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The nth Power of Generalized (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas Matrix Sequences and Some Combinatorial Properties

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Abstract − In this study, new formulas for the n^{th} power of (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas special matrix sequences are established by using determinant and trace of the matrices. By these formulas, some identities for (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas sequences are obtained. The formulas for finding the n^{th} power for classic Jacobsthal and Jacobsthal Lucas matrix sequences are also derivable if we choose $s = t = 1$.

Keywords − *Jacobsthal numbers, recurrence relations, special matrices*

Mathematics Subject Classification (2020) – 11B37, 11C20

1. Introduction

In the literature, the researchers investigated the n^{th} power of the matrices by different methods. In [1], Williams studied the n^{th} power of a 2×2 matrix. Laughlin found identities deriving from the n^{th} power of some matrices in [2,3]. Belbachir investigated linear recurrent sequences and powers of a square matrix in [4]. There are certainly new developments on special integer and matrix sequences by constructing recurrence relation. In [5], the authors studied sums and products for recurring sequences. Halıcı and Akyuz derived combinatorial identities by using the trace, the determinant and the n^{th} power of a special matrix whose entries are Horadam numbers [6,7]. Among these integer sequences, the Jacobsthal and Jacobsthal Lucas numbers have been studied extensively in the last decade years in [8-12]. The Jacobsthal numbers j_n are terms of the sequence $\{0,1,1,3,5,11,\cdots\}$, defined by the recurrence relation, $j_n = j_{n-1} + 2j_{n-2}$, for $n \geq 2$, beginning with the values $j_0 = 0$, $j_1 = 1$. The Jacobsthal Lucas numbers c_n are the terms of the sequence $\{2,1,5,7,17,\cdots\}$, defined by the recurrence relation, $c_n = c_{n-1} + 2c_{n-2}$, for $n \geq 2$, beginning with the values $c_0 = 2$ and $c_1 = 1$ in [13]. (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas sequences are defined by using the following recurrence relation,

$$j_n(s,t) = sj_{n-1}(s,t) + 2t\, j_{n-2}(s,t) \quad (j_0(s,t) = 0 \text{ and } j_1(s,t) = 1) \tag{1}$$

and

$$c_n(s,t) = c_{n-1}(s,t) + 2c_{n-2}(s,t) \quad (c_0(s,t) = 2 \text{ and } c_1(s,t) = 1)$$

where $s > 0, t \neq 0$ and $s^2 + 8t > 0$ [8].

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The Binet formula enables us to state (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas number easily. It can be clearly obtained from the roots r_1 and r_2 of the characteristic equation as the form $x^2 = sx + 2t$, where

$$r_1 = \frac{s+\sqrt{s^2+8t}}{2} \text{ and } r_2 = \frac{s-\sqrt{s^2+8t}}{2}$$

The Binet formula for (s,t)-Jacobsthal numbers and (s,t)-Jacobsthal Lucas numbers are given, respectively, by

$$j_n(s,t) = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \text{ and } c_n(s,t) = {r_1}^n + {r_2}^n$$

In [9], for any integer $n \geq 1$, (s,t)-Jacobsthal matrix sequence is defined as

$$J_n(s,t) = s\,J_{n-1}(s,t) + 2t\,J_{n-2}\,(s,t) \tag{2}$$

with initial conditions $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J_1 = \begin{pmatrix} s & 2 \\ t & 0 \end{pmatrix}$ and (s,t)-Jacobsthal Lucas matrix sequence is defined as

$$C_n(s,t) = s\,C_{n-1}(s,t) + 2t\,C_{n-2}\,(s,t) \tag{3}$$

with initial conditions $C_0 = \begin{pmatrix} s & 4 \\ 2t & -s \end{pmatrix}$ and $C_1 = \begin{pmatrix} s^2+4t & 2s \\ st & 4t \end{pmatrix}$. Some important properties for (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas matrix sequences are given as in [9]

a) $J_n = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ t\,j_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}$

b) $C_n = \begin{pmatrix} c_{n+1}(s,t) & 2c_n(s,t) \\ t\,c_n(s,t) & 2tc_{n-1}(s,t) \end{pmatrix}$

c) $J_{m+n} = J_m\,J_n$

d) $J_n = {J_1}^n$

e) $C_{n+1} = C_1\,J_n$

2. The n^{th} Power of Generalized (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Matrix Sequences and Some Combinatorial Properties

In [1], Williams gave a well-known formula for any integer $n \geq 1$, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^n = \begin{cases} \dfrac{{r_1}^n - {r_2}^n}{r_1 - r_2} A - \dfrac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2, & r_1 \neq r_2 \\ n r^{n-1} A - (n-1)\det(A)\, r^{n-2} I_2, & r_1 = r_2 \end{cases} \tag{4}$$

where r_1, r_2 being the roots of the associated characteristic equation $r^2 - (a+d)r + \det(A) = 0$ of the matrix A and I_2 is the identity matrix 2×2.

Corollary 1. For any integer $n \geq 1$, the n^{th} power of $J_1(s,t)$ and $C_1(s,t)$ are

$${J_1}^n(s,t) = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \begin{pmatrix} s & 2 \\ t & 0 \end{pmatrix} - \frac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2 \tag{5}$$

where $r_1 = \frac{s+\sqrt{s^2+8t}}{2}$ and $r_2 = \frac{s-\sqrt{s^2+8t}}{2}$

$$C_1{}^n(s,t) = \frac{s_1{}^n - s_2{}^n}{s_1 - s_2}\begin{pmatrix} s^2+4t & 2s \\ st & 4t \end{pmatrix} - \frac{s_1{}^{n-1} - s_2{}^{n-1}}{s_1 - s_2} I_2 \tag{6}$$

where $s_1 = \frac{s^2+8t+s\sqrt{s^2+8t}}{2}$ and $s_2 = \frac{s^2+8t-s\sqrt{s^2+8t}}{2}$.

If we choose $s = t = 1$ in (5) and (6), we get the n^{th} power of classic Jacobsthal and Jacobsthal Lucas matrix sequences:

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n = \frac{2^n - (-1)^n}{3}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - \frac{2^{n-1} - (-1)^{n-1}}{3} I_2$$

and

$$\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}^n = \frac{2^n - (-1)^n}{3}\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix} - \frac{2^{n-1} - (-1)^{n-1}}{3} I_2$$

PROOF. The proof is obtained by using the eigenvalues of $J_1(s,t)$ and $C_1(s,t)$ and (2), (3), (4). □

Corollary 2. For any integer $n \geq 1$, the determinants of $J_1{}^n(s,t)$ and $C_1{}^n(s,t)$ are

$$\det\left(J_1{}^n(s,t)\right) = (-2t)^n \text{ and } \det\left(C_1{}^n(s,t)\right) = (2t)^n(s^2+8t)^n$$

PROOF. By using the property of the determinant of a matrix is the product of eigenvalues of this matrix, we get the determinant of $J_1(s,t)$ and $C_1(s,t)$ is $-2t$ and $(2t)(s^2+8t)$, respectively. The determinant of the n^{th} power of a matrix is the n^{th} power of the product of the eigenvalues. So, the results are easily seen.
□

If we choose $s = t = 1$, we get classic Jacobsthal and Jacobsthal Lucas matrix sequences, and the determinant of them are obtained as

$$\det(J_1{}^n) = (-2)^n, \ \det(C_1{}^n) = (18)^n$$

Laughlin, in [2,3] gave if A is a 2×2 matrix as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the nth power of A is given by

$$A^n = \begin{pmatrix} x_n - dx_{n-1} & bx_{n-1} \\ cx_{n-1} & x_n - ax_{n-1} \end{pmatrix} \tag{7}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i$, T is the trace of A, and D is the determinant of A.

Corollary 3. The n^{th} power of $J_1(s,t)$ and $C_1(s,t)$ are

$$J_1{}^n(s,t) = \begin{pmatrix} x_n & 2x_{n-1} \\ tx_{n-1} & x_n - sx_{n-1} \end{pmatrix} \tag{8}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i}(2t)^i$, and

$$C_1{}^n(s,t) = \begin{pmatrix} y_n - 4t\, y_{n-1} & 2s\, y_{n-1} \\ st\, y_{n-1} & y_n - (s^2+4t)\, y_{n-1} \end{pmatrix} \tag{9}$$

such that $y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (s^2+4t)^{n-i}(2t)^i$.

If we choose $s = t = 1$ in (8) and (9), we get the n^{th} power of classic Jacobsthal and Jacobsthal Lucas matrix sequences,

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} x_n & 2x_{n-1} \\ x_{n-1} & x_n - x_{n-1} \end{pmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (2)^i$ and

$$\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}^n = \begin{pmatrix} y_n - 4\,y_{n-1} & 2\,y_{n-1} \\ y_{n-1} & y_n - 5\,y_{n-1} \end{pmatrix}$$

such that $y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 9^{n-i} 2^i$.

Proof. The proof is obtained by (2), (3), and (7). □

Corollary 4. The n^{th} element of (s,t)-Jacobsthal Lucas matrix sequence is given as

$$C_n(s,t) = C_1(s,t) J_{n-1}(s,t) = \begin{pmatrix} (s^2+4t)x_{n-1} + 2st\,x_{n-2} & 2(s\,x_{n-1} + 4t\,x_{n-2}) \\ t(s\,x_{n-1} + 4t\,x_{n-2}) & 2t(x_{n-1} - s\,x_{n-2}) \end{pmatrix}$$

or

$$C_n(s,t) = s\,J_n(s,t) + 4t\,J_{n-1}(s,t) = \begin{pmatrix} s\,x_n + 4t\,x_{n-1} & 2(s\,x_{n-1} + 4t\,x_{n-2}) \\ t(s\,x_{n-1} + 4t\,x_{n-2}) & x_n - s\,x_{n-1} + 4t(x_{n-1} - s\,x_{n-2}) \end{pmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i} (2t)^i$, for any integer $n \geq 1$.

Proof. By (d, e), (2-3), (7), the proofs are easily obtained. □

Theorem 5. For any integer $n \geq 1$, the following property is satisfied,

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n-i}{i} s^{n-2i} (2t)^i = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i} (s^2+8t)^i \tag{10}$$

Proof. The eigenvalues of $J_1(s,t)$ are $r_1 = \frac{s+\sqrt{s^2+8t}}{2}$ and $r_2 = \frac{s-\sqrt{s^2+8t}}{2}$. The eigenvalues of ${J_1}^n(s,t)$ are ${r_1}^n$ and ${r_2}^n$. By using (8), it is obtained that ${J_1}^n(s,t) = \begin{pmatrix} x_n & 2x_{n-1} \\ tx_{n-1} & x_n - sx_{n-1} \end{pmatrix}$. The trace of ${J_1}^n(s,t)$ is $\text{tr}({J_1}^n(s,t)) = 2x_n - sx_{n-1}$. Because the sum of the eigenvalues is equal to the trace of the matrix, ${r_1}^n + {r_2}^n = 2x_n - sx_{n-1}$

$$2x_n - s\,x_{n-1} = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i} (-D)^i - s \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} T^{n-1-2i} (-D)^i$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i} (2t)^i \left(\frac{i}{n-i}\right)$$

By binomial expansion

$${r_1}^n + {r_2}^n = \left(\frac{s+\sqrt{s^2+8t}}{2}\right)^n + \left(\frac{s-\sqrt{s^2+8t}}{2}\right)^n$$

$$= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i} (s^2+8t)^i$$

The equality of the results completes the proof. □

If we choose $s=t=1$ in (10), we get the same result for classic Jacobsthal and Jacobsthal Lucas matrix sequences as

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n}{n-i}\, 2^i = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\, 9^i$$

By the Binet formula of (s,t)-Jacobsthal Lucas sequence, the following is obtained,

$$c_n(s,t) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\, s^{n-2i}(s^2+8t)^i$$

and

$$c_n(s,t) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n}{n-i}\, s^{n-2i}(2t)^i$$

Corollary 6. The n^{th} element of Jacobsthal matrix sequence is also demonstrated by using the elements of (s,t)-Jacobsthal sequences,

$$J_n(s,t) = j_n(s,t)\, J_1 - j_{n-1}(s,t)\, I_2 \tag{11}$$

Proof. By (a, d) and Binet formulas, we get

$$\begin{aligned} J_n(s,t) &= \begin{pmatrix} j_{n-1}(\mathrm{s,t}) & 2\, j_n(\mathrm{s,t}) \\ t\, j_n(\mathrm{s,t}) & 2t\;\, j_{n-1}(\mathrm{s,t}) \end{pmatrix} = J_1^n(s,t) \\ &= \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} J_1(s,t) - \frac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2 \\ &= j_n(\mathrm{s,t})\, J_1(s,t) - j_{n-1}(\mathrm{s,t})\, I_2 \end{aligned}$$

□

If we choose $s=t=1$ in (11), we get the same result for classic Jacobsthal and Jacobsthal Lucas matrix sequences as

$$J_n = j_{n-1} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - j_{n-1}\; I_2$$

By binomial expansion, the following is derived,

$$\frac{{r_1}^n - {r_2}^n}{r_1 - r_2} = \frac{1}{\sqrt{s^2+8t}} \left[\left(\frac{s+\sqrt{s^2+8t}}{2} \right)^n - \left(\frac{s-\sqrt{s^2+8t}}{2} \right)^n \right] = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}\, s^{n-2i-1}\, (s^2+8t)^i$$

and

$$\begin{aligned} J_n(s,t) &= \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} J_1(s,t) - \frac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2 \\ &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}\, s^{n-1-2i}\, (s^2+8t)^i J_1(s,t) \\ &\quad - \frac{1}{2^{n-2}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{2i+1}\, s^{n-2-2i}\, (s^2+8t)^i\, I_2 \end{aligned}$$

Lemma 7. In [2], for all $g \epsilon \mathbb{R}$ or $\mathbb{Z}$, for any integer $n \geq 1$, if

$$A = \frac{1}{g^2 + Tg + D}(A + gI)(gA + DI) \tag{12}$$

then

$$A^n = \left(\frac{gD}{g^2 + Tg + D}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{D}{g^2}\right)^i \left(\frac{g}{D}\right)^r A^r \tag{13}$$

Corollary 8. For all $g \epsilon \mathbb{R}$ or $\mathbb{Z}$, for any integer $n \geq 1$,

$$J_1{}^n(s,t) = \left(\frac{-2tg}{g^2 + sg - 2t}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{(-2t)^{i-r}}{g^{2i-r}}\right) J_1{}^r\ (s,t)$$

and

$$C_1{}^n(s,t) = \left(\frac{2t(s^2+8t)g}{g^2 + (s^2+8t)g + 2t(s^2+8t)}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{(2t(s^2+8t))^{i-r}}{g^{2i-r}}\right) C_1{}^r(s,t)$$

Example 9. If $s = t = 1$, we get classic the Jacobsthal and Jacobsthal Lucas matrix sequences. For $n = 4$, the following is obtained,

$$J_1{}^4 = \begin{pmatrix} J_5 & 2J_4 \\ J_4 & 2J_3 \end{pmatrix} = \left(\frac{-2tg}{g^2 + g - 2}\right)^4 \sum_{r=0}^{s} \sum_{i=0}^{r} \binom{4}{i} \binom{4}{r-i} (-2)^{i-r} g^{r-2i} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

and

$$C_1{}^4 = \begin{pmatrix} c_5 & 2c_4 \\ c_4 & 2c_3 \end{pmatrix} = \left(\frac{18g}{g^2 + 9g + 18}\right)^4 \sum_{r=0}^{2n} \sum_{i=0}^{r} \binom{n}{i} \binom{n}{r-i} \left(\frac{18^{i-r}}{g^{r-2i}}\right) \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}^r$$

Theorem 10. For any integer $n \geq 1$,

$$j_{nk}(\mathrm{s,t}) = j_n(\mathrm{s,t}) \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-i}{i} c_n{}^{k-1-2i}(\mathrm{s,t})\ (2t)^{in} \tag{14}$$

Proof. By using the property $j_{n+1}(s,t) + 2t\, j_{n-1}(s,t) = c_n(s,t)$ and the Binet formula of $(\mathrm{s,t})$-Jacobsthal sequence, the following is obtained,

$$(J_1{}^n)^k = J_1{}^{nk} = J_{nk} = \begin{pmatrix} j_{nk+1}(s,t) & 2j_{nk}(s,t) \\ tj_{nk}(s,t) & 2tj_{nk-1}(s,t) \end{pmatrix}$$

$$(J_1{}^n)^k = (J_n)^k = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ tj_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}^k$$

$$= \begin{pmatrix} x_k - 2t\, j_{n-1}(s,t)x_{k-1} & 2j_n(s,t)x_{k-1} \\ tj_{nk}(s,t)x_{k-1} & x_k - j_{n+1}(s,t)x_{k-1} \end{pmatrix}$$

where

$$x_k = 2 \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} T^{k-2i} (-D)^i$$

$$= 2\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} \begin{bmatrix} \left(j_{n+1}(s,t) + 2t\, j_{n-1}(s,t)\right)^{k-2i} \\ (\,2t\,(j_{n+1}(s,t)\, j_{n-1}(s,t) - j_n{}^2(s,t))^i \end{bmatrix}$$

$$= 2\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} (c_n(s,t))^{k-2i}(-2t)^{in}$$

By the equality of the matrices, the proof is completed. □

Theorem 11.

$$j_{nk+r}(s,t) = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i} \left(c_n(s,t)\right)^{k-2i}(-2t)^{in}\left[j_r(s,t) + \frac{k-2i}{k-i}\frac{(-2t)^r\, j_{n-1}(s,t)}{c_n(s,t)}\right] \tag{15}$$

Proof. By using (a), (10) and [9], we get

$$J_1{}^n(s,t) = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ tj_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}$$

Then,

$$J_1{}^{nk+r}(s,t) = \begin{pmatrix} j_{nk+r+1}(s,t) & 2j_{nk+r}(s,t) \\ tj_{nk+r}(s,t) & 2tj_{nk+r-1}(s,t) \end{pmatrix}$$

and

$$J_1{}^{nk+r}(s,t) = \begin{pmatrix} j_{n+1}(s,t) & 2j_n(s,t) \\ tj_n(s,t) & 2tj_{n-1}(s,t) \end{pmatrix}^k \begin{pmatrix} j_{r+1}(s,t) & 2j_r(s,t) \\ tj_r(s,t) & 2tj_{r-1}(s,t) \end{pmatrix}$$

$$= \begin{pmatrix} x_k - 2t\, j_{n-1}(s,t)x_{k-1} & 2j_n(s,t)x_{k-1} \\ tj_n(s,t)x_{k-1} & x_k - j_{n+1}(s,t)x_{k-1} \end{pmatrix} \begin{pmatrix} j_{r+1}(s,t) & 2j_r(s,t) \\ tj_r(s,t) & 2tj_{r-1}(s,t) \end{pmatrix}$$

where

$$x_k = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i}\; T^{k-2i}\,(-D)^i$$

$$= 2\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i}\; (c_n(s,t))^{k-2i}(-2t)^{in}$$

By the equality of the matrices,

$$j_{nk+r}(s,t) = (\,x_k - 2t\, j_{n-1}(s,t)x_{k-1})j_r(s,t) + 2tj_n(s,t)x_{k-1}j_{r-1}(s,t)$$

$$= j_r(s,t)x_k - 2t(\,j_{n-1}(s,t)j_r(s,t) - j_n(s,t)j_{r-1}(s,t))\, x_{k-1}$$

$$= j_r(s,t)x_k - (-2t)^r\, j_{n-r-1}(s,t)\, x_{k-1}$$

$$= j_r(s,t) \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-i}{i}\; (c_n(s,t))^{k-2i}(-1)^{in-i}(2t)^{in}$$

$$+(-2t)^r\, j_{n-r-1}(s,t) \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-i}{i}\; (c_n(s,t))^{k-1-2i}(-1)^{in-i}(2t)^{in}$$

$$= \sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \binom{k-i}{i} \left[\begin{array}{c} \left(c_n(s,t)\right)^{k-2i} (-1)^{in-i} (2t)^{in} \\ \left[j_r(s,t) + \frac{k-2i}{k-i} \frac{(-2t)^r \, j_{n-r-1}(s,t)}{c_n(s,t)} \right] \end{array} \right] \qquad \square$$

If we choose $s = t = 1$ in (15), we get the property of the classic Jacobsthal sequences,

$$j_{nk+r} = \sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \binom{k-i}{i} (c_n)^{k-2i} (2)^{in} (-1)^{in-i} \left[j_r + \frac{k-2i}{k-i} \frac{(-2)^r \, j_{n-r-1}}{c_n} \right]$$

3. Conclusion

The paper aims to find the nth power of 2×2 special matrices whose entries are the elements of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas number sequences. From the results, some properties of the (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences are established. We develop new methods for finding the nth element of (s,t)-Jacobsthal sequences.

Conflict of Interest

The authors declare no conflict of interest.

References

[1] K. S. Williams, *The* n^{th} *Power of a* 2×2 *Matrix*, Mathematics Magazine 65(5) (1992) 336-336.

[2] J. Mc Laughlin, *Combinatorial Identities Deriving from the* n^{th} *Power of a* 2×2 *Matrix*, Integers: Electronic Journal of Combinatorial Number Theory 4 (2004) 1-15.

[3] J. Mc Laughlin, B. Sury, *Powers of Matrix and Combinatorial Identities*, Integers: Electronic Journal of Combinatorial Number Theory 5 (2005) 1-9.

[4] H. Belbachir, *Linear Recurrent Sequences and Powers of a Square Matrix*, Integers: Electronic Journal of Combinatorial Number Theory 6 (2006) 1-17.

[5] G. E. Bergum, V. E. Hoggatt Jr., Sums and products for recurring sequences, The Fibonacci Quarterly, 13(2) (1975) 115-120.

[6] Z. Akyüz, S. Halıcı, *On Some Combinatorial Identities Involving the Terms of Generalized Fibonacci and Lucas Sequences*, Hacettepe Journal of Mathematics and Statistics 42(4) (2013) 431-435.

[7] Z. Akyüz, S. Halıcı, *Some Identities Deriving from the* n^{th} *Power of a Special Matrix*. Advances in Difference Equations 1 (2012) 1-6.

[8] S. Uygun, *The* (s,t)*-Jacobsthal and* (s,t)*-Jacobsthal Lucas Sequences*, Applied Mathematical Sciences 70(9) (2015) 3467-3476.

[9] K. Uslu, S. Uygun, *The* (s,t)*-Jacobsthal and* (s,t)*-Jacobsthal-Lucas Matrix Sequences*, ARS Combinatoria 108 (2013) 13-22.

[10] S. Uygun, *Some Sum Formulas of* (s,t)*-Jacobsthal and* (s,t)*-Jacobsthal Lucas Matrix Sequences*, Applied Mathematics 7 (2016) 61-69.

[11] S. Halıcı, M. Uysal, *A Study on Some identities involving* (s_k,t)*-Jacobsthal Numbers*, Notes on Number Theory and Discrete Mathematics 26(4) (2020) 4 74-79. DOI: 10.7546/nntdm.2020.26.4.74-79

[12] A. A. Wani, P. Catarino, S. Halıcı, *On a Study of* (s,t)*-generalized Pell Sequence and Its Matrix Sequence,* Journal of Mathematics 51(9) (2019) 17-32.

[13] A. F. Horadam, *Jacobsthal Representation Numbers*, The Fibonacci Quarterly 34(1) (1996) 40-54.