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On New Fractional Hermite-Hadamard Type Inequalities for (α^*, m)-Convex Functions

Erhan Set[1*], M. Emin Özdemir[2] and Nejla Korkut[1]

[1] *Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey*
[2] *Department of Elementary Education, Faculty of Education, Uludağ University, Bursa, Turkey*
[*] *Corresponding author E-mail: erhanset@yahoo.com*

Abstract

The aim of the present paper is to investigate some new Hermite-Hadamard type integral inequalities for (α^*, m)-convex functions via Riemann-Liouville fractional integrals.

Keywords: *Fractional Hermite-Hadamard inequalities, m-convex function, (α^*, m)-convex function.*
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1. Introduction

In recent years, theory of convex functions has received special attention by many researchers because of its importance in different fields of pure and applied sciences such as optimization and economics. Consequently the classical concepts of convex functions has been extended and generalized in different directions using novel and innovative ideas, see [4, 7, 8, 10, 12, 17, 18]. A functions $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. One of the most famous inequality for convex functions is so called Hermite-Hadamard inequality as follows:
Let I be an interval of real numbers and $a, b \in I$ with $a < b$. If $f : I \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The Hermite-Hadamard inequality usually stated as a results valid for convex functions only, actually holds for some different type of convexity. For recent results concerning Hermite-Hadamard inequalities obtained via different type of convexity, see ([1]-[20]).
In [17], G.Toader defined $m-convexity$ as the following:

Definition 1. *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be $m-convex$, where $m \in [0,1]$, if we have*

$$f(tx + m(1-t)y) \leq t f(x) + m(1-t) f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m(b)$ the set of the $m-convex$ functions on $[0,b]$ for which $f(0) \leq 0$. In [10], V.G.Miheşan defined $(\alpha^*, m)-convexity$ as the following:

Definition 2. *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be $(\alpha^*, m)-convex$, where $(\alpha^*, m) \in [0,1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha^*} f(x) + m(1-t^{\alpha^*}) f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Email addresses: erhanset@yahoo.com (E. Set), eminozdemir@uludag.edu.tr (M. E. Özdemir), neclakrkt63@gmail.com (N. Korkut)

Denote by $K_m^{\alpha^*}(b)$ the class of all $(\alpha^*,m)-convex$ functions on $[0,b]$ for which $f(0)\leq 0$. It can be easily seen that for $(\alpha^*,m)=(1,m)$, $(\alpha^*,m)-convexity$ reduces to $m-convexity$; $(\alpha^*,m)=(\alpha^*,1)$, $(\alpha^*,m)-convexity$ reduces to $\alpha^*-convexity$ and for $(\alpha^*,m)=(1,1)$, $(\alpha^*,m)-convexity$ reduces to the concept of usual convexity defined on $[0,b]$, $b>0$. For recent results and generalizations concerning $(\alpha^*,m)-convex$ functions, see [1, 2, 14].
We give some necessary definitions and mathematical preliminary of fractional calculus theory which are used throughout this paper. The symbols ${}_{RL}J_{a^+}^{\alpha}f$ and ${}_{RL}J_{b^-}^{\alpha}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\in\mathbb{R}^+$ (see [9]) defined by

$$\begin{aligned}({}_{RL}J_{a^+}^{\alpha}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \;\; (0\leq a<x\leq b),\\ ({}_{RL}J_{b^-}^{\alpha}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \;\; (0\leq a<x\leq b),\end{aligned}$$

respectively, where $\Gamma(x)=\int_0^\infty e^{-t}u^{x-1}du$. Here is $J_{a^+}^0 f=J_{b^-}^0 f=f(x)$. In the case of $\alpha=1$, the fractional integral reduces to the classical integral. Some recent result and properties concerning the operator can be found [3, 5, 11, 15].
In the present note, we establish new Hermite-Hadamard type inequalities for (α^*,m)-convex function via Riemann-Liouville fractional integral. An intersting feature of our results is that they provide new estimate, on these types of inequalities for fractional integrals.

2. Main Results

In order to prove our results we need the following Lemmas:

Lemma 1. *Let $f:I\to\mathbb{R}$, $I\subset\mathbb{R}$, be a differentiable mapping on I^o(interior of I), $m\in[0,1]$ and $a,b\in I$, where $0\leq a<b<\infty$. If $f'\in L[ma,b]$, $ma\in I$, then*

$$\begin{aligned}&\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\\ &-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\,{}_{RL}J_{x^-}^{\alpha}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\,{}_{RL}J_{mx^+}^{\alpha}f(b)\right)\\ =\;&\frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^\alpha+x)f'(tx+m(1-t)a)dt\\ &-\frac{(b-mx)^2}{(b-a)^2}\int_0^1(b(1-t)^\alpha+x)f'(tb+m(1-t)x)dt\end{aligned}$$

for $\alpha>0$ and $x\in(a,b)$.

Proof. It suffices to note that

$$\begin{aligned}I &= \frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^\alpha+x)f'(tx+m(1-t)a)dt\\ &\quad-\frac{(b-mx)^2}{(b-a)^2}\int_0^1(b(1-t)^\alpha+x)f'(tb+m(1-t)x)dt\\ &= I_1-I_2. \end{aligned}\tag{2.1}$$

Integrating by parts

$$\begin{aligned}I_1 &= \frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^\alpha+x)f'(tx+m(1-t)a)dt\\ &= \frac{(x-ma)^2}{(b-a)^2}\left[\frac{(at^\alpha+x)f(tx+m(1-t)a)}{(x-ma)}\Bigg|_0^1-\int_0^1\frac{a\alpha t^{\alpha-1}f(tx+m(1-t)a)}{(x-ma)}dt\right]\\ &= \frac{(x-ma)^2}{(b-a)^2}\left(\frac{(a+x)f(x)-xf(ma)}{(x-ma)}-\frac{a\alpha}{(x-ma)}\int_0^1 t^{\alpha-1}f(tx+m(1-t)a)dt\right)\\ &= \frac{(x-ma)^2[(a+x)f(x)-xf(ma)]}{(b-a)^2(x-ma)}-\frac{a\alpha}{(b-a)^2}\int_{ma}^x\left(\frac{\tau-ma}{x-ma}\right)^{\alpha-1}f(\tau)d\tau\\ &= \frac{(x-ma)[(a+x)f(x)-xf(ma)]}{(b-a)^2}-\frac{a\Gamma(\alpha+1)}{(b-a)^2(x-ma)^{\alpha-1}}\,{}_{RL}J_{x^-}^{\alpha}f(ma) \end{aligned}\tag{2.2}$$

and similary we get

$$
\begin{aligned}
I_2 &= \frac{(b-mx)^2}{(b-a)^2}\int_0^1 (b(1-t)^\alpha + x) f'(tb+m(1-t)x)dt \\
&= \frac{(b-mx)^2}{(b-a)^2}\left[\frac{(b(1-t)^\alpha + x) f(tb+m(1-t)x)}{(b-mx)}\bigg|_0^1 \right. \\
&\quad \left. -\int_0^1 \frac{(b\alpha(1-t)^{\alpha-1}) f(tb+m(1-t)x)}{(b-mx)}dt \right] \\
&= \frac{(b-mx)^2}{(b-a)^2}\left(\frac{[xf(b)-(b+x)f(mx)]}{(b-mx)} \right. \\
&\quad \left. -\frac{b\alpha}{b-mx}\int_0^1 (1-t)^{\alpha-1} f(tb+m(1-t)x)dt \right) \\
&= \frac{(b-mx)^2[(x)f(b)-(b+x)f(mx)]}{(b-a)^2(b-mx)} - \frac{b\alpha}{(b-a)^2}\int_{mx}^b \left(\frac{b-\tau}{b-mx}\right)^{\alpha-1} f(\tau)d\tau \\
&= \frac{(b-mx)[(x)f(b)-(b+x)f(mx)]}{(b-a)^2} - \frac{b\Gamma(\alpha+1)}{(b-a)^2(b-mx)^{\alpha-1}}\ {}_{RL}J^\alpha_{mx^+} f(b) \qquad (2.3)
\end{aligned}
$$

Submitting (2.2) and (2.3) to (2.1), we have

$$
\begin{aligned}
I &= \frac{(x-ma)[(a+x)f(x)-xf(ma)]-(b-mx)[(b+x)f(mx)-xf(b)]}{(b-a)^2} \\
&- \frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^\alpha_{x^-} f(ma) + \frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^\alpha_{mx^+} f(b) \right).
\end{aligned}
$$

Lemma 2. *Let $f: I \to \mathbb{R}$, $I \subset \mathbb{R}$, be twice differentiable mapping on I^o(interior of I), $m \in [0,1]$ and $a,b \in I$, where $0 \le a < b < \infty$. If $f'' \in L[ma,mb]$, $ma,mb \in I$, then*

$$
\begin{aligned}
& \frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)} + \frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)} \right] \\
& -\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^\alpha_{x^+} f(mb) + \frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^\alpha_{mx^-} f(a) \right) \\
=\ & \frac{(x-mb)}{(b-a)^2}\int_0^1 (t^{\alpha+1}x+tb) f''(tx+m(1-t)b)dt \\
& + \frac{(a-mx)}{(b-a)^2}\int_0^1 (b(1-t)^\alpha + (1-t)x) f''(ta+m(1-t)x)dt
\end{aligned}
$$

for $\alpha > 0$ and $x \in (a,b)$.

Proof. It suffices to note that

$$
\begin{aligned}
I &= \frac{(x-mb)}{(b-a)^2}\int_0^1 (t^{\alpha+1}x+tb) f''(tx+m(1-t)b)dt \\
&= \frac{(a-mx)}{(b-a)^2}\int_0^1 (b(1-t)^\alpha + (1-t)x) f''(ta+m(1-t)x)dt \\
&= I_3 + I_4. \qquad (2.4)
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
I_3 &= \frac{(x-mb)}{(b-a)^2}\int_0^1 (t^{\alpha+1}x+tb) f''(tx+m(1-t)b)dt \\
&= \frac{(x-mb)}{(b-a)^2}\left[\frac{(t^{\alpha+1}x+tb) f'(tx+m(1-t)b)}{(x-mb)}\bigg|_0^1 \right. \\
&\quad \left. -\int_0^1 \frac{f'(tx+m(1-t)b)((\alpha+1)t^\alpha x+b)}{(x-mb)}dt \right]
\end{aligned}
$$

(2.5)

$$\begin{aligned}
&= \frac{(x+b)f'(x)}{(b-a)^2} - \frac{1}{(b-a)^2}\int_0^1 ((\alpha+1)t^{\alpha}x+b)f'(tx+m(1-t)b)dt \\
&= \frac{(x+b)f'(x)}{(b-a)^2} - \left(\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(x-mb)(b-a)^2}\right. \\
&\quad \left. + \frac{\alpha(\alpha+1)x}{(x-mb)(b-a)^2}\int_{mb}^{x}\left(\frac{\tau-mb}{x-mb}\right)^{\alpha-1} f(\tau)\frac{1}{x-mb}d\tau\right) \\
&= \frac{(x+b)f'(x)}{(b-a)^2} + \frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)(b-a)^2} \\
&\quad - \frac{x(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb) \qquad (2.6)
\end{aligned}$$

and similary we get

$$\begin{aligned}
I_4 &= \frac{(a-mx)}{(b-a)^2}\int_0^1 ((1-t)^{\alpha+1}b+(1-t)x)f''(ta+m(1-t)x)dt \\
&= \frac{(a-mx)}{(b-a)^2}\left[\frac{((1-t)^{\alpha+1}b+(1-t)x)f'(ta+m(1-t)x)}{a-mx}\Bigg|_0^1\right. \\
&\quad \left. - \int_0^1 \frac{((\alpha+1)(1-t)^{\alpha}b+x)f'(ta+m(1-t)x)}{a-mx}dt\right] \\
&= -\frac{[(b+x)f'(mx)]}{(b-a)^2} \\
&\quad - \frac{(a-mx)}{(b-a)^2}\int_0^1 \frac{((\alpha+1)(1-t)^{\alpha}b+x)f'(ta+m(1-t)x)dt}{a-mx} \\
&= -\frac{(b+x)f'(mx)}{(b-a)^2} + \left(\frac{(xf(a)-((\alpha+1)b+x)f(mx))}{(b-a)^2(mx-a)}\right. \\
&\quad \left. + \frac{\alpha(\alpha+1)b}{(b-a)^2(a-mx)}\int_{mx}^{a}\left(\frac{a-\tau}{a-mx}\right)^{\alpha-1} f(\tau)\frac{1}{a-mx}d\tau\right) \\
&= -\frac{(b+x)f'(mx)}{(b-a)^2} + \frac{(xf(a)-((\alpha+1)b+x)f(mx))}{(b-a)^2(mx-a)} \\
&\quad - \frac{b(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a). \qquad (2.7)
\end{aligned}$$

Submitting (2.6) and (2.7) to (2.4), we have

$$\begin{aligned}
I: &= \frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)} + \frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right] \\
&\quad - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb) + \frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right).
\end{aligned}$$

Now, we show some new fractional Hermite-Hadamard type inequalities involving left-sided and right-sided Riemann-Liouville fractional integrals using our equalities in Lemma 1 and Lemma 2 via Definition 1 and Definition 2. Also, we will need the following elemantary inequality in proofs of some theorems:

Lemma 3. *[19, 20] For $A>0,\ B>0$, when $\theta \geq 1$ it holds*

$$A^{\theta}+B^{\theta} \leq (A+B)^{\theta} \leq 2^{\theta-1}(A^{\theta}+B^{\theta}).$$

Now we are ready to present the main results in this paper.

Theorem 1. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L[ma,b]$, where $ma\in I$ with $0\leq a<b<\infty$. If $|f'|$ is $(\alpha^*,m)-$convex on $[ma,b]$ for some fixed $\alpha^*,m\in[0,1]$, then for any $0<\alpha$ and $x\in(a,b)$, the* *following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\\
&-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{mx^+}f(b)\right)\Bigg|\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\left[\left(\frac{a}{\alpha+\alpha^*+1}+\frac{x}{\alpha^*+1}\right)|f'(x)|\right.\\
&\left.+\left(\frac{am}{\alpha+1}-\frac{am}{\alpha+\alpha^*+1}+xm-\frac{xm}{\alpha^*+1}\right)|f'(a)|\right]\\
&+\frac{(b-mx)^2}{(b-a)^2}\left[\left(\frac{b\Gamma(\alpha+1)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+2)}+\frac{x}{\alpha^*+1}\right)|f'(b)|\right.\\
&\left.+\left(\frac{bm}{\alpha+1}-\frac{bm\Gamma(\alpha+1)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+2)}+xm-\frac{xm}{\alpha^*+1}\right)|f'(x)|\right].
\end{aligned}$$

Proof. Using Lemma 1 and $|f'|$ is $(\alpha^*,m)-convex$ function, we have,

$$\begin{aligned}
&\Bigg|\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\\
&-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{mx^+}f(b)\right)\Bigg|\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^{\alpha}+x)|f'(tx+m(1-t)a)|dt\\
&+\frac{(b-mx)^2}{(b-a)^2}\int_0^1(b(1-t)^{\alpha}+x)|f'(tb+m(1-t)x)|dt\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^{\alpha}+x)\left[t^{\alpha^*}|f'(x)|+m(1-t^{\alpha^*})|f'(a)|\right]dt\\
&+\frac{(b-mx)^2}{(b-a)^2}\int_0^1(b(1-t)^{\alpha}+x)\left[t^{\alpha^*}|f'(b)|+m(1-t^{\alpha^*})|f'(x)|\right]dt
\end{aligned}$$

$$\begin{aligned}
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\left[\int_0^1(at^{\alpha}+x)t^{\alpha^*}|f'(x)|dt+\int_0^1(at^{\alpha}+x)m(1-t^{\alpha^*})|f'(a)|dt\right]\\
&+\frac{(b-mx)^2}{(b-a)^2}\left[\int_0^1(b(1-t)^{\alpha}+x)t^{\alpha^*}|f'(b)|dt+\int_0^1(b(1-t)^{\alpha}+x)m(1-t^{\alpha^*})|f'(x)|dt\right]\\
=\ &\frac{(x-ma)^2}{(b-a)^2}\left[\left(\frac{a}{\alpha+\alpha^*+1}+\frac{x}{\alpha^*+1}\right)|f'(x)|\right.\\
&\left.+\left(\frac{am}{\alpha+1}-\frac{am}{\alpha+\alpha^*+1}+xm-\frac{xm}{\alpha^*+1}\right)|f'(a)|\right]\\
&+\frac{(b-mx)^2}{(b-a)^2}\left[\left(\frac{b\Gamma(\alpha+1)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+2)}+\frac{x}{\alpha^*+1}\right)|f'(b)|\right.\\
&\left.+\left(\frac{bm}{\alpha+1}-\frac{bm\Gamma(\alpha+1)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+2)}+xm-\frac{xm}{\alpha^*+1}\right)|f'(x)|\right].
\end{aligned}$$

This completes the proof.

Corollary 1. *In Theorem 1, if we choose* $(\alpha^*,m)=(1,1)$ *and* $x=\frac{a+b}{2}$, *we have*

$$\begin{aligned}
&\left|\frac{(a+b)\left(4f\left(\frac{a+b}{2}\right)-f(a)-f(b)\right)}{4(b-a)}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(a\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^-}f(a)+b\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^+}f(b)\right)\right|\\
\leq\ &\frac{1}{4}\left[\left(\frac{a}{\alpha+2}+\frac{a+b}{4}\right)\left|f'\left(\frac{a+b}{2}\right)\right|+\left(\frac{a}{\alpha+1}-\frac{a}{\alpha+2}+\frac{a+b}{4}\right)|f'(a)|\right]\\
&+\frac{1}{4}\left[\left(\frac{b\Gamma(\alpha+1)}{\Gamma(\alpha+3)}+\frac{a+b}{4}\right)|f'(b)|+\left(\frac{b}{\alpha+1}-\frac{b\Gamma(\alpha+1)}{\Gamma(\alpha+3)}+\frac{a+b}{4}\right)\left|f'\left(\frac{a+b}{2}\right)\right|\right].
\end{aligned}$$

Theorem 2. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L[ma,b]$, where $ma\in I$ with $0\le a<b<\infty$. If $|f'|^q$ is $(\alpha^*,m)-convex$ on $[ma,b]$ for some fixed $\alpha^*,m\in[0,1]$, $q>1$, then for any $0<\alpha$ and $x\in(a,b)$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{mx^+}f(b)\right)\right|\\
\le\ &\frac{(x-ma)^2}{(b-a)^2}\left(\frac{2^{p-1}a^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{p}}\left(\frac{|f'(x)|^q}{\alpha^*+1}+m|f'(a)|^q-\frac{m|f'(a)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}\\
&+\frac{(b-mx)^2}{(b-a)^2}\left(\frac{2^{p-1}b^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{\alpha^*+1}+m|f'(x)|^q-\frac{m|f'(x)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1, Lemma 3, Hölder's inequality and the fact that $|f'|^q$ is $(\alpha^*,m)-convex$ function, one has

$$\begin{aligned}
&\left|\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{mx^+}f(b)\right)\right|\\
\le\ &\frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^{\alpha}+x)|f'(tx+m(1-t)a)|dt\\
&+\frac{(b-mx)^2}{(b-a)^2}\int_0^1(b(1-t)^{\alpha}+x)|f'(tb+m(1-t)x)|dt\\
\le\ &\frac{(x-ma)^2}{(b-a)^2}\left(\int_0^1(at^{\alpha}+x)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+m(1-t)a)|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-mx)^2}{(b-a)^2}\left(\int_0^1(b(1-t)^{\alpha}+x)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tb+m(1-t)x)|^q dt\right)^{\frac{1}{q}}\\
\le\ &\frac{(x-ma)^2}{(b-a)^2}\left(\int_0^1 2^{p-1}(a^p t^{p\alpha}+x^p)dt\right)^{\frac{1}{p}}\left(\int_0^1\left[t^{\alpha^*}|f'(x)|^q+m(1-t^{\alpha^*})|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{(b-mx)^2}{(b-a)^2}\left(\int_0^1 2^{p-1}(b^p(1-t)^{p\alpha}+x^p)dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[t^{\alpha^*}|f'(b)|^q+m(1-t^{\alpha^*})|f'(x)|^q\right]dt\right)^{\frac{1}{q}}\\
=\ &\frac{(x-ma)^2}{(b-a)^2}\left(\frac{2^{p-1}a^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{p}}\left(\frac{|f'(x)|^q}{\alpha^*+1}+m|f'(a)|^q-\frac{m|f'(a)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}\\
&+\frac{(b-mx)^2}{(b-a)^2}\left(\frac{2^{p-1}b^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{\alpha^*+1}+m|f'(x)|^q-\frac{m|f'(x)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}
\end{aligned}$$

Remark 1. *In Theorem 2, if we choose $(\alpha^*,m)=(1,1)$, we obtain*

$$\begin{aligned}
&\left|\frac{(x-a)[(a+x)f(x)-xf(a)]+(b-x)[(x+b)f(x)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-a)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(a)+\frac{b}{(b-x)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^+}f(b)\right)\right|\\
\le\ &\frac{(x-a)^2}{(b-a)^2}\left(\frac{2^{p-1}a^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{p}}\left(\frac{|f'(x)|^q}{2}+\frac{|f'(a)|^q}{2}\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2}{(b-a)^2}\left(\frac{2^{p-1}b^p}{p\alpha+1}+2^{p-1}x^p\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{2}+\frac{|f'(x)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

which is a special case of Theorem 3.6 for $s=1$ in [6].

Theorem 3. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L[ma,b]$, where $ma\in I$ with $0\le a<b<\infty$. If $|f'|^q$ is $(\alpha^*,m)-convex$ on $[ma,b]$ for some fixed $\alpha^*,m\in[0,1]$, $q\ge1$, then for any $0<\alpha$ and $x\in(a,b)$, the* *following inequality holds:*

$$\begin{aligned}
&\left|\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{mx^+}f(b)\right)\right|\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\left(\frac{a}{\alpha+1}+x\right)^{1-\frac{1}{q}}\psi(\alpha,m,x)+\frac{(b-mx)^2}{(b-a)^2}\left(\frac{b}{\alpha+1}+x\right)^{1-\frac{1}{q}}\omega(\alpha,m,x)
\end{aligned}$$

where

$$\begin{aligned}
\psi(\alpha,m,x) &= \left[\left(\frac{a}{\alpha+\alpha^*+1}+\frac{x}{\alpha^*+1}\right)|f'(x)|^q\right.\\
&\quad\left.+\left(\frac{am}{\alpha+1}-\frac{am}{\alpha+\alpha^*+1}+xm-\frac{xm}{\alpha^*+1}\right)|f'(a)|^q\right]^{\frac{1}{q}}\\
\omega(\alpha,m,x) &= \left[\left(\frac{b\Gamma(\alpha+1)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+2)}+\frac{x}{\alpha^*+1}\right)|f'(b)|^q\right.\\
&\quad\left.+\left(\frac{bm}{\alpha+1}-\frac{mb\Gamma(\alpha+1)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+2)}+xm-\frac{xm}{\alpha^*+1}\right)|f'(x)|^q\right]^{\frac{1}{q}}
\end{aligned}$$

Proof. Using Lemma 1, the well-known power mean inequality and $|f'|^q$ is $(\alpha^*,m)-convex$ function, we have:

$$\begin{aligned}
&\left|\frac{(x-ma)[(a+x)f(x)-xf(ma)]+(b-mx)[(x+b)f(mx)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{a}{(x-ma)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{x^-}f(ma)+\frac{b}{(b-mx)^{\alpha-1}}\ {}_{RL}J^{\alpha}_{mx^+}f(b)\right)\right|\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\int_0^1(at^{\alpha}+x)|f'(tx+m(1-t)a)|dt\\
&+\frac{(b-mx)^2}{(b-a)^2}\int_0^1(b(1-t)^{\alpha}+x)|f'(tb+m(1-t)x)|dt\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\left(\int_0^1(at^{\alpha}+x)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(at^{\alpha}+x)|f'(tb+m(1-t)x)|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-mx)^2}{(b-a)^2}\left(\int_0^1(b(1-t)^{\alpha}+x)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(b(1-t)^{\alpha}+x)|f'(tb+m(1-t)x)|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\frac{(x-ma)^2}{(b-a)^2}\left(\int_0^1(at^{\alpha}+x)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(at^{\alpha}+x)\left[t^{\alpha^*}|f'(x)|^q+m(1-t^{\alpha^*})|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{(b-mx)^2}{(b-a)^2}\left(\int_0^1(b(1-t)^{\alpha}+x)dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(b(1-t)^{\alpha}+x)\left[t^{\alpha^*}|f'(b)|^q+m(1-t^{\alpha^*})|f'(x)|^q\right]\right)^{\frac{1}{q}}\\
=\ &\frac{(x-ma)^2}{(b-a)^2}\left(\frac{a}{\alpha+1}+x\right)^{1-\frac{1}{q}}\psi(\alpha,m,x)+\frac{(b-mx)^2}{(b-a)^2}\left(\frac{b}{\alpha+1}+x\right)^{1-\frac{1}{q}}\omega(\alpha,m,x)
\end{aligned}$$

This completes the proof.

Corollary 2. *In Theorem 3, if we choose $(\alpha^*,m)=(1,1)$ and $x=\frac{a+b}{2}$, we have*

$$\begin{aligned}
&\left|\frac{(a+b)\left(4f\left(\frac{a+b}{2}\right)-f(a)-f(b)\right)}{4(b-a)}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(a\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^-}f(a)+b\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^+}f(b)\right)\right|\\
\leq\ &\frac{1}{4}\left(\frac{a}{\alpha+1}+\frac{a+b}{2}\right)^{1-\frac{1}{q}}\psi(\alpha,m,x)+\frac{1}{4}\left(\frac{b}{\alpha+1}+\frac{a+b}{2}\right)^{1-\frac{1}{q}}\omega(\alpha,m,x)
\end{aligned}$$

where

$$\begin{aligned}
\psi(\alpha,m,x) &= \left[\left(\frac{a}{\alpha+2}+\frac{a+b}{4}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(\frac{a}{\alpha+1}-\frac{a}{\alpha+2}+\frac{a+b}{4}\right)|f'(a)|^q\right]^{\frac{1}{q}}\\
\omega(\alpha,m,x) &= \left[\left(\frac{b\Gamma(\alpha+1)}{\Gamma(\alpha+3)}+\frac{a+b}{4}\right)|f'(b)|^q+\left(\frac{b}{\alpha+1}-\frac{b\Gamma(\alpha+1)}{\Gamma(\alpha+3)}+\frac{a+b}{4}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ twice be a differentiable function on I such that $f''\in L[ma,mb]$, where $ma,mb\in I$ with $0\le a<b<\infty$. If $|f''|$ is $(\alpha^*,m)-convex$ on $[ma,mb]$ for some fixed $\alpha^*,m\in[0,1]$, then for any $0<\alpha$ and $x\in(a,b)$, the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)}+\frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right]\\
&-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb)+\frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right)\Bigg|\\
\le\ &\frac{(x-mb)}{(b-a)^2}\Bigg[\left(\frac{x}{\alpha+\alpha^*+2}+\frac{b}{\alpha^*+2}\right)|f''(x)|\\
&+\left(\frac{mx}{\alpha+2}-\frac{mx}{\alpha+\alpha^*+2}+\frac{bm}{2}-\frac{bm}{\alpha^*+2}\right)|f''(b)|\Bigg]\\
&+\frac{(a-mx)}{(b-a)^2}\Bigg[\left(\frac{b\Gamma(\alpha^*+1)\Gamma(\alpha+2)}{\Gamma(\alpha+\alpha^*+3)}+\frac{x}{\alpha^*+1}-\frac{x}{\alpha^*+2}\right)|f''(a)|\\
&+\left(\frac{bm}{\alpha+2}-\frac{bm\Gamma(\alpha^*+1)\Gamma(\alpha+2)}{\Gamma(\alpha+\alpha^*+3)}+\frac{mx}{2}-\frac{mx}{\alpha^*+1}+\frac{mx}{\alpha^*+2}\right)|f''(x)|\Bigg]
\end{aligned}$$

Proof. Using Lemma 2 via $|f''|$ is $(\alpha^*,m)-convex$ function, we have,

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)}+\frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right]\\
&-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb)+\frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right)\Bigg|\\
\le\ &\frac{(x-mb)}{(b-a)^2}\int_0^1(xt^{\alpha+1}+tb)|f''(tx+m(1-t)b)|dt\\
&+\frac{(a-mx)}{(b-a)^2}\int_0^1((1-t)^{\alpha+1}b+(1-t)x)|f''(ta+m(1-t)x)|dt\\
\le\ &\frac{(x-mb)}{(b-a)^2}\int_0^1(xt^{\alpha+1}+tb)\left[t^{\alpha^*}|f''(x)|+m(1-t^{\alpha^*})|f''(b)|\right]dt\\
&+\frac{(a-mx)}{(b-a)^2}\int_0^1((1-t)^{\alpha+1}b+(1-t)x)\left[t^{\alpha^*}|f''(a)|+m(1-t^{\alpha^*})|f''(x)|\right]dt\\
=\ &\frac{(x-mb)}{(b-a)^2}\Bigg[\left(\frac{x}{\alpha+\alpha^*+2}+\frac{b}{\alpha^*+2}\right)|f''(x)|\\
&+\left(\frac{mx}{\alpha+2}-\frac{mx}{\alpha+\alpha^*+2}+\frac{bm}{2}-\frac{bm}{\alpha^*+2}\right)|f''(b)|\Bigg]\\
&+\frac{(a-mx)}{(b-a)^2}\Bigg[\left(\frac{b\Gamma(\alpha^*+1)\Gamma(\alpha+2)}{\Gamma(\alpha+\alpha^*+3)}+\frac{x}{\alpha^*+1}-\frac{x}{\alpha^*+2}\right)|f''(a)|\\
&+\left(\frac{bm}{\alpha+2}-\frac{bm\Gamma(\alpha^*+1)\Gamma(\alpha+2)}{\Gamma(\alpha+\alpha^*+3)}+\frac{mx}{2}-\frac{mx}{\alpha^*+1}+\frac{mx}{\alpha^*+2}\right)|f''(x)|\Bigg]
\end{aligned}$$

This completes the proof.

Corollary 3. *In Theorem 4, if we choose $(\alpha^*,m)=(1,1)$ and $x=\frac{a+b}{2}$, we have*

$$\begin{aligned}
&\Bigg|\frac{((a+3b)(\alpha+1)+(a+3b))f\left(\frac{a+b}{2}\right)-2bf(a)-(a+b)f(a)}{(b-a)^3}\\
&-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{2^{\alpha}(a+b)}{(b-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^+}f(b)+\frac{2^{\alpha+1}b}{(b-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^-}f(a)\right)\Bigg|\\
\le\ &\frac{1}{2(a-b)}\left[\left(\frac{x}{\alpha+3}+\frac{b}{3}\right)\left|f''\left(\frac{a+b}{2}\right)\right|+\left(\frac{1}{\alpha+2}-\frac{1}{\alpha+3}+\frac{b}{6}\right)|f''(b)|\right]\\
&+\frac{1}{2(a-b)}\Bigg[\left(\frac{b\Gamma(\alpha+2)}{\Gamma(\alpha+4)}+\frac{a+b}{12}\right)|f''(a)|\\
&+\left(\frac{b}{\alpha+2}-\frac{b\Gamma(\alpha+2)}{\Gamma(\alpha+4)}+\frac{a+b}{12}\right)\left|f''\left(\frac{a+b}{2}\right)\right|\Bigg].
\end{aligned}$$

Theorem 5. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ twice be a differentiable function on I such that $f''\in L[ma,mb]$, where $ma,mb\in I$ with $0\le a<b<\infty$. If $|f''|^q$ is $(\alpha^*,m)-convex$ on $[ma,mb]$ for some fixed $\alpha^*,m\in[0,1]$, $q>1$, then for any $0<\alpha$ and $x\in(a,b)$, the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)}+\frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right]\\
&-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb)+\frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right)\Bigg|\\
\le\ &\frac{(x-mb)}{(b-a)^2}\left(\frac{2^{p-1}x^p}{p(\alpha+1)+1}+\frac{2^{p-1}b^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q}{\alpha^*+1}+m|f''(b)|^q-\frac{m|f''(b)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}\\
&+\frac{(a-mx)}{(b-a)^2}\left(\frac{2^{p-1}b^p}{p(\alpha+1)+1}+\frac{2^{p-1}x^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q}{\alpha^*+1}+m|f''(x)|^q-\frac{m|f''(x)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2, Lemma 3, Hölder's inequality and $|f''|^q$ is $(\alpha^*,m)-convex$ function, we have,

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)}+\frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right]\\
&-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb)+\frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right)\Bigg|\\
\le\ &\frac{(x-mb)}{(b-a)^2}\int_0^1(t^{\alpha+1}x+tb)|f''(tx+m(1-t)b)|dt\\
&+\frac{(a-mx)}{(b-a)^2}\int_0^1((1-t)^{\alpha+1}b+(1-t)x)|f''(ta+m(1-t)x)|dt\\
\le\ &\frac{(x-mb)}{(b-a)^2}\left(\int_0^1\left(t^{\alpha+1}x+tb\right)^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+m(1-t)b)\right|^qdt\right)^{\frac{1}{q}}\\
&+\frac{(a-mx)}{(b-a)^2}\left(\int_0^1\left((1-t)^{\alpha+1}b+(1-t)x\right)^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(ta+m(1-t)x)\right|^qdt\right)^{\frac{1}{q}}\\
\le\ &\frac{(x-mb)}{(b-a)^2}\left(\int_0^1 2^{p-1}(x^pt^{p(\alpha+1)}+t^pb^p)dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[t^{\alpha^*}|f''(x)|^q+m(1-t^{\alpha^*})|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{(a-mx)}{(b-a)^2}\left(\int_0^1 2^{p-1}((1-t)^{p(\alpha+1)})b^p+(1-t)^px^p)dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[t^{\alpha^*}|f''(a)|^q+m(1-t^{\alpha^*})|f''(x)|^q\right]dt\right)^{\frac{1}{q}}\\
=\ &\frac{(x-mb)}{(b-a)^2}\left(\frac{2^{p-1}x^p}{p(\alpha+1)+1}+\frac{2^{p-1}b^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q}{\alpha^*+1}+m|f''(b)|^q-\frac{m|f''(b)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}\\
&+\frac{(a-mx)}{(b-a)^2}\left(\frac{2^{p-1}b^p}{p(\alpha+1)+1}+\frac{2^{p-1}x^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q}{\alpha^*+1}+m|f''(x)|^q-\frac{m|f''(x)|^q}{\alpha^*+1}\right)^{\frac{1}{q}}
\end{aligned}$$

This completes the proof.

Remark 2. *In Theorem 5, if we choose $(\alpha^*,m)=(1,1)$, we obtain*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(b)}{(b-x)}+\frac{(b(\alpha+1)+x)f(x)-xf(a)}{(x-a)}\right]\\
&-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(b-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(b)+\frac{b}{(x-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^-}f(a)\right)\Bigg|\\
\le\ &\frac{(x-b)}{(b-a)^2}\left[\left(\frac{2^{p-1}x^p}{p(\alpha+1)+1}+\frac{2^{p-1}b^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q}{2}+\frac{|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right]\\
&+\frac{(a-x)}{(b-a)^2}\left[\left(\frac{2^{p-1}b^p}{p(\alpha+1)+1}+\frac{2^{p-1}x^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q}{2}+\frac{|f''(x)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which is a special case of Theorem 3.13 for $s=1$ in [6].

Theorem 6. *Let I be an open real interval such that $[0,\infty) \subset I$ and $f : I \to \mathbb{R}$ twice be a differentiable function on I such that $f'' \in L[ma,mb]$, where $ma,mb \in I$ with $0 \le a < b < \infty$. If $|f''|^q$ is $(\alpha^*,m)-convex$ on $[ma,mb]$ for some fixed $\alpha^*,m \in [0,1]$, $q \ge 1$, then for any $0 < \alpha$ and $x \in (a,b)$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)}+\frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right]\right.\\
&\left.-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb)+\frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right)\right|\\
\le\ &\frac{(x-mb)}{(b-a)^2}\left(\frac{x}{\alpha+2}+\frac{b}{2}\right)^{1-\frac{1}{q}}\psi(\alpha,m,x)+\frac{(a-mx)}{(b-a)^2}\left(\frac{b}{\alpha+2}+\frac{x}{2}\right)^{1-\frac{1}{q}}\omega(\alpha,m,x)
\end{aligned}$$

where

$$\begin{aligned}
\psi(\alpha,m,x) &= \left[\left(\frac{x}{\alpha+\alpha^*+2}+\frac{b}{\alpha^*+2}\right)|f'(x)|^q\right.\\
&\left.+\left(\frac{mx}{\alpha+2}-\frac{mx}{\alpha+\alpha^*+2}+\frac{mb}{2}-\frac{mb}{\alpha^*+2}\right)|f'(b)|^q\right]^{\frac{1}{q}}\\
\omega(\alpha,m,x) &= \left[\left(\frac{b\Gamma(\alpha+2)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+3)}+\frac{x}{\alpha^*+1}-\frac{x}{\alpha^*+2}\right)|f'(b)|^q\right.\\
&\left.+\left(\frac{mb}{\alpha+2}-\frac{mb\Gamma(\alpha+2)\Gamma(\alpha^*+1)}{\Gamma(\alpha+\alpha^*+3)}+\frac{mx}{2}-\frac{mx}{\alpha^*+1}+\frac{mx}{\alpha^*+2}\right)|f'(x)|^q\right]^{\frac{1}{q}}
\end{aligned}$$

Proof. Using Lemma 2, the well known the power mean inequality and $|f''|^q$ is $(\alpha^*,m)-$*convex* function, we have,

$$\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+1)+b)f(x)-bf(mb)}{(mb-x)}+\frac{(b(\alpha+1)+x)f(mx)-xf(a)}{(mx-a)}\right]\right.\\
&\left.-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{x}{(mb-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{x^+}f(mb)+\frac{b}{(mx-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{mx^-}f(a)\right)\right|\\
\le\ &\frac{(x-mb)}{(b-a)^2}\int_0^1(t^{\alpha+1}x+tb)|f''(tx+m(1-t)b)|dt\\
&+\frac{(a-mx)}{(b-a)^2}\int_0^1((1-t)^{\alpha+1}b+(1-t)x)|f''(ta+m(1-t)x)|dt\\
\le\ &\frac{(x-mb)}{(b-a)^2}\left(\int_0^1(t^{\alpha+1}x+tb)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(t^{\alpha+1}x+tb)\left|f''(tx+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(a-mx)}{(b-a)^2}\left(\int_0^1((1-t)^{\alpha+1}b+(1-t)x)dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1((1-t)^{\alpha+1}b+(1-t)x)\left|f''(ta+m(1-t)x)\right|^q dt\right)^{\frac{1}{q}}\\
\le\ &\frac{(x-mb)}{(b-a)^2}\left(\int_0^1(t^{\alpha+1}x+tb)dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(t^{\alpha+1}x+tb)\left[t^{\alpha^*}|f''(x)|^q+m(1-t^{\alpha^*})|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{(a-mx)}{(b-a)^2}\left(\int_0^1((1-t)^{\alpha+1}b+(1-t)x)dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1((1-t)^{\alpha+1}b+(1-t)x)\left[t^{\alpha^*}|f''(b)|^q+m(1-t^{\alpha^*})|f''(x)|^q\right]dt\right)^{\frac{1}{q}}\\
=\ &\frac{(x-mb)}{(b-a)^2}\left(\frac{x}{\alpha+2}+\frac{b}{2}\right)^{1-\frac{1}{q}}\psi(\alpha,m,x)+\frac{(a-mx)}{(b-a)^2}\left(\frac{b}{\alpha+2}+\frac{x}{2}\right)^{1-\frac{1}{q}}\omega(\alpha,m,x)
\end{aligned}$$

This completes the proof.

Corollary 4. *In Theorem 6, if we choose $(\alpha^*,m)=(1,1)$ and $x=\frac{a+b}{2}$, we have*

$$\begin{aligned}
&\left|\frac{((a+3b)(\alpha+1)+(a+3b))f\left(\frac{a+b}{2}\right)-2bf(a)-(a+b)f(a)}{(b-a)^3}\right.\\
&\left.-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^2}\left(\frac{2^{\alpha}(a+b)}{(b-x)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^+}f(b)+\frac{2^{\alpha+1}b}{(b-a)^{\alpha+1}}\ {}_{RL}J^{\alpha}_{\frac{a+b}{2}^-}f(a)\right)\right|\\
\le\ &\frac{(a-b)}{2(b-a)^2}\left(\frac{a+b}{2\alpha+4}+\frac{b}{2}\right)^{1-\frac{1}{q}}\psi(\alpha,m,x)+\frac{(a-b)}{2(b-a)^2}\left(\frac{b}{\alpha+2}+\frac{a+b}{4}\right)^{1-\frac{1}{q}}\omega(\alpha,m,x)
\end{aligned}$$

where

$$\psi(\alpha,m,x) = \left[\left(\frac{a+b}{2\alpha+6}+\frac{b}{3}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(\frac{a+b}{2\alpha+4}-\frac{a+b}{2\alpha+6}+\frac{b}{6}\right)|f'(b)|^q\right]^{\frac{1}{q}}$$

$$\omega(\alpha,m,x) = \left[\left(\frac{b\Gamma(\alpha+2)}{\Gamma(\alpha+4)}+\frac{a+b}{12}\right)|f'(b)|^q + \left(\frac{b}{\alpha+2}-\frac{b\Gamma(\alpha+2)}{\Gamma(\alpha+4)}+\frac{a+b}{6}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.$$

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