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A Note on Projective Coordinate Spaces Over Modules

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Abstract

In this article, we deal with a special class of local rings and determine some of its properties. Later, several properties of the (left) modules constructed over the class are examined, and a projective coordinate space over the (left) modules is constructed. In a 3-dimensional projective coordinate space, all points of a line given with the incidence matrix and, dually, the incidence matrix for the line passing through two points are obtained by the help of the Maple programme.

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1. Introduction

Recently, algebraic structures with fewer conditions and the geometric structures coordinated by them have been substantially studied. Local rings constitute an important class of these structures. The goal of this article is primarily to investigate a special class of local rings and the projective spaces over these classes.

In [1], Erdogan et. al. examined some properties of the (left) modules constructed over the real plural algebra of order n. Later, in [2], Ciftci and Erdogan obtained an n-dimensional projective coordinate space over $(n+1)$-dimensional (left) module constructed by the help of this real plural algebra. For more detailed information on the real plural algebra, see [3, 4].

In the present article, we will study the algebra $A := \mathbb{F}\eta_0 + \mathbb{F}\eta_1 + \mathbb{F}\eta_2 + \mathbb{F}\eta_3$ with the basis $\{\eta_0 = 1, \eta_1, \eta_2, \eta_3\}$ for $\eta_1, \eta_2, \eta_3 \notin \mathbb{F}$, where $\mathbb{F}$ is a field, so the ones similar to almost all of the results that are obtained in [1, 2] will also be available on A. Moreover, we can state that the results obtained here are richer and more complex, although we are studying with an algebra of order 4 instead of the algebra of order n used in [2, 1].

The remaining part of the article is structured as follows:

Section 2 gives some properties of the local ring A. Section 3 introduces some properties of the modules constructed over A, and a projective coordinate space over the module is presented in Section 4. This article has been finalized with that result. In a 3-dimensional projective coordinate space, all points of a line given with the incidence matrix and the incidence matrix for a line passing through two points are obtained by the help of the Maple programme.

2. Some Properties of A

In this section, first of all, we will start by giving a definition of a local ring: A ring with an identity element is called local if the set of its non-units form an ideal. Now, let us take a closer look at the local ring A and determine some properties of A.

Let $\mathbb{F}$ be a field. Consider $A := \mathbb{F}\eta_0 + \mathbb{F}\eta_1 + \mathbb{F}\eta_2 + \mathbb{F}\eta_3$ with componentwise addition and multiplication as follows:

$$\begin{aligned} a \cdot b &= (a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_3) \cdot (b_0 + b_1\eta_1 + b_2\eta_2 + b_3\eta_3) \\ &= a_0b_0 + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\eta_1 + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)\eta_2 \\ &\quad + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\eta_3 \end{aligned}$$

where

$\cdot$	η_1	η_2	η_3
η_1	0	η_3	-η_2
η_2	-η_3	0	η_1
η_3	η_2	-η_1	0

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with the property $\eta_i \cdot \eta_j = \frac{\eta_i\eta_j - \eta_j\eta_i}{2}$ for $1 \le i, j \le 3$ and the set $\{1, \eta_1, \eta_2, \eta_3\}$ is a basis of A. Then, A is a unital local ring with the maximal ideal

$$\mathbf{I} = \eta\, A = \{a_1\eta_1 + a_2\eta_2 + a_3\eta_3 \,|a_i \in \mathbb{F},\ \ 1 \le i \le 3\}.$$

Note that $\boldsymbol{A}$ is neither commutative nor associative. Therefore, we reach the following result stating which elements in A have an inverse.

Proposition 2.1. *An element $\alpha = a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_3 \in A$ is a unit if and only if $a_0 \neq 0$.*

Proof. We must find an element $\beta = b_0 + b_1\eta_1 + b_2\eta_2 + b_3\eta_3 \in A$ such that $\alpha \cdot \beta = 1 = \beta \cdot \alpha$ in A. From $\alpha \cdot \beta = 1$, we can write the following equations:

$$\begin{aligned} a_0b_0 &= 1 \\ a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 &= 0 \\ a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 &= 0 \\ a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 &= 0. \end{aligned}$$

From the first equation, it is obvious that $b_0 = a_0^{-1}$. By putting this result in other equations, we have the following system of linear equations:

$$\begin{aligned} a_0b_1 + a_2b_3 - a_3b_2 &= -a_1a_0^{-1} \\ a_0b_2 - a_1b_3 + a_3b_1 &= -a_2a_0^{-1} \\ a_0b_3 + a_1b_2 - a_2b_1 &= -a_3a_0^{-1}. \end{aligned}$$

The determinant of coefficients of the system is

$$\triangle = \begin{vmatrix} a_0 & -a_3 & a_2 \\ a_3 & a_0 & -a_1 \\ -a_2 & a_1 & a_0 \end{vmatrix} = a_0\left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right).$$

If $\triangle \neq 0$ (that is, $a_0 \neq 0$), then the system is Cramer and has a unique solution. In this case, the solutions are $b_i = -a_0^{-2}a_i$ for $1 \le i \le 3$. Therefore, we uniquely find that

$$\beta = \alpha^{-1} = a_0^{-1} - a_0^{-2}a_1\eta_1 - a_0^{-2}a_2\eta_2 - a_0^{-2}a_3\eta_3.$$

This completes the proof. □

Now, we can give the following result related to zero divisors of A, as an analogue to Theorem 6 in [1].

Proposition 2.2. *None of the units of A are zero divisors, namely for every $\alpha, \beta \in A$; $\alpha = a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_3$, $a_0 \neq 0$ and $\beta = b_0 + b_1\eta_1 + b_2\eta_2 + b_3\eta_3$ if $\alpha \cdot \beta = 0$ or $\beta \cdot \alpha = 0$, so $\beta = 0$. Moreover, for $1 \le k \le 3$ and $\alpha = a_k\eta_k + \cdots + a_3\eta_3$, $a_k \neq 0$ if $\alpha \cdot \beta = 0$ or $\beta \cdot \alpha = 0$, so $\beta = b_1\eta_1 + b_2\eta_2 + b_3\eta_3$.*

Proof. From $\alpha \cdot \beta = 0 = \beta \cdot \alpha$, we have $a_0b_0 = 0 \Rightarrow b_0 = 0$ since $a_0 \neq 0$, and

$$\begin{aligned} a_0b_1 + a_2b_3 - a_3b_2 &= 0 \\ a_0b_2 - a_1b_3 + a_3b_1 &= 0 \\ a_0b_3 + a_1b_2 - a_2b_1 &= 0. \end{aligned}$$

Then, it is clear that b_1,b_2 and $b_3 = 0$ with the help of the $\triangle$ in the proof of Proposition 2.1. Therefore, we find $\beta = 0$. Now, let $a_1 \neq 0$ for $k = 1$ while $\alpha = a_1\eta_1 + a_2\eta_2 + a_3\eta_3$. In this case, from $\alpha \cdot \beta = 0 = \beta \cdot \alpha$, we obtain

$$\begin{aligned} a_1b_0 + a_2b_3 - a_3b_2 &= 0 \\ -a_1b_3 + a_2b_0 + a_3b_1 &= 0 \\ a_1b_2 - a_2b_1 + a_3b_0 &= 0, \end{aligned}$$

a non-Cramer linear system. The matrix of coefficients of the system is

$$\begin{bmatrix} a_1 & 0 & -a_3 & a_2 \\ a_2 & a_3 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{bmatrix}$$

and as the principal determinant of the system we can choose

$$\delta_3 = \begin{vmatrix} a_1 & -a_3 & a_2 \\ a_2 & 0 & -a_1 \\ a_3 & a_1 & 0 \end{vmatrix} = a_1\left(a_1^2 + a_2^2 + a_3^2\right) \neq 0$$

since $a_1 \neq 0$. Then we can rearrange the system in the following manner according to δ_3:

$$\begin{aligned} a_1b_0 - a_3b_2 + a_2b_3 &= 0 \\ a_2b_0 - a_1b_3 &= -a_3b_1 \\ a_3b_0 + a_1b_2 &= a_2b_1. \end{aligned}$$

The last system is Cramer and its solutions are

$$\begin{aligned} b_0 &= 0 \\ b_2 &= a_1^{-1} a_2 b_1 \\ b_3 &= a_1^{-1} a_3 b_1 \end{aligned}$$

depending on b_1. It can be seen that $b_0 = 0$ by similar calculations to those in the cases $a_2 \neq 0$ or $a_3 \neq 0$. □

Now, we can give the following result without proof as an analogue to Proposition 7 in [1]. The result implies that there exists a matrix algebra that is isomorphic to the local ring A.

Proposition 2.3. *Let $\mathbf{K} = M_{4x4}(\mathbb{F})$ be the (linear) algebra of a matrix*

$$\mathbf{k} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}$$

which also can be stated in the form

$$\mathbf{k} = a_0 I_4 + a_1 \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\eta_1} + a_2 \underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\eta_2} + a_3 \underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\eta_3}$$

and $\eta_i \cdot \eta_j = \frac{\eta_i \eta_j - \eta_j \eta_i}{2}$ for $1 \leq i, j \leq 3$. Then, the map $f : A \to \mathbf{K}$ which is defined by $f(\mathbf{a}) = \mathbf{k}$ for every $\mathbf{a} = a_0 + a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 \in A$ is an isomorphism.
Thus we have that the set $\{\eta_0 = I_4, \eta_1, \eta_2, \eta_3\}$ is a basis of $\mathbf{K}$ with the property $\eta_i \cdot \eta_j = \frac{\eta_i \eta_j - \eta_j \eta_i}{2}$ for $1 \leq i, j \leq 3$, see the following table for the operation

$\cdot$	η_1	η_2	η_3
η_1	0	η_3	$-\eta_2$
η_2	$-\eta_3$	0	η_1
η_3	η_2	$-\eta_1$	0

Moreover, $\{\eta_1, \eta_2, \eta_3\}$ is the canonical basis of the Lie algebra sp (1) of the matrix Lie group SP (1) with the similar multiplication table, [5, p. 340].
The local ring we will study is considered the vector space. Throughout this article, we restrict ourselves to the local ring A or the algebra $\mathbf{K}$, which is isomorphic to it.

3. *A*-Modules

In this section, we will investigate some properties of the modules constructed over A, which are called an A-module. Therefore, we can give the following definition from [4, p. 69].

Definition 3.1. *Let A be a local ring. Let M be a finitely generated A-module. Then M is an A-space of finite dimension if there exists $E_1, E_2, ..., E_n$ in M where*

i. *$M = AE_1 \oplus AE_2 \oplus ... \oplus AE_n$;*
ii. *the map $A \to AE_i$ defined by $x \to xE_i$ is an isomorphism for $1 \leq i \leq n$.*

Now, we will construct a module M over the algebra $\mathbf{K}$ in the following proposition, as obtained in Proposition 8 of [1]. Thanks to this, we will obtain a basis of M.

Proposition 3.2. *$M = \mathbb{F}_m^4$ is a left module over the linear algebra of a matrix $\mathbf{K} = M_{4x4}(\mathbb{F})$. Then the following set is a basis of the $\mathbf{K}$-module M.*

$$\left\{ E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{4xm}, E_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{4xm}, \ldots, \; E_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{4xm} \right\}$$

Proof. Linear independence of this set is obvious. Moreover, for every $X \in M$, X can be written as follows:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix}_{4\times m} = \begin{pmatrix} x_{11} & -x_{21} & -x_{31} & -x_{41} \\ x_{21} & x_{11} & -x_{41} & x_{31} \\ x_{31} & x_{41} & x_{11} & -x_{21} \\ x_{41} & -x_{31} & x_{21} & x_{11} \end{pmatrix}_{4x4} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{4\times m}$$
$$+ \begin{pmatrix} x_{12} & -x_{22} & -x_{32} & -x_{42} \\ x_{22} & x_{12} & -x_{42} & x_{32} \\ x_{32} & x_{42} & x_{12} & -x_{22} \\ x_{42} & -x_{32} & x_{22} & x_{12} \end{pmatrix}_{4x4} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{4\times m}$$
$$+ \cdots + \begin{pmatrix} x_{1m} & -x_{2m} & -x_{3m} & -x_{4m} \\ x_{2m} & x_{1m} & -x_{4m} & x_{3m} \\ x_{3m} & x_{4m} & x_{1m} & -x_{2m} \\ x_{4m} & -x_{3m} & x_{2m} & x_{1m} \end{pmatrix}_{4x4} \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{4\times m}.$$

Thus, $[E_1, E_2, \cdots, E_m] = M$. Consequently, the set $\{E_1, E_2, \cdots, E_m\}$ is a basis of the $\mathbf{K}$-module M. □

Now, from [6] or [2, p. 943], we recall a definition which will be used in the next section.

Definition 3.3. *Let R be a local ring, R_0 be the maximal ideal of R, and M be a free module with unity over R. Let S be a non-empty subset of the module M. Let M_0 be a submodule of M constructed over R_0. For $x_1, x_2, \cdots, x_k \in S$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in R$, if*

$$\sum_{i=1}^{k} \alpha_i x_i \in M_0 \Rightarrow \alpha_i \in R_0 \text{ for every } i$$

holds, then S is called $R-$independent. Otherwise, S is called an $R-$dependent subset.

Finally, we would like to complete this section by giving two results on A-spaces without proof. They are the analogues of Theorem 9 and Proposition 10 in [1], respectively.

Proposition 3.4. *Let $M = A^n$. Then, for $u_1, u_2, ..., u_k \in A \setminus \mathbf{I}$ and $x_{ij} \in \mathbf{I}$, there are linearly independent vectors such that $\alpha_1 = (u_1, x_{21}, x_{31}, ..., x_{n1})$, $\alpha_2 = (x_{12}, u_2, x_{32}, ..., x_{n2})$, $\alpha_3 = (x_{13}, x_{23}, u_3, ..., x_{n3}), ..., \alpha_k = (x_{1k}, x_{2k}, x_{3k}, ..., u_k)$. For $k = n$, the set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a basis for M.*

Proposition 3.5. *An A-module M over a local ring A is an A-space if and only if it is a free finitely dimensional module.*

4. Construction of a Projective Coordinate Space

In this section, an $(m-1)$-dimensional projective coordinate space over the left module obtained in the previous section will be constructed with the help of equivalence classes, by the similar method given in [2]. Therefore, the points and lines of this projective space are determined and the points are classified.

We know from the previous section that the set $M = \mathbb{F}_m^4$ is an m-dimensional module over the local ring $\mathbf{K} = M_{4x4}(\mathbb{F})$ and that the set $\{E_1, E_2, \ldots, E_m\}$ is a basis of M. Each element of the $\mathbf{K}$-module M can be expressed uniquely as a linear combination of $E_1, E_2, \ldots, E_m$. Furthermore, a maximal ideal of $\mathbf{K}$ is denoted by

$$\mathbf{I} = \left\{ \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -a_3 & a_2 \\ a_2 & a_3 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{pmatrix} \middle| \, a_i \in \mathbb{F}, 1 \leq i \leq 3 \right\}.$$

Now let us define the set

$$M_0 = \left\{ \sum_{i=1}^{m} A_i E_i \,|\, A_i \in \mathbf{I},\ 1 \leq i \leq m \right\}.$$

Then, we have

$$M_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix} \,|\, x_{ij} \in \mathbb{F} \right\}.$$

Now, we consider the equivalence relation on the elements of

$$M^* = M \setminus M_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix} |1 \leq i \leq m,\ \exists\, x_{1i} \neq 0 \right\}$$

whose equivalence classes are the one-dimensional left submodules of M with the set M_0 deleted. Thus, if $X, Y \in M^*$, then X is equivalent to Y if $Y = \lambda X$ for $\lambda \in \mathbf{K}^* = \mathbf{K} \setminus \mathbf{I}$. The set of equivalence classes is denoted by $P(M)$. Then $P(M)$ is called an $(m-1)$-dimensional projective coordinate space and the elements of $P(M)$ are called points; the equivalence class of vector X is the point $\overline{X}$. Consequently, X is called a coordinate vector for $\overline{X}$ or that X is a vector representing $\overline{X}$. In this case, λX with $\lambda \in \mathbf{K}^*$ also represents $\overline{X}$; that is, by $\overline{\lambda X} = \overline{X}$. Thus, $\overline{X}$ can be expressed as follows:

$$\begin{aligned}
\overline{X} &= \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}_{4x4} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix}_{4xm} \\
&= \begin{pmatrix} z_{11} & z_{12} & z_{13} & \cdots & z_{1m} \\ z_{21} & z_{22} & z_{23} & \cdots & z_{2m} \\ z_{31} & z_{32} & z_{33} & \cdots & z_{3m} \\ z_{41} & z_{42} & z_{43} & \cdots & z_{4m} \end{pmatrix}_{4\times m}
\end{aligned}$$

where $a_0 \neq 0 \wedge 1 \leq i \leq m, \exists x_{1i} \neq 0$. Note that $\exists z_{1i} \neq 0$.

Let $\overline{X}, \overline{Y}, \cdots$ be $p+1$ points such that any two of them are **K**-independent. Then the set $\Pi_p = Sp\{\overline{X}, \overline{Y}, \cdots\} \backslash M_0$ is called a subspace of dimension p or p-space.

In $P(M)$, a point is a subspace of dimension 0 and a line is a subspace of dimension 1.

For $X \in M^*$, the set $\overline{X} = \{\lambda X \mid \lambda \in \mathbf{K}^*\}$ is a 0-dimensional subspace of $P(M)$. Therefore, $\overline{X}$ is a point of $P(M)$.

Now, we investigate the condition of being **K**-independent for two different points $\overline{X}$ and $\overline{Y}$ of $P(M)$.

Firstly, let us denote the coordinate vectors for the points $\overline{X}$ and $\overline{Y}$ by X and Y, respectively. We form a linear combination as

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix} + \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1m} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2m} \\ y_{31} & y_{32} & y_{33} & \cdots & y_{3m} \\ y_{41} & y_{42} & y_{43} & \cdots & y_{4m} \end{pmatrix}.$$

If this linear combination is an element of M_0, then we can write

$$\begin{aligned}
a_0x_{11} - a_1x_{21} - a_2x_{31} - a_3x_{41} + b_0y_{11} - b_1y_{21} - b_2y_{31} - b_3y_{41} &= 0 \\
a_0x_{12} - a_1x_{22} - a_2x_{32} - a_3x_{42} + b_0y_{12} - b_1y_{22} - b_2y_{32} - b_3y_{42} &= 0 \\
&\vdots \\
a_0x_{1m} - a_1x_{2m} - a_2x_{3m} - a_3x_{4m} + b_0y_{1m} - b_1y_{2m} - b_2y_{3m} - b_3y_{4m} &= 0.
\end{aligned} \tag{4.1}$$

Therefore, a homogeneous system of linear equations, which involve m equations and eight variables a_0, a_1, a_2, a_3, b_0, b_1, b_2, and b_3, is obtained. Let us denote the coefficient matrix of (4.1) by

$$A = \begin{pmatrix} x_{11} & -x_{21} & -x_{31} & -x_{41} & y_{11} & -y_{21} & -y_{31} & -y_{41} \\ x_{12} & -x_{22} & -x_{32} & -x_{42} & y_{12} & -y_{22} & -y_{32} & -y_{42} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1m} & -x_{2m} & -x_{3m} & -x_{4m} & y_{1m} & -y_{2m} & -y_{3m} & -y_{4m} \end{pmatrix}_{mx8}.$$

Now, we would like to interpret solutions of the system according to a_0 and b_0:

1. If $rankA = 8$, then we have $a_0 = a_1 = a_2 = a_3 = b_0 = b_1 = b_2 = b_3 = 0$. Therefore, this shows that

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbf{I}.$$

In that case, the coordinate vectors X and Y for the points $\overline{X}$ and $\overline{Y}$, respectively, are **K**-independent if and only if the rank of the coefficient matrix is equal to 8.

2. If $rankA = 7$, then we have $a_0 = rt$ and $b_0 = st$ where t is a parameter. There are four cases for $a_0 = b_0 = 0$:

- **i.** If $r = s = 0$, then t can be arbitrarily chosen,
- **ii.** If $r = 0$ and $s \neq 0$, then t must be chosen zero,
- **iii.** If $r \neq 0$ and $s = 0$, then t must be chosen zero,
- **iv.** If $r \neq 0$ and $s \neq 0$, then t must be chosen zero.

In this case, we have the result that the coordinate vectors X and Y for the points $\overline{X}$ and $\overline{Y}$, respectively, are **K**-independent if and only if at least one of the conditions i-iv is satisfied. Similarly, it is possible to determine the conditions of **K**-independent for the cases $2 \leq rankA \leq 6$.

Let the set $Sp\{\overline{X}, \overline{Y}\} = \{\lambda X + \gamma Y \mid \exists\, \lambda, \gamma \in \mathbf{K}^*\}$ be a 1-dimensional subspace of $P(M)$ such that $\overline{X}$ and $\overline{Y}$ are **K**-independent elements. Then $Sp\{\overline{X}, \overline{Y}\}$ is a line of $P(M)$. It is denoted by

$$Sp\{\overline{X}, \overline{Y}\} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix} + \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1m} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2m} \\ y_{31} & y_{32} & y_{33} & \cdots & y_{3m} \\ y_{41} & y_{42} & y_{43} & \cdots & y_{4m} \end{pmatrix}$$

where $a_0 \neq 0 \wedge 1 \leq i \leq m$, $\exists\, x_{1i} \neq 0$ or $b_0 \neq 0 \wedge 1 \leq i \leq m$, $\exists y_{1i} \neq 0$.

We know that, for every coordinate vector $X \in M^*$ of the point $\overline{X} \in P(M)$, X can be written uniquely as a linear combination of the vectors $E_1, E_2, \cdots, E_m$. Therefore, the matrix X is expressed as $X = \sum_{i=1}^{m} X_i E_i$ or as

$$X = (X_1, X_2, \cdots, X_m) \in \mathbf{K}^m$$

where

$$X_1 = \begin{pmatrix} x_{11} & -x_{21} & -x_{31} & -x_{41} \\ x_{21} & x_{11} & -x_{41} & x_{31} \\ x_{31} & x_{41} & x_{11} & -x_{21} \\ x_{41} & -x_{31} & x_{21} & x_{11} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} & -x_{22} & -x_{32} & -x_{42} \\ x_{22} & x_{12} & -x_{42} & x_{32} \\ x_{32} & x_{42} & x_{12} & -x_{22} \\ x_{42} & -x_{32} & x_{22} & x_{12} \end{pmatrix} \cdots, X_m = \begin{pmatrix} x_{1m} & -x_{2m} & -x_{3m} & -x_{4m} \\ x_{2m} & x_{1m} & -x_{4m} & x_{3m} \\ x_{3m} & x_{4m} & x_{1m} & -x_{2m} \\ x_{4m} & -x_{3m} & x_{2m} & x_{1m} \end{pmatrix}.$$

There are two cases:
Case 1: For the first component of the coordinate vector X of the point $\overline{X}$, if $x_{11} \neq 0$, then $X_1 \notin \mathbf{I}$ and X_1 is a unit element, so there is an inverse of X_1. If we multiply both sides of the equation with the inverse matrix X_1^{-1}, we have

$$X = (I_4, X_2, \cdots, X_m) = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1m} \\ 0 & x_{22} & x_{23} & \cdots & x_{2m} \\ 0 & x_{32} & x_{33} & \cdots & x_{3m} \\ 0 & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix}.$$

Thus, these points are called proper points.
Case 2: For the first component of the coordinate vector X of the point $\overline{X}$, if $x_{11} = 0$, then $X_1 \in \mathbf{I}$. Therefore, the inverse of the matrix X_1 does not exist. Thus, we call the points of $P(M)$ whose coordinate vectors are in the form

$$\begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix}$$

as ideal points.
Now, by giving a definition which is an analogue of the definition in [2, p. 947], we will handle a special example related to the definition.

Definition 4.1. *An s-space is the set of points whose representing vectors*

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{4m} \end{pmatrix} = (X_1, X_2, \cdots, X_m)$$

of the points $\overline{X}$ satisfy the equations $XA = 0$, where A is an $m \times ((m-1)-s)$ matrix of rank $(m-1)-s$ with coefficients in $\mathbf{K}$.

Now let us take $m = 4$ and $n = 2$, so we study an example of a 3-dimensional projective coordinate space $P(M)$. For the 3-dimensional projective coordinate space, first we will determine all points of a line whose incidence matrix is given and we will then determine the incidence matrix of a line that goes through the given points.

Example 4.2. *In the 3-dimensional projective coordinate space $P(M)$, any line, a 1-dimensional subspace Π_1, is the set of points whose representing vectors $\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = (X_1, X_2, X_3, X_4)$ of the points $\overline{X}$ satisfy the equations $XA = 0$, where A is a 4×2 matrix of rank 2 with coefficients in $\mathbf{K}$. Thus, $\Pi_1 = \left\{ \overline{X} \mid XA = 0, A \in \mathbf{K}_2^4 \diagdown \mathbf{I}_2^4 \right\}$ is obtained. Now, we identify all points of a line whose incidence matrix is*

$$\begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} & \begin{pmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_3 & e_0 & -e_1 \\ e_3 & -e_2 & e_1 & e_0 \end{pmatrix} \\ \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} & \begin{pmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & -f_3 & f_2 \\ f_2 & f_3 & f_0 & -f_1 \\ f_3 & -f_2 & f_1 & f_0 \end{pmatrix} \\ \begin{pmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & -c_3 & c_2 \\ c_2 & c_3 & c_0 & -c_1 \\ c_3 & -c_2 & c_1 & c_0 \end{pmatrix} & \begin{pmatrix} g_0 & -g_1 & -g_2 & -g_3 \\ g_1 & g_0 & -g_3 & g_2 \\ g_2 & g_3 & g_0 & -g_1 \\ g_3 & -g_2 & g_1 & g_0 \end{pmatrix} \\ \begin{pmatrix} d_0 & -d_1 & -d_2 & -d_3 \\ d_1 & d_0 & -d_3 & d_2 \\ d_2 & d_3 & d_0 & -d_1 \\ d_3 & -d_2 & d_1 & d_0 \end{pmatrix} & \begin{pmatrix} h_0 & -h_1 & -h_2 & -h_3 \\ h_1 & h_0 & -h_3 & h_2 \\ h_2 & h_3 & h_0 & -h_1 \\ h_3 & -h_2 & h_1 & h_0 \end{pmatrix} \end{bmatrix} \in \mathbf{K}_2^4 \diagdown \mathbf{I}_2^4.$$

As a consequence of the incidence matrix, it is trivial to see that $\exists a_0, b_0, c_0, d_0, e_0, f_0, g_0, h_0 \neq 0$.
For $XA = 0$, *we have the following cases:*

Case 1: *For the coordinate vector X of the point* $\overline{X}$, *if* $x_{11} \neq 0$, *then* $X = (I_4, X_2, X_3, X_4) \in \mathbf{K}^4$. *Thus, we obtain* 32 *equations with* 12 *variables from* $XA = 0$. *If we solve this system of linear equations by using the Maple programme, we have the following solutions:*

$$\begin{aligned}
x_{12} &= a' + b'x_{14} + c'x_{24} + d'x_{34} + e'x_{44}\\
x_{22} &= a'' + b''x_{14} + c''x_{24} + d''x_{34} + e''x_{44}\\
x_{32} &= a''' + b'''x_{14} + c'''x_{24} + d'''x_{34} + e'''x_{44}\\
x_{42} &= a^{iv} + b^{iv}x_{14} + c^{iv}x_{24} + d^{iv}x_{34} + e^{iv}x_{44}\\
x_{13} &= a^{v} + b^{v}x_{14} + c^{v}x_{24} + d^{v}x_{34} + e^{v}x_{44}\\
x_{23} &= a^{vi} + b^{vi}x_{14} + c^{vi}x_{24} + d^{vi}x_{34} + e^{vi}x_{44}\\
x_{33} &= a^{vii} + b^{vii}x_{14} + c^{vii}x_{24} + d^{vii}x_{34} + e^{vii}x_{44}\\
x_{43} &= a^{viii} + b^{viii}x_{14} + c^{viii}x_{24} + d^{viii}x_{34} + e^{viii}x_{44}\\
x_{14} &= x_{14},\ x_{24} = x_{24},\ x_{34} = x_{34}, x_{44} = x_{44}
\end{aligned}$$

where $a', b', c', d', e', a'', b'', c'', d'', e'', a''', b''', c''', d''', e''', a^{iv}, b^{iv}, c^{iv}, d^{iv}, e^{iv}, a^{v}, b^{v}, c^{v}, d^{v}, e^{v}, a^{vi}, b^{vi}, c^{vi}, d^{vi}, e^{vi}, a^{vii}, b^{vii}, c^{vii}, d^{vii}, e^{vii}, a^{viii}, b^{viii}, c^{viii}, d^{viii}, e^{viii} \in \mathbb{F}$.

Case 2: *For the coordinate vector X of the point* $\overline{X}$, *if* $x_{11} = 0$, *then* $\overline{X}$ *is an ideal point of the form*

$$\left(X_1 = \begin{pmatrix} 0 & -x_{21} & -x_{31} & -x_{41}\\ x_{21} & 0 & -x_{41} & x_{31}\\ x_{31} & x_{41} & 0 & -x_{21}\\ x_{41} & -x_{31} & x_{21} & 0 \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} & -x_{22} & -x_{32} & -x_{42}\\ x_{22} & x_{12} & -x_{42} & x_{32}\\ x_{32} & x_{42} & x_{12} & -x_{22}\\ x_{42} & -x_{32} & x_{22} & x_{12} \end{pmatrix} \right.$$
$$\left. X_3 = \begin{pmatrix} x_{13} & -x_{23} & -x_{33} & -x_{43}\\ x_{23} & x_{13} & -x_{43} & x_{33}\\ x_{33} & x_{43} & x_{13} & -x_{23}\\ x_{43} & -x_{33} & x_{23} & x_{13} \end{pmatrix}, X_4 = \begin{pmatrix} x_{14} & -x_{24} & -x_{34} & -x_{44}\\ x_{24} & x_{14} & -x_{44} & x_{34}\\ x_{34} & x_{44} & x_{14} & -x_{24}\\ x_{44} & -x_{34} & x_{24} & x_{14} \end{pmatrix} \right).$$

Here, we know that $\exists\, x_{12}, x_{13}, x_{14} \neq 0$. *Thus, we reach the followings solutions from* $XA = 0$ *by using the Maple programme:*
x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42} *are written as depending on the parameters* $x_{13} = t_1$, $x_{14} = t_2$, $x_{23} = t_3$, $x_{24} = t_4$, $x_{33} = t_5$, $x_{34} = t_6$, $x_{43} = t_7$, $x_{44} = t_8$. *Therefore,* x_{11} *can be written as*
$x_{11} = u_1t_1 + u_2t_2 + u_3t_3 + u_4t_4 + u_5t_5 + u_6t_6 + u_7t_7 + u_8t_8$ *where* $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 \in \mathbb{F}$. *Then, we have the following situations:*

(a) If $\forall t_i = 0$, *then* $x_{11} = x_{12} = x_{21} = x_{22} = x_{31} = x_{32} = x_{41} = x_{42} = 0$. *Therefore,* $\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \in M_0$ *is obtained. However, this is a contradiction because the zero matrix does not represent a point. Therefore,* $\exists t_i \neq 0$.

(b) If $\forall u_i = 0$, *then* $x_{11} = 0$. *In this situation, from (a), we can choose either* $\exists t_1 = 1$ *or* $\exists t_2 = 1$. *The ideal point is then* $\begin{pmatrix} 0 & x_{12} & 1 & t_2\\ x_{21} & x_{22} & t_3 & t_4\\ x_{31} & x_{32} & t_5 & t_6\\ x_{41} & x_{42} & t_7 & t_8 \end{pmatrix} \in M^*$ *or* $\begin{pmatrix} 0 & x_{12} & t_1 & 1\\ x_{21} & x_{22} & t_3 & t_4\\ x_{31} & x_{32} & t_5 & t_6\\ x_{41} & x_{42} & t_7 & t_8 \end{pmatrix} \in M^*$.

(c) For $\exists k \in \{1, 2, 3, ..., 8\}$, *if* $\exists u_k \neq 0$, *then from* $x_{11} = u_1t_1 + u_2t_2 + \cdots + u_kt_k + \cdots + u_8t_8$ *we have*

$$u_kt_k = -\sum_{i \neq k} u_it_i, \tag{4.2}$$

so $x_{11} = 0$. *Here, there are two cases:*
i) If $t_i = 0$ *for* $\forall i$ *where* $i \neq k$ *in (4.2), then* $u_kt_k = 0$. *Here* $t_k = 0$ *since* $u_k \neq 0$. *Thus,* $\forall t_i = 0$ *for* $1 \leq i \leq 8$ *is obtained. This is a contradiction of (a). Therefore,* $t_i \neq 0$ *for* $\exists i$, $i \neq k$. *In that case, if* $k = 1$, *then by choosing* $\exists t_2 = 1 \neq 0$ *in (4.2), the ideal point* $\begin{pmatrix} 0 & x_{12} & t_1 & 1\\ x_{21} & x_{22} & t_3 & t_4\\ x_{31} & x_{32} & t_5 & t_6\\ x_{41} & x_{42} & t_7 & t_8 \end{pmatrix} \in M^*$ *is obtained. If* $k = 2$, *then by choosing* $\exists t_1 = 1 \neq 0$ *in (4.2), the ideal point* $\begin{pmatrix} 0 & x_{12} & 1 & t_2\\ x_{21} & x_{22} & t_3 & t_4\\ x_{31} & x_{32} & t_5 & t_6\\ x_{41} & x_{42} & t_7 & t_8 \end{pmatrix} \in M^*$ *is obtained. If* $3 \leq k \leq 8$, *then by choosing either* $\exists t_1 = 1 \neq 0$ *or* $\exists t_2 = 1 \neq 0$ *in (4.2) the ideal point can be found easily.*
ii) If $u_i = 0$ *for* $\forall i$,$i \neq k$ *in (4.2), then* $t_k = 0$ *since* $u_kt_k = 0$ *and* $u_k \neq 0$. *Therefore, if* $k = 1$, *then* $t_1 = 0$ *and* $\exists t_i \neq 0$ *from (a) by choosing* $t_2 = 1$ *in (4.2), the ideal point* $\begin{pmatrix} 0 & x_{12} & 0 & 1\\ x_{21} & x_{22} & t_3 & t_4\\ x_{31} & x_{32} & t_5 & t_6\\ x_{41} & x_{42} & t_7 & t_8 \end{pmatrix} \in M^*$ *is obtained. If* $k = 2$, *then* $t_2 = 0$ *and* $\exists t_i \neq 0$ *from (a) by choosing* $t_1 = 1$ *in*

(4.2). The ideal point $\begin{pmatrix} 0 & x_{12} & 1 & 0 \\ x_{21} & x_{22} & t_3 & t_4 \\ x_{31} & x_{32} & t_5 & t_6 \\ x_{41} & x_{42} & t_7 & t_8 \end{pmatrix} \in M^*$ *is obtained. If* $k = 3,4,5,6,7,8$*, then* $t_k = 0$ *and* $\exists t_i \neq 0$ *from (a) by choosing either* $t_1 = 1$ *or* $t_2 = 1$ *in (4.2). The ideal point can be found easily.*

Now conversely, we have a new situation. We determine the incidence matrix of a line whose points are given.

Let us take the general coordinate vectors

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{pmatrix}$$

of the points $\overline{X}$ *and* $\overline{Y}$*, respectively. Then we search the incidence matrix of the form*

$$\begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} & \begin{pmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_3 & e_0 & -e_1 \\ e_3 & -e_2 & e_1 & e_0 \end{pmatrix} \\ \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} & \begin{pmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & -f_3 & f_2 \\ f_2 & f_3 & f_0 & -f_1 \\ f_3 & -f_2 & f_1 & f_0 \end{pmatrix} \\ \begin{pmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & -c_3 & c_2 \\ c_2 & c_3 & c_0 & -c_1 \\ c_3 & -c_2 & c_1 & c_0 \end{pmatrix} & \begin{pmatrix} g_0 & -g_1 & -g_2 & -g_3 \\ g_1 & g_0 & -g_3 & g_2 \\ g_2 & g_3 & g_0 & -g_1 \\ g_3 & -g_2 & g_1 & g_0 \end{pmatrix} \\ \begin{pmatrix} d_0 & -d_1 & -d_2 & -d_3 \\ d_1 & d_0 & -d_3 & d_2 \\ d_2 & d_3 & d_0 & -d_1 \\ d_3 & -d_2 & d_1 & d_0 \end{pmatrix} & \begin{pmatrix} h_0 & -h_1 & -h_2 & -h_3 \\ h_1 & h_0 & -h_3 & h_2 \\ h_2 & h_3 & h_0 & -h_1 \\ h_3 & -h_2 & h_1 & h_0 \end{pmatrix} \end{bmatrix} \in \mathbf{K}_2^4 \diagdown \mathbf{I}_2^4.$$

We know that the coordinate vectors of these points are as follows:

$$\left(X_1 = \begin{pmatrix} x_{11} & -x_{21} & -x_{31} & -x_{41} \\ x_{21} & x_{11} & -x_{41} & x_{31} \\ x_{31} & x_{41} & x_{11} & -x_{21} \\ x_{41} & -x_{31} & x_{21} & x_{11} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} & -x_{22} & -x_{32} & -x_{42} \\ x_{22} & x_{12} & -x_{42} & x_{32} \\ x_{32} & x_{42} & x_{12} & -x_{22} \\ x_{42} & -x_{32} & x_{22} & x_{12} \end{pmatrix} \right.$$
$$\left. X_3 = \begin{pmatrix} x_{13} & -x_{23} & -x_{33} & -x_{43} \\ x_{23} & x_{13} & -x_{43} & x_{33} \\ x_{33} & x_{43} & x_{13} & -x_{23} \\ x_{43} & -x_{33} & x_{23} & x_{13} \end{pmatrix}, X_4 = \begin{pmatrix} x_{14} & -x_{24} & -x_{34} & -x_{44} \\ x_{24} & x_{14} & -x_{44} & x_{34} \\ x_{34} & x_{44} & x_{14} & -x_{24} \\ x_{44} & -x_{34} & x_{24} & x_{14} \end{pmatrix} \right).$$

and

$$\left(Y_1 = \begin{pmatrix} y_{11} & -y_{21} & -y_{31} & -y_{41} \\ y_{21} & y_{11} & -y_{41} & y_{31} \\ y_{31} & y_{41} & y_{11} & -y_{21} \\ y_{41} & -y_{31} & y_{21} & y_{11} \end{pmatrix}, Y_2 = \begin{pmatrix} y_{12} & -y_{22} & -y_{32} & -y_{42} \\ y_{22} & y_{12} & -y_{42} & y_{32} \\ y_{32} & y_{42} & y_{12} & -y_{22} \\ y_{42} & -y_{32} & y_{22} & y_{12} \end{pmatrix} \right.$$
$$\left. Y_3 = \begin{pmatrix} y_{13} & -y_{23} & -y_{33} & -y_{43} \\ y_{23} & y_{13} & -y_{43} & y_{33} \\ y_{33} & y_{43} & y_{13} & -y_{23} \\ y_{43} & -y_{33} & y_{23} & y_{13} \end{pmatrix}, Y_4 = \begin{pmatrix} y_{14} & -y_{24} & -y_{34} & -y_{44} \\ y_{24} & y_{14} & -y_{44} & y_{34} \\ y_{34} & y_{44} & y_{14} & -y_{24} \\ y_{44} & -y_{34} & y_{24} & y_{14} \end{pmatrix} \right).$$

Thus, we obtain 64 *equations with* 32 *variables from* $XA = 0$ *and* $YA = 0$*:*

If we solve this linear equation system by using the Maple programme, we have the following solutions:

$$
\begin{aligned}
a_0 &= \lambda_1+\lambda_2c_0+\lambda_3c_1+\lambda_4c_2+\lambda_5c_3+\lambda_6d_0+\lambda_7d_1+\lambda_8d_2+\lambda_9d_3\\
a_1 &= \lambda_{10}+\lambda_{11}c_0+\lambda_{12}c_1+\lambda_{13}c_2+\lambda_{14}c_3+\lambda_{15}d_0+\lambda_{16}d_1+\lambda_{17}d_2+\lambda_{18}d_3\\
a_2 &= \lambda_{19}+\lambda_{20}c_0+\lambda_{21}c_1+\lambda_{22}c_2+\lambda_{23}c_3+\lambda_{24}d_0+\lambda_{25}d_1+\lambda_{26}d_2+\lambda_{27}d_3\\
a_3 &= \lambda_{28}+\lambda_{29}c_0+\lambda_{30}c_1+\lambda_{31}c_2+\lambda_{32}c_3+\lambda_{33}d_0+\lambda_{34}d_1+\lambda_{35}d_2+\lambda_{36}d_3\\
b_0 &= \lambda_{37}+\lambda_{38}c_0+\lambda_{39}c_1+\lambda_{40}c_2+\lambda_{41}c_3+\lambda_{42}d_0+\lambda_{43}d_1+\lambda_{44}d_2+\lambda_{42}d_3\\
b_1 &= \lambda_{46}+\lambda_{47}c_0+\lambda_{48}c_1+\lambda_{49}c_2+\lambda_{50}c_3+\lambda_{51}d_0+\lambda_{52}d_1+\lambda_{53}d_2+\lambda_{54}d_3\\
b_2 &= \lambda_{55}+\lambda_{56}c_0+\lambda_{57}c_1+\lambda_{58}c_2+\lambda_{59}c_3+\lambda_{60}d_0+\lambda_{61}d_1+\lambda_{62}d_2+\lambda_{63}d_3\\
b_3 &= \lambda_{64}+\lambda_{65}c_0+\lambda_{66}c_1+\lambda_{67}c_2+\lambda_{68}c_3+\lambda_{69}d_0+\lambda_{70}d_1+\lambda_{71}d_2+\lambda_{72}d_3\\
e_0 &= \lambda_{73}+\lambda_{74}g_0+\lambda_{75}g_1+\lambda_{76}g_2+\lambda_{77}g_3+\lambda_{78}h_0+\lambda_{79}h_1+\lambda_{80}h_2+\lambda_{81}h_3\\
e_1 &= \lambda_{82}+\lambda_{83}g_0+\lambda_{84}g_1+\lambda_{85}g_2+\lambda_{86}g_3+\lambda_{87}h_0+\lambda_{88}h_1+\lambda_{89}h_2+\lambda_{90}h_3\\
e_2 &= \lambda_{91}+\lambda_{92}g_0+\lambda_{93}g_1+\lambda_{94}g_2+\lambda_{95}g_3+\lambda_{96}h_0+\lambda_{97}h_1+\lambda_{98}h_2+\lambda_{99}h_3\\
e_3 &= \lambda_{100}+\lambda_{101}g_0+\lambda_{102}g_1+\lambda_{103}g_2+\lambda_{104}g_3+\lambda_{105}h_0+\lambda_{106}h_1+\lambda_{107}h_2+\lambda_{108}h_3\\
f_0 &= \lambda_{109}+\lambda_{110}g_0+\lambda_{111}g_1+\lambda_{112}g_2+\lambda_{113}g_3+\lambda_{114}h_0+\lambda_{115}h_1+\lambda_{116}h_2+\lambda_{117}h_3\\
f_1 &= \lambda_{118}+\lambda_{119}g_0+\lambda_{120}g_1+\lambda_{121}g_2+\lambda_{122}g_3+\lambda_{123}h_0+\lambda_{124}h_1+\lambda_{125}h_2+\lambda_{126}h_3\\
f_2 &= \lambda_{127}+\lambda_{128}g_0+\lambda_{129}g_1+\lambda_{130}g_2+\lambda_{131}g_3+\lambda_{132}h_0+\lambda_{133}h_1+\lambda_{134}h_2+\lambda_{135}h_3\\
f_3 &= \lambda_{136}+\lambda_{137}g_0+\lambda_{138}g_1+\lambda_{139}g_2+\lambda_{140}g_3+\lambda_{141}h_0+\lambda_{142}h_1+\lambda_{143}h_2+\lambda_{144}h_3\\
c_0 &= c_0,\ c_1=c_1,\ c_2=c_2,\ c_3=c_3,\ d_0=d_0,\ d_1=d_1,\ d_2=d_2,\ d_3=d_3\\
g_0 &= g_0,\ g_1=g_1,\ g_2=g_2,\ g_3=g_3,\ h_0=h_0,\ h_1=h_1,\ h_2=h_2,\ h_3=h_3
\end{aligned}
$$

where $\lambda_i \in \mathbb{F}$, $1 \le i \le 144$. *Moreover, the following special cases for the general two points X and Y can be also examined. These are as follows:*
1. X and Y are proper points.
2. One is a proper point and the other is an ideal point.
3. X and Y are ideal points.

Finally, we have reached all of the results we have targeted in view of the papers in [1, 2].

5. Conclusion

In this study, we deal with the Lie algebra sp(1) of the matrix Lie group SP(1). It is found that the (left) modules were constructed over the algebra (which is also a local ring); therefore, an (m-1)-dimensional projective coordinate space was constructed over the m-dimensional module. As a concrete example, in a 3-dimensional projective coordinate space, all points of a line given with the incidence matrix and, dually, the incidence matrix for the line going through two points are obtained by the help of the Maple programme since it is difficult to do operations manually.

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