Konuralp Journal of Mathematics, 7 (1) (2019) 25-32

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Faber Polynomial Coefficients Estimates of Bi-univalent Functions Associated with Generalized Salagean q-Differential Operator

Sibel Yalçın[1*], Shahid Khan[2] and Saqib Hussain[3]

[1]*Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Bursa, Turkey*
[2]*Department of Mathematics, Riphah International University, Islamabad, Pakistan*
[3]*Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan*
[*]*Corresponding author E-mail: syalcin@uludag.edu.tr*

Abstract

In this paper, we introduce a new subclass of analytic and bi-univalent functions by using generalized Salagean q-differential operator in open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. By using Faber polynomial expansions and $q-$analysis to find a general coefficient bounds $|a_n|$, for $n \geq 3$, of class of bi-subordinate functions, also find initial coefficients bounds. We also highlight some known consequences of our main results.

Keywords: *Bi-univalent function; Faber polynomial expansions; Generalized Salagean q-differential operator.*
2010 Mathematics Subject Classification: *30C45; 30C50.*

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized under the conditions

$$f(0) = 0,$$

$$f'(0) = 1.$$

Further, by S we shall denote the class of all functions in A which are univalent in E.
Let $f \in A$ given by (1.1) and $g \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in E),$$

we define the convolution product (or Hadamard) of f and g as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} b_n a_n z^n \quad (z \in E). \tag{1.2}$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in E. Given functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $u \in \Lambda$, where

$$\Lambda = \{u : u(0) = 0, \; |u(z)| = |u_1 z + \ldots + u_n z^n + \ldots| < 1, \; z \in E\},$$

Email addresses: syalcin@uludag.edu.tr (Sibel Yalçın), shahidmath761@gmail.com (Shahid Khan), saqib_math@yahoo.com (Saqib Hussain)

such that

$$f(z) = g(u(z)) \qquad (z \in E).$$

We denote this subordination by

$$f \prec g \text{ or } f(z) \prec g(z) \quad (z \in E).$$

In particular, if the function g is univalent in E, the above subordination is equivalent to

$$f(0) = g(0), \quad f(E) \subset g(E).$$

For the Schwarz function $u(z)$, $|u_n| \leq 1$, $n \in \mathbb{N}$, see [15].

The Koebe-one quarter theorem [15] shows that the image of E under every univalent function $f \in A$ contains a disc $\{w : |w| < \frac{1}{4}\}$ of radius $\frac{1}{4}$. Every univalent function f has an inverse f^{-1} defined on some disc containing the disc $\{w : |w| < \frac{1}{4}\}$ and satisfying:

$$f^{-1}(f(z)) = z \ \ (z \in E),$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \ \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots. \tag{1.3}$$

A function $f \in \Sigma$ is said to be bi-univalent on E if both f and $g = f^{-1}$ are univalent on E.
Lewin [29] studied the class of bi-univalent functions, obtained the bound $|a_2| \leq 1.51$. Netanyahu [31] showed that Max $|a_2| = \frac{4}{3}$. Brannan and Clunie [12] conjectured that $|a_2| \leq \sqrt{2}$. Ali et al. [7], Altınkaya and Yalçın [8, 9, 10], Frasin and Aouf [17], Hamidi and Jahangiri [20, 22, 27, 28], Srivastava et al. [35, 36] and Bulut [13] investigate the coefficients bounds for the subclasses of bi-univalent functions.
The theory of $q-$analysis in the recent past has been applied in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, $q-$difference and $q-$integral equations and in $q-$transform analysis. The $q-$theory has wide applications in special functions and quantum physics which makes the study interesting and pertinent in this field. Note that the $q-$difference operator plays an important role in the theory of hypergeometric series and quantum theory, number theory, statistical mechanics, etc. At the beginning of the last century studies on $q-$difference equations appeared in intensive works especially by Jackson [25], Carmichael [14], Mason [30], Adams [1] and Trjitzinsky [37]. Research work in connection with function theory and $q-$theory together was first introduced by Ismail et al. [24]. Till now only non-significant interest in this area was shown although it deserves more attention.
For any non-negative integer n, the q-integer number n denoted by $[n]_q$, is define as:

$$[n]_q = \frac{1-q^n}{1-q}, \quad [0]_q = 0.$$

For non-negative integer n the q-number shift factorial is defined by

$$[n]_q! = [1]_q[2]_q[3]_q \ldots [n]_q, \quad \left([0]_q! = 1\right).$$

We note that when $q \to 1^-$, $[n]_q!$ reduces to classical definition of factorial. In general, for a non-integer number t, $[t]_q$ is defined by $[t]_q = \frac{1-q^t}{1-q}$, $[0]_q = 0$. Throughout in this paper, we will assume q to be a fixed number between 0 and 1.
The q-difference operator related to the q-calculus was introduced by Andrews et al. (see in [2] CH 10).

Definition 1.1. *[2] For $f \in A$, the q-derivative operator or q-difference operator is defined as:*

$$D_qf(z) = \frac{f(qz) - f(z)}{z(q-1)} \quad (z \in E).$$

It can easily verify that.

$$D_qf(z) \to f'(z) \quad \text{as } q \to 1^-.$$

Definition 1.2. *[18] For $f \in A$, let the Salagean q-differential operator be defined by*

$$\begin{aligned}
S_q^0(f(z)) &= f(z), \\
S_q^1(f(z)) &= zD_qf(z), \\
S_q^2(f(z)) &= S_q^1\left(S_q^1 f(z)\right) = zD_q(zD_qf(z)), \\
&\vdots \\
S_q^p(f(z)) &= zD_q(S_q^{p-1}(f(z)).
\end{aligned}$$

A simple calculation implies

$$S_q^p(f(z)) = f(z) * G_{q,p}(z) \quad (z \in E,\ p \in \mathbb{N} \cup \{0\} = \mathbb{N}_0),$$

where

$$\begin{aligned} G_{q,p}(z) &= z + \sum_{n=2}^{\infty} [n]_q^p z^n, \\ S_q^p(f(z)) &= z + \sum_{n=2}^{\infty} [n]_q^p a_n z^n. \end{aligned}$$

The symbol "$*$" stands for Hadamard product (or convolution).
In this article we define Generalized Salagean q-differential operator by using the same technique of [18].

Definition 1.3. *For $f \in A$, let the Generalized Salagean q-differential operator be defined as:*

$$\begin{aligned} D_{q,\alpha}^0(f(z)) &= f(z), \\ D_{q,\alpha}^1(f(z)) &= (1-\alpha)f(z) + \alpha z D_q f(z), \quad \alpha \geq 0, \\ &\vdots \\ D_{q,\alpha}^p(f(z)) &= D_{q,\alpha}\left(D_{q,\alpha}^{p-1} f(z)\right). \end{aligned}$$

A simple calculation implies

$$D_{q,\alpha}^p(f(z)) = f(z) * G_{q,\alpha}^p(z), \quad (z \in E,\ p \in \mathbb{N}_0), \tag{1.4}$$

where

$$G_{q,\alpha}^p(z) = z + \sum_{n=2}^{\infty} \left(1 + \alpha([n]_q - 1)\right)^p z^n. \tag{1.5}$$

Making the use of (1.4), (1.5) the power series of $D_{q,\alpha}^p(f(z))$ for $f(z)$ of the form (1.1) is given by

$$D_{q,\alpha}^p(f(z)) = z + \sum_{n=2}^{\infty} \left(1 + \alpha([n]_q - 1)\right)^p a_n z^n. \tag{1.6}$$

Note that
(i) For $\alpha = 1$, we get Salagean q-differential operator introduced by Govindaraj and Sivasubramanian in [18].
(ii) For $q \to 1^-$, $\alpha = 1$, we get Salagean differential operator introduced by Salagean in [32].
The Faber polynomials introduced by Faber [16] play an important role in various areas of mathematical sciences, especially in geometric function theory see also [19, 33, 34]. Not much is known about the bounds on general coefficients $|a_n|$, for $n \geq 4$ of bi-univalent functions as Ali et al. [7] also declared the bounds for the $n-th$ $(n \geq 4)$ coefficients of bi-univalent functions an open problem. In the literature only a few work determining the general coefficient $|a_n|$, for $n \geq 4$ for the analytic bi-univalent function given by (1.1). For more study see [3, 4, 21, 23, 26, 38].
Motivated by the works of Altınkaya and Yalçın [11], we define new subclass of bi-univalent functions with the theory of q-calculus. we determine estimates for the general coefficient bounds $|a_n|$ for $n \geqq 3$, by using Faber polynomial expansions.

Definition 1.4. *A function $f \in \Sigma$ is said to be in the class*

$$B_\Sigma(q, p, \lambda, \alpha, A, B)$$

$$(-1 \leq B < A \leq 1,\ q \in (0,1), p \in \mathbb{N}_0,\ \lambda \geq 0,\ \alpha \geq 0;\ z, w \in E),$$

if the following subordinations are satisfied:

$$\frac{(1-\lambda)D_{q,\alpha}^p f(z) + \lambda D_{q,\alpha}^{p+1} f(z)}{z} \prec \frac{1+Az}{1+Bz}$$

$$\frac{(1-\lambda)D_{q,\alpha}^p g(w) + \lambda D_{q,\alpha}^{p+1} g(w)}{w} \prec \frac{1+Aw}{1+Bw}$$

where the function g is given by (1.3).

Special case
i) For $q \to 1^-$, $\alpha = 1, A = 1$ and $B = -1$, then the class $B_\Sigma(q, p, \lambda, \alpha, A, B)$ reduce into the class $B_\Sigma(p, \lambda, \varphi)$ introduced by Altınkaya and Yalçın [11].
ii) For $q \to 1^-$, $\alpha = 1$, $p = 0$, $A = 1$ and $B = -1$, then the class $B_\Sigma(q, p, \lambda, \alpha, A, B)$ reduce into the class $B_\Sigma(\varphi, \lambda)$ introduced by Frasin and Aouf [17].

2. Main Results

By using the Faber polynomial expansion of functions f of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ are given by,

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-5)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2 \right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4 \right] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

and $g = f^{-1}$ given by (1.3), V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, \ldots.. |a_n|$ [5]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -(5a_2^3 - 5a_2 a_3 + a_4). \end{aligned}$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^{p} [4] is,

$$K_{n-1}^{p} = p a_n + \frac{p(p-1)}{2} E_{n-1}^2 + \frac{p!}{(p-3)!3!} E_{n-1}^3 + \ldots + \frac{p!}{(p-n+1)!(n-1)!} E_{n-1}^{n-1},$$

where $E_{n-1}^{p} = E_{n-1}^{p}(a_2, a_3 \ldots.)$ [6] given by

$$E_{n-1}^{m}(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ldots \mu_{n-1}!}, \quad for\ m \leq n.$$

While $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, \ldots, \mu_n$ satisfying:

$$\mu_1 + \mu_2 + \ldots + \mu_n = m,$$

and

$$\mu_1 + 2\mu_2 + \ldots + (n-1)\mu_{n-1} = n-1.$$

Evidently, $E_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$, [3], or equivalently,

$$E_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!}, \quad \text{for } m \leq n,$$

again $a_1 = 1$, and the taking the sum over all nonnegative integer $\mu_1, \ldots, \mu_n$ satisfying:

$$\begin{aligned} \mu_1 + \mu_2 + \ldots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ldots + n\mu_n &= n. \end{aligned}$$

It is clear that $E_n^n(a_1, \ldots, a_n) = E_1^n$ the first and last polynomials are $E_n^n = a_1^n$, and $E_n^1 = a_n$.

Theorem 2.1. *For $\lambda \geq 0$, $\alpha \geq 0$, $q \in (0,1)$, $-1 \leq B < A \leq 1$, and $p \in \mathbb{N}_0$, let $f \in B_\Sigma(q, p, \lambda, \alpha, A, B)$. If $a_m = 0$; $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{A - B}{\left(1 + \alpha([n]_q - 1)\right)^p \left\{1 + \alpha\lambda\left([n]_q - 1\right)\right\}}; \qquad n \geq 4.$$

Proof. Let f be given by (1.1), we have

$$\frac{(1-\lambda)D^{p}_{q,\alpha}f(z)+\lambda D^{p+1}_{q,\alpha}f(z)}{z}=1+\sum_{n=2}^{\infty}\left(1+\alpha([n]_q-1)\right)^{p}\left\{1+\alpha\lambda\left([n]_q-1\right)\right\}a_n z^{n-1}, \tag{2.1}$$

and for its inverse map $g=f^{-1}$, we have

$$\begin{aligned}\frac{(1-\lambda)D^{p}_{q,\alpha}g(w)+\lambda D^{p+1}_{q,\alpha}g(w)}{w} &= 1+\sum_{n=2}^{\infty}\left\{\begin{array}{c}\left(1+\alpha([n]_q-1)\right)^{p}\left\{1+\alpha\lambda\left([n]_q-1\right)\right\}\\ \times\frac{1}{[n]_q}K^{-n}_{n-1}(a_2,a_3....,a_n)w^{n-1}\end{array}\right\}\\ &= 1+\sum_{n=2}^{\infty}\left(1+\alpha([n]_q-1)\right)^{p}\left\{1+\alpha\lambda\left([n]_q-1\right)\right\}b_n w^{n-1}, \end{aligned}\tag{2.2}$$

where $b_n=\frac{1}{[n]_q}K^{-n}_{n-1}(a_2,a_3....,a_n)$.

Since, both functions f and its inverse map $g=f^{-1}$ are in $B_{\Sigma}(q,p,\lambda,\alpha,A,B)$, by the definition of subordination, for $z,w\in E$ there exist two Schwarz functions

$$\psi(z)=\sum_{n=1}^{\infty}c_n z^n$$

and

$$\Phi(w)=\sum_{n=1}^{\infty}d_n w^n,$$

such that

$$\frac{(1-\lambda)D^{p}_{q,\alpha}f(z)+\lambda D^{p+1}_{q,\alpha}f(z)}{z}=\frac{1+A(\psi(z))}{1+B(\psi(z))}, \tag{2.3}$$

and

$$\frac{(1-\lambda)D^{p}_{q,\alpha}g(w)+\lambda D^{p+1}_{q,\alpha}g(w)}{w}=\frac{1+A(\Phi(w))}{1+B(\Phi(w))}, \tag{2.4}$$

where

$$\frac{1+A(\psi(z))}{1+B(\psi(z))}=1-\sum_{n=1}^{\infty}(A-B)K^{-1}_{n}(c_1,c_2,...,c_n,B)z^n, \tag{2.5}$$

and

$$\frac{1+A(\Phi(w))}{1+B(\Phi(w))}=1-\sum_{n=1}^{\infty}(A-B)K^{-1}_{n}(d_1,d_2,...,d_n,B)w^n. \tag{2.6}$$

In general [3, 4] for any $p\in\mathbb{N}$ and $n\geq 2$, an expansion of $K^{p}_{n}(k_1,k_2,...,k_n,B)$,

$$\begin{aligned}K^{p}_{n}(k_1,k_2,...,k_n,B) &= \frac{p!}{(p-n)!n!}k_1^{n}B^{n-1}+\frac{p!}{(p-n+1)!(n-2)!}k_1^{n-2}k_2B^{n-2}\\ &+\frac{p!}{(p-n+2)!(n-3)!}\times k_1^{n-3}k_3B^{n-3}\\ &+\frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}\left[k_4B^{n-4}+\frac{p-n+3}{2}k_3^{2}B\right]\\ &+\frac{p!}{(p-n+4)!(n-5)!}k_1^{n-5}\left[k_5B^{n-5}+(p-n+4)k_3k_4B\right]\\ &+\sum_{j\geq 6}k_1^{n-1}X_j,\end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1,k_2,...,k_n$.

Comparing the corresponding coefficients of (2.3) and (2.5) yields

$$\left(1+\alpha([n]_q-1)\right)^{p}\{1+\alpha\lambda(n-1)\}a_n=-(A-B)K^{-1}_{n-1}(c_1,c_2,...,c_{n-1},B) \tag{2.7}$$

and similarly, from (2.4) and (2.6) yields

$$\left(1+\alpha([n]_q-1)\right)^{p}\{1+\alpha\lambda(n-1)\}b_n=-(A-B)K^{-1}_{n-1}(d_1,d_2,...,d_{n-1},B). \tag{2.8}$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$ we have $b_n = -a_n$, and so

$$\left(1+\alpha([n]_q-1)\right)^p \{1+\alpha\lambda(n-1)\} a_n = -(A-B)c_{n-1}, \quad (2.9)$$

$$\left(1+\alpha([n]_q-1)\right)^p \{1+\alpha\lambda(n-1)\} a_n = (A-B)d_{n-1}. \quad (2.10)$$

Now taking the absolute values of equation (2.9) and (2.10) and using the fact that $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| = \frac{|-(A-B)c_{n-1}|}{\left|\left(1+\alpha([n]_q-1)\right)^p \left\{1+\alpha\lambda\left([n]_q-1\right)\right\}\right|} = \frac{|(A-B)d_{n-1}|}{\left|\left(1+\alpha([n]_q-1)\right)^p \left\{1+\alpha\lambda\left([n]_q-1\right)\right\}\right|}$$

$$\leq \frac{A-B}{\left(1+\alpha([n]_q-1)\right)^p \left\{1+\alpha\lambda\left([n]_q-1\right)\right\}}.$$

□

For $q \to 1^-$, $\alpha = 1$, $A = 1$ and $B = -1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.2. *[11] For $\lambda \geq 0$ and $p \in \mathbb{N}_0$, let $f \in B_\Sigma(p,\lambda,\varphi)$. If $a_m = 0$; $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{2}{n^p\{1+\lambda(n-1)\}}; \qquad n \geq 4.$$

Theorem 2.3. *Let $f \in B_\Sigma(q,p,\lambda,\alpha,A,B)$, $q \in (0,1)$, $-1 \leq B < A \leq 1$, $\lambda \geq 0$, and $\alpha \geq 0$. Then*

$$|a_2| \leq \min\left\{\frac{A-B}{(1+\alpha q)^p(1+\alpha\lambda q)}, \sqrt{\frac{(A-B)\{1+|B|\}}{(1+\alpha(q+q^2))^p(1+\alpha\lambda(q+q^2))}}\right\},$$

$$|a_3| \leq \min\left\{\frac{(A-B)^2}{(1+\alpha q)^{2p}(1+\alpha\lambda q)^2} + \frac{A-B}{(1+\alpha(q+q^2))^p(1+\alpha\lambda(q+q^2))},\right.$$

$$\left.\frac{(A-B)\{2+|B|\}}{(1+\alpha((q+q^2))^p(1+\alpha\lambda(q+q^2))}\right\},$$

$$|a_3 - a_2^2| \leq \frac{A-B}{2\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda(q+q^2)}.$$

Proof. Replacing n by 2 and 3 in (2.7) and (2.8), respectively, we find that

$$(1+\alpha q)^p(1+\alpha\lambda q)a_2 = -(A-B)c_1, \quad (2.11)$$

$$\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda(q+q^2))a_3 = (A-B)c_2 + B(B-A)c_1^2, \quad (2.12)$$

$$(1+\alpha q)^p(1+\alpha\lambda q)a_2 = (A-B)d_1, \quad (2.13)$$

$$\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda(q+q^2)(2a_2^2-a_3) = (A-B)d_2 + B(B-A)d_1^2. \quad (2.14)$$

From (2.11) and (2.13) we obtain

$$|a_2| = \frac{|-(A-B)c_1|}{(1+\alpha q)^p(1+\alpha\lambda q)} = \frac{|(A-B)d_1|}{(1+\alpha q)^p(1+\alpha\lambda q)}$$

$$\leq \frac{A-B}{(1+\alpha q)^p(1+\alpha\lambda q)}. \quad (2.15)$$

Adding (2.12) and (2.14) implies

$$2\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda(q+q^2))a_2^2 = (A-B)(c_2+d_2) + B(B-A)(c_1^2+d_1^2),$$

or equivalently,

$$|a_2| \leq \sqrt{\frac{(A-B)\{1+|B|\}}{(1+\alpha(q+q^2))^p(1+\alpha\lambda(q+q^2))}}. \quad (2.16)$$

Now from (2.12), one can easily see that

$$|a_3| = \frac{\left|(A-B)c_2 + B(B-A)c_1^2\right|}{(1+\alpha(q+q^2))^p(1+\alpha\lambda(q+q^2))} \leq \frac{(A-B)\{1+|B|\}}{(1+\alpha(q+q^2))^p(1+\alpha\lambda(q+q^2))}.$$

Next in order to find the bound on the coefficient $|a_3|$,we subtract (2.14) from (2.12) we thus obtain

$$2\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda(q+q^2))(a_3-a_2^2) = (A-B)(c_2-d_2) + B(B-A)(c_1^2-d_1^2) \quad (2.17)$$

Using the fact that $c_1^2 = d_1^2$ and taking the absolute values of both sides of the equation (2.17), we obtain the desired inequality

$$\begin{aligned} |a_3| &= |a_2|^2 + \frac{|(A-B)(c_2-d_2)|}{2\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda((q+q^2))} \\ &\leq |a_2|^2 + \frac{A-B}{\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda((q+q^2))}. \end{aligned} \tag{2.18}$$

Substituting the value of a_2^2 from (2.15) into (2.18), we obtain

$$|a_3| \leq \frac{(A-B)^2}{(1+\alpha q)^{2p}(1+\alpha\lambda q)^2} + \frac{A-B}{\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda((q+q^2))}. \tag{2.19}$$

Additionaly, substituting the value of a_2^2 from (2.16) into (2.18), we obtain

$$|a_3| \leq \frac{(A-B)\{2+|B|\}}{\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda((q+q^2))}. \tag{2.20}$$

Solving the equation (2.17) for $a_3 - a_2^2$, we get the desired inequality as

$$\begin{aligned} \left|a_3-a_2^2\right| &= \left|\frac{(A-B)(c_2-d_2)+B(B-A)(c_1^2-d_1^2)}{2\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda((q+q^2))}\right| \\ &\leq \frac{A-B}{2\left(1+\alpha(q+q^2)\right)^p(1+\alpha\lambda((q+q^2))}. \end{aligned} \tag{2.21}$$

□

For $q \to 1^-, \alpha = 1, A = 1$ and $B = -1$ in Theorem 2.3 and from equation (2.14), we have the following Corollary.

Corollary 2.4. *[11] Let $f \in B_\Sigma(p,\lambda,\varphi)$, $-1 \leq B < A \leq 1$, $\lambda \geq 0$. Then*

$$|a_2| \leq \min\left\{\frac{1}{2^{p-1}(1+\lambda)}, \sqrt{\frac{4}{3^p(1+2\lambda)}}\right\} = \frac{1}{2^{p-1}(1+\lambda)}$$

$$|a_3| \leq \min\left\{\frac{1}{2^{2p-2}(1+\lambda)^2} + \frac{2}{3^p(1+2\lambda)}, \frac{2}{3^{p-1}(1+2\lambda)}\right\}$$

$$= \frac{1}{2^{2p-2}(1+\lambda)^2} + \frac{2}{3^p(1+2\lambda)},$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{4}{3^p(1+2\lambda)},$$

$$\left|a_3 - a_2^2\right| \leq \frac{2}{3^p(1+2\lambda)}.$$

Remark 2.5. ***(i)*** *For $q \to 1^-$, $p = 0$, $A = 1$, $\alpha = 1$ and $B = -1$ in Theorem 2.3 we obtain the bounds on $|a_2|$ and $|a_3|$ are improvement of the estimates given in Frasin and Aouf [17].*
(ii) *For $q \to 1^-$, $p = 0$, $\lambda = 1$, $\alpha = 1$, $A = 1$ and $B = -1$ in Theorem 2.3 we obtain the bounds on $|a_2|$ and $|a_3|$ are improvement of the estimates given in Srivastava et al.[35].*

References

[1] C. R. Adams, On the linear partial q-difference equation of general type, Trans. Amer. Math. Soc., 31 (1929), 360–371.
[2] G. E. Andrews, G. E. Askey and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[3] H. Airault, Remarks on Faber polynomials, Int. Math. Forum., 3 (9–12) (2008), 449–456.
[4] H. Airault, A. Bouali, Differential calculus on the Faber polynomials, Bull. Sci. Math., 130 (3) (2006), 179–222.
[5] H. Airault, J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, Bull. Sci. Math., 126 (5) (2002), 343–367.
[6] H. Airault. Symmetric sums associated to the factorizations of Grunsky coefficients, in: Conference, Groups and Symmetries Montreal Canada, April 2007.
[7] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma–Minda starlike and convex functions, Appl. Math. Lett., 25 (3) (2012), 344–351.
[8] Ş. Altınkaya, S. Yalçın, Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points, J. Funct. Spaces, (2015), 145242, 5 pp.
[9] Ş. Altınkaya, S. Yalçın, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Acta Univ. Apulensis, Mat. Inform., 40 (2014), 347–354.
[10] Ş. Altınkaya, S. Yalçın, Initial coefficient bounds for a general class of bi-univalent functions, Int. J. Anal., (2014), 867871, 4 pp.
[11] Ş. Altınkaya, S. Yalçın, Faber polynomial coefficient bounds for a subclass of bi-univalent functions. C. R. Acad. Sci. Paris Ser. I, 353 (2015), 1075-1080.
[12] D. A. Brannan, J. Clunie, Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Instute Held at University of Durham, New York, Academic Press, 1979.
[13] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, C. R. Acad. Sci. Paris Ser. I, 352 (6) (2014), 479–484.

[14] R. D. Carmichael, The general theory of linear q-difference equations, Amer. J. Math., 34 (1912), 147–168.
[15] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.
[16] G. Faber, Uber polynomische Entwickelungen, Math. Ann., 57 (3) (1903), 389–408.
[17] B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (9) (2011), 1569–1573.
[18] M. Govindaraj, S. Sivasubramanian, On a class of analytic functions related to conic domains involving $q-$ calculus, Analysis Math., 43 (3) (2017), 475-487.
[19] H. Grunsky, Koffizientenbedingungen fur schlict abbildende meromorphe funktionen, Math. Zeit., 45 (1939), 29-61.
[20] S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, C. R. Acad. Sci. Paris Ser. I, 352 (1) (2014), 17–20.
[21] S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficients of bi-subordinate functions, C. R. Acad. Sci. Paris Ser. I, 354 (2016), 365–370.
[22] S. G. Hamidi, J.M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by subordinations, Bull. Iran. Math. Soc., 41 (5) (2015), 1103–1119.
[23] S. Hussain, S. Khan, M. A. Zaighum, Maslina Darus, and Zahid Shareef, Coefficients Bounds for Certain Subclass of Biunivalent Functions Associated with Ruscheweyh q-Differential Operator, Journal of Complex Analysis, (2017), 2826514, 9 pp.
[24] M. E. H. Ismail, E. Merkes and D. Styer, A generalization of starlike functions, ComplexVariables Theory Appl., 14 (1990), 77–84.
[25] F. H. Jackson, On q-definite integrals, Quart. J. Pure Appl. Math., 41 (15) (1910), 193–203.
[26] J. M. Jahangiri, On the coefficients of powers of a class of Bazilevic functions, Indian J. Pure Appl. Math., 17 (9) (1986), 1140–1144.
[27] J. M. Jahangiri, S.G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, Int. J. Math. Math. Sci. (2013), 190560, 4 pp.
[28] J. M. Jahangiri, S.G. Hamidi, S. Abd Halim, Coefficients of bi-univalent functions with positive real part derivatives, Bull. Malays. Math. Soc., (2) 3 (2014), 633–640.
[29] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967), 63-68.
[30] T. E. Mason, On properties of the solution of linear q-difference equations with entire function coefficients, Amer. J. Math., 37 (1915), 439–444.
[31] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Archive for Rational Mechanics and Analysis, 32 (1969), 100-112.
[32] G. S. Salagean, Subclasses of univalent functions, in: Complex Analysis, fifth Romanian–Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Mathematics, 1013, Springer (Berlin, 1983), 362–372.
[33] M. Schiffer, A method of variation within the family of simple functions, Proc. London Math. Soc., 44 (1938), 432-449.
[34] A. C. Schaeffer, D. C. Spencer, The coefficients of schlict functions, Duke Math. J., 10 (1943), 611-635
[35] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (10) (2010), 1188–1192.
[36] H. M. Srivastava, S. S. Eker, R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat, 29 (8) (2015), 1839–1845.
[37] W. J. Trjitzinsky, Analytic theory of linear q-difference equations, Acta Math., 61 (1933), 1–38.
[38] P. G. Todorov, On the Faber polynomials of the univalent functions of class , J. Math. Anal. Appl., 162 (1) (1991), 268-276.