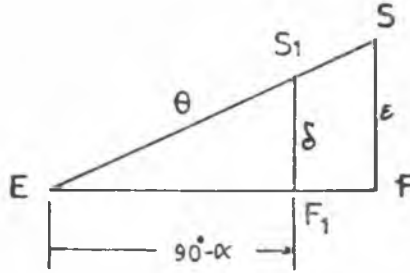


Fig. 4

$$\theta_1 = ES_1 = ES(\delta/\varepsilon) \approx \theta \sin (\lambda/R),$$

here Al-khwârizmî assumes $R = 150$.

§ 5. *Treatise in the determination of the azimuth of the Sun*

In chapter 2 Al-Khwârizmî finds the azimuth from the altitude of the sun, and in solving this problem he uses not plane but spherical trigonometry. We produce the rule of Al-Khwârizmî for the south zodiacal signs. Here is the Arabic passage :

معرفة السمّ من قبل الارتفاع للبروج الشماليّة

إذا اردت ذلك فخذ الارتفاع فالحه من تسعين، فما بقي فاجعله جيبا، فما كان فهو جيب تمام الارتفاع. فاقسم عليه جملة الجيب، فما خرج فهو الاصل الاول. ثم اقسام جيب عرض البلد الذي تريد على جملة تمامه، فما خرج فاضربه في جيب الارتفاع الذي تعمل له السمّ. فما خرج فهو الاصل الثاني. فزد عليه جيب سعة المشرق للبرج الذي تعمل له أو الدرجة، فما بلغ فاضربه في الاصل الاول. فما بلغ فهو السمّ.

Its English translation is as follows :

“If you want [to find] this, take the altitude and subtract from ninety, then turn the remaining into sine. The thing you get is the sine of the complement of the altitude. Divide the complete sine by it, the thing you get is the ‘first base.’ Then divide the sine of the

latitude of the city you are in by the sine of the complement [of it]. Multiply the thing you get by the sine of the altitude for which you find the azimuth, the thing you get is the 'second base.' Add to this the sine of the azimuth of rising of the zodiacal sign or the degree. Multiply the sum by the 'first base', the thing you get is the sine of the complement of the azimuth." (F. 182b).

For solving the same problem for the north zodiacal signs Al-Khwârizmî recommends in chapter 3 to find the same "first and second bases" for the cases of south zodiacal signs. Here is the Arabic passage :

معرفة السميت للبروج الشمالية

وهو أن تعمل الاصل الاول والثاني على ما أرينك . ثم خذ الاصل الثاني فانقصه من جيب سعة المشرق . فان كان أكثر من جيب سعة المشرق ، فالتجيب سعة المشرق منه . فما بقي بعد ذلك فاضربه في الاصل الاول ، فما بلغ فاقوسه . فما خرج فهو السميت . فاذا أردت أن تعلم جهة هذا السميت شمالي هو أم جنوبي فانظر (فان كان الاصل الثاني اكبر من جيب سعة المشرق فالسميت جنوبي) ، وان كان الاصل الثاني مثل جيب سعة المشرق سوا فالسميت على خط المشرق والمغرب ، وان كان الاصل الثاني اقل من جيب سعة المشرق فالسميت شمالي .

In English the passage reads as follows (leaving out the first sentence):

After that — Al-Khwârizmî writes — "take the 'second base' and subtract it from the sine of azimuth of sunrise; if the 'second base' is greater than the azimuth. Then subtract the sine of azimuth. Multiple the difference by the 'first base' and take the arc of the product, you get the [south] azimuth. . . . If the 'second base' is exactly equal to the sine of the azimuth of the sunrise, then the azimuth is on the east-west line. If the 'second base' is smaller than the azimuth of the sunrise, then the azimuth is northern." (F. 182b-183a).

If we consider the spherical triangle of the celestial sphere the vertices of which are the Sun S in one of the south zodiacal signs, i.e., on the arc of the ecliptic from the beginning of Libra up to the beginning of Aries, the north pole of the world P and the Zenith Z (fig. 5), then in this triangle $ZP = 90^\circ - \varphi$, $ZS = 90^\circ - h$, $PS = 90^\circ + \delta$ and by virtue of the spherical cosine theorem

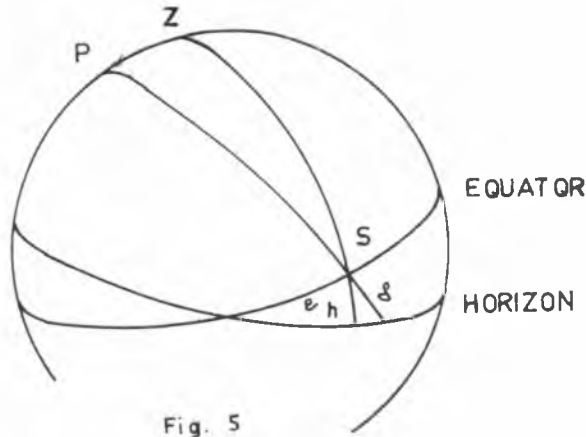


Fig. 5

$$-\sin\delta = \sinh \sin\varphi + \cosh \cos\varphi \cos A$$

and

$$|\cos A| = (\sinh \sin\varphi + \sin\delta) / \cosh \cos\varphi.$$

Al-Khwârizmî finds $\sin(90^\circ - h) = \text{Cosh}$, his "first base" is $R/\text{Cosh} = 1/\cos h$, his "second base" is $(\sin\varphi/\text{Cos}\varphi) \text{ Sinh} = \text{tg}\varphi \text{ Sinh}$. In finding the azimuth of the sunrise θ above, he found that $\text{Sin}\delta/\text{Cos}\varphi = \sin\theta$, therefore the rule of Al-Khwârizmî is equivalent to the rule we mentioned above which follows from the spherical cosine theorem.

In the case when the Sun is in one of the north zodiacal signs, i.e. on the arc of the ecliptic from the beginning of Aries up to the beginning of Libra (fig. 6), $\angle P = 90^\circ - \varphi$, $\angle S = 90^\circ - h$, $PS = 90^\circ - \delta$ and by virtue of the spherical cosine theorem of cosines for the triangle $\angle PS$

$$\sin\delta = \sinh \text{Sin}\varphi + \cosh \cos\varphi \text{Cos}A$$

and

$$|\cos A| = (\sin\delta - \sinh \sin\varphi) / (\cosh \cos\varphi)$$

which is equivalent of the rule of Al-Khwârizmî for this case. The case $h = 0$ corresponds to the case when the "second base" $\tan\varphi \sinh$

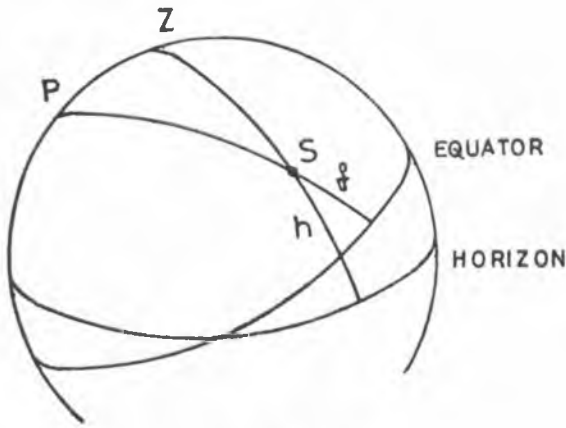


Fig. 6

together with $\sin \theta$ coincides with $\sin \theta$, the case $h > 0$ — to the case when $\tan \varphi \sinh + \sin \theta > \sin \theta$ and the case $h < 0$ — to the case when $\tan \varphi \sinh + \sin \theta < \sin \theta$. In these three cases the Sun is, respectively, on the horizon, above the horizon and below the horizon.

A rule equivalent to the spherical cosine theorem for the triangle ZPS was known to the Indian astronomer of the 5th c. Varâhamihira [13, p. 200-201] and apparently Al-Khwârizmî came to know this rule from an Indian source. Apparently Thâbit ibn Qurra (9th c.) knew this rule through Al-Khwârizmî and produced it in his “Book on the clock instruments called sundial” [6, p. 252-266) and from him Regiomontanus came to know it through the “Şâbian *zîj*” of Al-Battânî. Since the Europeans knew this theorem of spherical trigonometry through Al-Battânî, this theorem is often called in Europe “Albategnius’ theorem”.

The above given rule of Al-Khwârizmî for determining the azimuth of the Sun from the altitude of the Sun, the latitude of the city and the azimuth of the sunrise is only equivalent to the same theorem of spherical trigonometry, that was equivalent to the rule of Varâhamihira in which the altitude of the Sun is determined from the day arc, the midday altitude of the Sun in the given day and the hour angle at the moment for which the altitude of the Sun is deter-

mined; however Al-Khwārizmī gives in chapter 4 the rule of Varāhamihira itself. Here is the Arabic passage :

معرفة عمل السميت والظل والارتفاع

... نستخرج أولاً نصف قوس النهار لذلك اليوم، وجيب نصف قوس النهار المنكوس، وجيب ارتفاع نصف تهار درجة الشمس، وجيب تمام الارتفاع لتصف درجة الشمس لذلك اليوم، وجيب سعة مشرق درجة الشمس لذلك اليوم. فاذا عرفت هذه كله فأنظر الساعة التي تريد أن تعرف سمتها وارتفاعها وظلها. فان كانت مستوية فاضربها في $\frac{1}{15}$ ، وان كانت معوجة فاضربها في أجزاء ساعات ذلك اليوم. فما اجتمع فالقه من جيب نصف قوس النهار المنكوس فما بقي فهو الحصة، فاحفضه. ثم اضرب هذه الحصة في جيب ارتفاع نصف تهار درجة الشمس، فما بلغ فاقسمه على جيب قوس النهار المنكوس. فما خرج فهو جيب ارتفاع تلك الساعة. فقوسه، فما كان فهو الارتفاع.

Its English translation is as follows :

“First of all find half of the day arc, the sine of the midday altitude of the degree of the Sun, the sine of the complement of the midday altitude of the degree of the Sun on this day and the sine of the azimuth of the rising of the degree of the Sun in this day. If you had found all these, look at the hour for which you want to find its azimuth, its altitude and its shadow. If this hour is equal, multiply it by 15, if it is temporal, multiply it by the parts of hours of this day. Subtract the versed sine of the product from the versed sine of half the day arc. The thing you get is ‘the argument’, keep it in mind. Then multiply this ‘argument’ by the sine of the midday altitude of the degree of the Sun. Divide the product by the versed sine of half the day arc. The thing you get as quotient is the sine of the altitude of this hour. Take its arc you get the altitude.” (F. 183a).

“Equal hours” — astronomical hours that are equal to $\frac{1}{24}$ of the day, the product of these hours by 15 is the hour angle in degrees, since the celestial sphere rotates by 15° for each “equal hour”. “Temporal hours” are equal to $\frac{1}{12}$ of daylight or night. In the medieval East all the civil life and the Muslim prayers were determined in “temporal hours”. The word “part” in the expression “parts of the hours” means the degrees of the celestial equator, “the parts of the hour” — the number of degrees of the celestial equator by which

it rotates for 1 "temporal hour". The product of the number of "temporal hours" by "the parts of the hour" is also equal to the hour angle in degrees. "The versed sine" (in Europe sinus-versus) for angles $\alpha < 90^\circ$ is the complement of the cosine line Cosa to R , and for $\alpha > 90^\circ$ — the sum $R + \text{Cosa}$ to $2R$, "the day arc" — the arc of day circle of the sun above the horizon; it is equal to the number of degrees of the celestial equator corresponding to the day-light. Since the hour angle is counted from the meridian, therefore half of the day arc is equal to the maximum value of the hour angle. The rule of Varâhamihira himself can be written in our notation in the form

$$\text{sinh} = (\sin \text{versa} - \sin \text{vers } t) \cos \varphi \cos \delta.$$

From it we obtain the rule of Al-Khwârizmî, if we put

$$\sin \text{versa} = 1 + \tan \varphi \tan \delta.$$

but the tangent function is not mentioned in this rule

These rules are equivalent to the spherical cosine theorem for the same triangle $\angle PS$; and for the angle $t \leq \angle PS$ (not for the angle $A \leq P\angle S$).

§ 6. The second treatise on the determination of the rising amplitude

Al-Khwârizmî applies the rules equivalent to rules of spherical trigonometry for solving the problem of determining the azimuth of the sunries in chapter 32. Here is the Arabic passage :

عمل سعة أى مشرق شئت من البروج في أى عرض شئت بالهندسة

نخط دائرة عليها ايجد ونقسم قوس ا د ص جزأ ونعد من نقطة د على قوس الدائرة بقدر عرض البلد ونخرج منه خطا الى المركز وهو خط ه ز. ثم نعد من نقطة آ على قوس الدائرة بقدر ميل البرج الذى نريد أن نعمل له، ونخرج منه خطا يوازي خط آ ه، وهو خط ح ط ينتهى الى خط د ب الذى هو القطر. ونظرأين يقطع من خط ز ه، فكأنه قطعه على نقطة م. ثم نأخذ منه ونثبت رأس البركار في نقطة المركز وهى نقطة ه ونظر الرأس الآخر أين يقطع من خط د ه، فكأنه قطعه على نقطة ل. فنخرج من نقطة ل عمودا ينتهى الى قوس ا ز وهو عمود ل ع. فحيث انتهى فعد منه الى نقطة آ، فما كان فهو سعة المشرق للبرج الذى اردته.

Its English translation is as follows :

The geometric construction of any azimuth of sunrise you wish for any zodiacal sign in any latitude. “Draw a circle $ABCD$ (fig. 7) and divide its arc AD into 90° parts. From the point D on the circle lay off the arc DG having the magnitude of the latitude of [your] city and draw a line from this [point] G to the centre [of the circle], i.e., the line EG . Then from the point A on the circle lay off the arc AH having the magnitude of the declination of that zodiacal sign for which you want [to make] the construction. From [the end of] this [arc] draw a line parallel to the line AE . It is the line HF , that reaches the line BD which is a diameter. Then look where it cuts the line EG ; let it be the point M . Then take a pair of compasses with a spread ME , put its leg in the centre point, i.e. in the point E , and find the place where the other leg cuts the line EG . Let this be the point N . From the point N raise a perpendicular reaching the arc AG , it is the perpendicular ON . From [the point O] where [the perpendicular] reaches count out [the arc] to the point A . The thing you get is the azimuth of the sunrise for the zodiacal sign that you wanted.” (F. 193b).

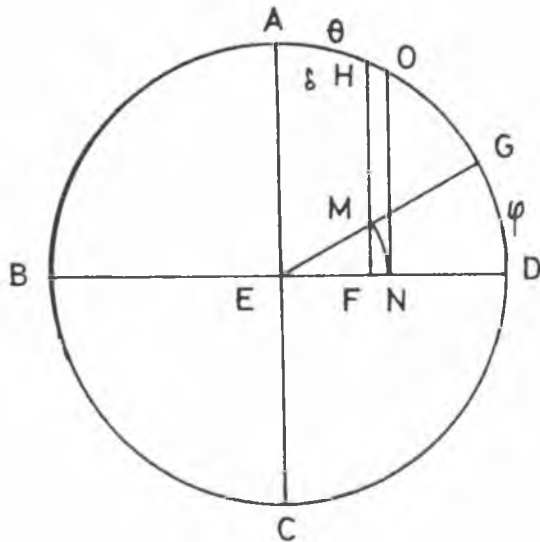


Fig. 7

for determining hours" is given. It is a conical sundial with a horizontal gnomon in its top part. The name *mukḥula* which literally means a vessel for storing antimony (kuḥl) can be explained by the shape of a truncated cone of the clock which is the same shape of this vessel. In chapter 31 a description of "the operation with the sundial that is called *miknasa*" is given. Here and in other places we translate the word *rukhâma* which literally means marble board by the word "sundial". A kind of plane sundial with a vertical gnomon is described here. The name *miknasa* which is from the word *kanîsa* - "a church, a pagan temple" points out the ritual meaning of this sundial. In chapter 36 a description of the operations with the horary quadrant is given. On one of the rectilinear sides of the quadrant two diopters are placed by the aid of which this side is directed to the Sun, in this case the thread with a load that is fixed in the centre of the quadrant cuts on its round side an arc equal to the altitude of the Sun and intersects an arc concentric with this side and corresponding to the month in a point on the hour line $t = \text{constant}$ that corresponds to the hour of observation. In chapters 52-59 the construction of the hour lines on the ordinary horizontal sundial with vertical gnomon is described. The tables concerning the sundial show the polar coordinates i.e., "the shadow" and "the azimuth" of the endpoint of the shadow of the gnomon on the plane of the sundial as a function of the time t of the day and the ecliptic longitude λ of the Sun that corresponds to different zodiacal signs (months). The lines $\lambda = \text{constant}$ are arcs of a hyperbola (for the equinoctial days — straight line segments), Al-Khwârizmî considers the line $t = \text{constant}$ "hour lines" straight. More detailed tables are given for the latitudes 33° (Baghdad) and 34° (Samarra).

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