Basic properties of certain class of non normal operators

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Abstract

Some properties of \(k\)-quasi-\(M\)-hyponormal are established in this paper. The ascent and an extension of the well-known Fuglede Putnam’s Theorem for such operators as well as other related results are also presented, which complete some results given in [7,12].

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1. Introduction

Let \(\mathcal{H}\) be an infinite dimensional complex separable Hilbert space, and let \(B(\mathcal{H})\) be the Banach algebra of all bounded linear operators on \(\mathcal{H}\). Denote by \(N(T)\) and \(R(T)\) respectively, for the null space and the range of an operator \(T\) in \(B(\mathcal{H})\). Operators \(T, S \in B(\mathcal{H})\) are said to be intertwined by an operator \(X \in B(\mathcal{H})\) if \(TX = XS\). The familiar Fuglede-Putnam’s Theorem asserts that if \(X \in B(\mathcal{H})\) intertwines two normal operators \(T, S \in B(\mathcal{H})\), then \(X\) intertwines their adjoints \(T^*\) and \(S^*\) too. Several extensions of this result for other classes of non normal operators have been studied by other authors, see [3], [5], [8] and [9]. An operator \(T \in B(\mathcal{H})\) is said to be dominant if \(R(T - \lambda) \subset R(T - \lambda)^*\) for all \(\lambda \in \mathbb{C}\), [11], \(M\)-hyponormal if there exists \(M > 0\) such that \(M(T(\lambda))^*(T(\lambda)) \geq T(T(\lambda))^*(T(\lambda))^k\) for all \(\lambda \in \mathbb{C}\), [2]. A 1-hyponormal operator is hyponormal. The operator \(T \in B(\mathcal{H})\) is said to be \(k\)-quasi-\(M\)-hyponormal for a positive integer \(k\), if there exists \(M > 0\) such that

\[T^kM(T(\lambda))^*(T(\lambda))^kT^k \geq T^k(T(\lambda))^*(T(\lambda))^kT^k\]
for all $\lambda \in \mathbb{C}$. \cite{7,12,13}.

This definition is equivalent to
\[
\left\| \sqrt{M(T - \lambda)}T^k x \right\| \geq \left\| (T - \lambda)^*T^k x \right\|
\]
for all $x \in \mathcal{H}$. If $k = 1$, $T$ is said to be quasi-$M$-hyponormal. Clearly,
\[
M\text{-hyponormal} \subset \text{quasi-}M\text{-hyponormal} \subset k\text{-quasi-}M\text{-hyponormal}
\]

**Example 1.1.** \cite{13} The matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$ is a 2-quasi-$M$-hyponormal but not $M$-hyponormal.

Properties of this class of operators have been presented in \cite{7,12}. In this paper, we add some complement results. We give the ascent of $T - \lambda$ where $T$ is an element of such class for every complex scalar $\lambda$. We present an extension of the well-known Fuglede-Putnam’s Theorem for $k$-quasi-$M$-hyponormal operators. The SVEP, Bishop’s and Dunford’s properties are also established.

2. Main Results

**Theorem 2.1.** Let $T \in B(\mathcal{H})$ be a $k$-quasi-$M$-hyponormal operator. If $T$ has dense range, then $T$ is $M$-hyponormal.

**Proof.** Let $x \in \mathcal{H}$. Since $\overline{R(T)} = \mathcal{H}$, there exists a sequence $(x_n)_n$ in $\mathcal{H}$ such that $x = \lim_{n \to \infty} T x_n$. By continuity of $T$, we get
\[
\lim_{n \to \infty} T^k x_n = \lim_{n \to \infty} T^{k-1} T x_n = T^{k-1} x
\]
Since $T$ is $k$-quasi-$M$-hyponormal,
\[
\left\| \sqrt{M(T - \lambda)}T^k x_n \right\| \geq \left\| (T - \lambda)^*T^k x_n \right\|
\]
for all $\lambda \in \mathbb{C}$. Thus,
\[
\left\| \sqrt{M(T - \lambda)}T^{k-1} x \right\| = \left\| \sqrt{M} \lim_{n \to \infty} (T - \lambda)^*T^k x_n \right\|
\]
\[
= \lim_{n \to \infty} \left\| \sqrt{M}(T - \lambda)^*T^k x_n \right\|
\]
\[
\geq \lim_{n \to \infty} \left\| (T - \lambda)^*T^k x_n \right\|
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\[
= \left\| \lim_{n \to \infty} (T - \lambda)^*T^k x_n \right\|
\]
\[
= \left\| (T - \lambda)^*T^{k-1} x \right\|
\]
Hence, $T$ is $(k - 1)$-quasi-$M$-hyponormal. Since $T$ has dense range, $T$ is $(k - 2)$-quasi-$M$-hyponormal. By iteration, $T$ is $M$-hyponormal.

**Corollary 2.1.** Let $T$ be a non-zero $k$-quasi-$M$-hyponormal operator, but not an $M$-hyponormal. Then $T$ admits at least, a non trivial closed invariant subspace.

**Proof.** Suppose that $T$ has no non trivial closed invariant subspace. Since $T \neq 0$, $N(T) \neq \mathcal{H}$ and $\overline{R(T)} \neq \{0\}$ are closed invariant subspaces for $T$. Thus, necessarily, $N(T) = \{0\}$ and $\overline{R(T)} = \mathcal{H}$. By Theorem 2.1, $T$ is $M$-hyponormal operator, which contradicts the hypothesis. \qed
Definition 2.1. [17] For $T \in B(H)$, the smallest integer $m$ such that $N(T^m) = N(T^{m+1})$ is said to be the ascent (length of the null chain) of $T$, and is denoted by $\alpha(T)$. If such integer does not exist, we shall write $\alpha(T) = \infty$.

Example 2.1. Since an $M$-hyponormal operator is dominant, and according to [17] Lemma 2.1, $\alpha(T) = 1$ for an $M$-hyponormal operator $T \in B(H)$.

Definition 2.2. [17] The smallest integer $m$ such that $R(T^m) = R(T^{m+1})$ is said to be the descent (length of the range chain) of $T$, and is denoted by $\delta(T)$. If no such integer exists, we set $\delta(T) = \infty$.

According to [1], $\alpha(T) = \delta(T)$ whenever $\alpha(T)$ and $\delta(T)$ are both finite.

In [13], F. Zuo and H. Zuo showed that $k$-quasi-$M$-hyponormal operators have finite ascent. Now, we give the value of this ascent for all complex scalar $\lambda$.

Theorem 2.2. Let $T$ be a $k$-quasi-$M$-hyponormal operator. Then:

1. $N(T^k) = N(T^{k+1})$
2. $N((T - \lambda)^2) = N(T - \lambda)$, for all $\lambda \in \mathbb{C}, \lambda \neq 0$.

Or equivalent, $\alpha(T) = k$ and $\alpha(T - \lambda) = 1$,

Proof. (1). It is enough to show that $N(T)^{k+1} \subset N(T)^k$ since clearly $N((T)^k) \subset N(T)^{k+1}$. Let $x$ be in $N(T^{k+1})$. Then $T^{k+1}x = 0$. Since $T$ is $k$-quasi-$M$-hyponormal, there exists $M > 0$ such that

$$0 = \| \sqrt{MT^{k+1}}x \| \geq \| T^*T^kx \|$$

So, $x \in N(T^*T^k)$. Thus, for all $z \in H$

$$\langle T^*T^kx, z \rangle = 0$$

i.e.,

$$\langle T^kx, z \rangle = 0$$

for all $z \in H$. Therefore, $T^kx \in R(T)^\perp$. Since $R(T^k) \subset R(T)$,

$$T^kx \in R(T^k)^\perp \cap R(T^k) = \{0\}$$

and so $x \in N(T^k)$.

(2). Let $x \in N((T - \lambda)^2)$. Since $N(T - \lambda) \subset N(T - \lambda)^*$ by [13] Lemma 2.2, $N(T - \lambda)$ reduces $(T - \lambda)$. Hence, according to the decomposition

$$H = (N(T - \lambda))^\perp \oplus N(T - \lambda)$$

we can write $x = x_1 + x_2$, where $x_1 \in (N(T - \lambda))^\perp$ and $x_2 \in (N(T - \lambda))$. It follows that

$$(T - \lambda)^2x = 0 = (T - \lambda)^2x_1 = (T - \lambda)((T - \lambda)x_1)$$

Thus, $(T - \lambda)x_1 \in N(T - \lambda)$ and $(T - \lambda)x_1 \in (N(T - \lambda))^\perp$. Therefore, $(T - \lambda)x_1 = 0$, and then $x_1 \in N(T - \lambda)$. So $x_1 = 0$. Finally, $x = x_2 \in N(T - \lambda)$.

\qed
Definition 2.3. [1] For an operator $T \in B(H)$ and $x \in H$, the local resolvent set of $T$ at $x$ denoted by $\rho_T(x)$, is defined to consist of complex elements $z_0$ such that there exists an analytic function $f(z)$ defined in a neighborhood of $z_0$, with values in $H$, for which $(T - z)f(z) = x$. The set $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ is called the local spectrum of $T$ at $x$.

Definition 2.4. [1] For every subset $F$ of $\mathbb{C}$, we define the local spectral subspace of $T$ by $H_T(F) = \{ x \in H : \sigma_T(x) \subseteq F \}$.

Definition 2.5. [1] An operator $T \in B(H)$ is said to have Dunford’s property (C) if $H_T(F)$ is closed for each closed subset $F$ of $\mathbb{C}$.

Definition 2.6. [1] An operator $T \in B(H)$ is said to be polaroid, if every isolated point of the spectrum $\sigma(T)$ of $T$ is a pole of the resolvent of $T$, or equivalent, if $\lambda \in \text{iso}(T)$, then $\alpha(T - \lambda)$ and $\delta(T - \lambda)$ are finite.

Definition 2.7. [1] $T \in B(H)$ is said to have Bishop’s property (B) if for each open subset $U \subset \mathbb{C}$ and every sequence of analytic functions $f_n : U \to H$ for which $(T - \lambda)f_n(\lambda) \to 0$ as $n \to \infty$ locally uniformly on each compact subset of $U$, $f_n(\lambda) \to 0$ as $n \to \infty$ again locally uniformly on $U$.

Definition 2.8. [1] An operator $T$ in $B(H)$ is said to have the single valued extension property, briefly SVEP at a complex number $\alpha$, if for each open neighborhood $V$ of $\alpha$, the operator $(T - \lambda)$ is one-to-one for all $\lambda \in V$.

If furthermore, $T$ has SVEP at every $\alpha \in \mathbb{C}$, then $T$ is said to have SVEP.

According to [1],

$$\text{Bishop’s property (B)} \Rightarrow \text{Dunford’s property (C)} \Rightarrow \text{SVEP} \quad (1)$$

F. Zuo and S. Mecheri in [12] proved that $k$-quasi-$M$-hyponormal operators have Bishop’s property $(B)$. Using this result, we present in the sequel, an extension of the Fuglede-Putnam’s Theorem for such type of operators. We’ve then

Proposition 2.1. The Fuglede-Putnam’s Theorem holds for $k$-quasi-$M$-hyponormal operators $T$ and $S^*$ in $B(H)$.

Proof. Operators $T$ and $S^*$ are reduced by their eigenspaces according to [13] Theorem 5, polaroid and having Bishop’s property by [12]. Thus, our result holds by [6] Theorem 2.4. \hfill \Box

Lemma 2.1. [13] Let $T$ in $B(H)$ and $S$ in $B(K)$. Then, the following assertions are equivalent :

(1) The pair $(T, S)$ satisfies the Fuglede-Putnam’s Theorem.

(2) If $TX = XS$ for some $X$ in $B(K, H)$, then $R(X)$ reduces $T$, $(N(X))^\bot$ reduces $S$, and the restrictions $T|_{R(X)}$, $S|(N(X))^\bot$ are unitarily equivalent normal operators.

Corollary 2.2. Let $T \in B(H)$ be a pure $k$-quasi-$M$-hyponormal operator, and let $S^* \in B(H)$ be $k$-quasi-$M$-hyponormal. Then, equation $TX = XS$ implies $X = 0$. 
Proof. Equations $TX = XS$ and $T^*X = XS^*$ hold by the previous Proposition. Hence, restrictions $T\left|\mathcal{R}(X)\right., S\left|\left(N(X)\right)\right.$ are unitarily equivalent normal operators by Lemma 2.1. Since $T$ is pure, $X = 0$ necessarily.

**Definition 2.9.** An operator $T \in B(\mathcal{H})$ is said to be bounded below if there exists $c > 0$ such that $\|x\| \leq c\|Tx\|$ for all $x \in \mathcal{H}$.

Note that such an operator is one-to-one. We’ve then

**Proposition 2.2.** Let $T \in B(\mathcal{H})$ be a k-quasi-M-hyponormal operator, and let $S \in B(\mathcal{H})$ be such that the pair $(T, S)$ satisfies the Fuglede-Putnam’s Theorem. If $X \in B(\mathcal{H})$ intertwines $T$ and $S$, then:

(i) If $X$ is one-to-one, then $S$ has SVEP.

(ii) If $X$ is an isometry, then $S$ has Dunford’s property (C).

(iii) If $X$ is bounded below, then $S$ has Bishop’s property ($\beta$).

Proof. Since $T$ has Bishop’s property ($\beta$) by [7], $T$ has SVEP and Dunford’s property (C) by (1). Thus, assertions (ii), (i) and (iii) hold by [11] Theorem 2.8.

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**References**