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On $\beta_1 - \mathscr{I} -$ Paracompact Spaces

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Abstract

In this paper, our aim is to introduce the class of β_1-paracompact spaces in ideal topological spaces. Then, some fundamental properties of $\beta_1 - \mathscr{I}-$paracompact spaces are given. Also, the relationships between $\beta_1 - \mathscr{I}-$paracompact spaces and other types of paracompact spaces are studied.

Keywords: *paracompact, $\beta-$open set, β_1-paracompact, $\mathscr{I}-$paracompact, $\beta_1 - \mathscr{I}-$paracompact.*
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1. Introduction

In 1944, Dieudonne [9] introduced the paracompact spaces. In 1948, Stone [28] proved the fundamental theorem that every metric space is a paracompact space. Since then, a lot of works has been done on paracompact spaces and many interesting results have been obtained [1, 2, 7, 8, 13, 23].
The notion of an ideal topological space was studied independently by Kuratowski [17] and Vaidyanathaswamy [29]. Hamlet and Jankovic [15] investigated further properties of ideal topological spaces.
Zahid [31] introduced the concept of paracompactness with respect to an ideal. Later, $\mathscr{I}-$paracompactness studied by Hamlet et al. [14] and Sathiyasundari and Renukadevi [26]. Also, Sanabria et al. [25] studied this concept to define $S-$ paracompactness in ideal topological spaces. In recent years, the use of ideals has taken a significant role in the generalization of some topological notions such as regularity, compactness, paracompactness, semi-paracompactness and $\beta-$paracompactness [22, 24].
In this work, we introduce and study a stronger version of $\mathscr{I}-$paracompact space called $\beta_1 - \mathscr{I}-$paracompact space which is defined on an ideal space. Then, we investigate the relationships between $\beta_1 - \mathscr{I}-$paracompact spaces and the other types of paracompactness. Moreover, we obtain various properties, examples and counterexamples concerning $\beta_1 - \mathscr{I}-$paracompactness.

2. Preliminaries

Throughout the present paper, (X,τ) denotes a topological space. If F is a subset of X, then the closure of F and the interior of F will be denoted by $cl(F)$ and $int(F)$, respectively. Also, we denote the class of all subsets of X by $\mathscr{P}(X)$.

Definition 2.1. *[17, 29] An ideal $\mathscr{I} \subseteq \mathscr{P}(X)$ on a set X is a nonempty collection of subsets of X which satisfies*

(i) If $A \in \mathscr{I}$ and $B \subseteq A$, then $B \in \mathscr{I}$,

(ii) If $A \in \mathscr{I}$ and $B \in \mathscr{I}$, then $A \cup B \in \mathscr{I}$.

In this paper, we denote a topological space (X,τ) together with an ideal $\mathscr{I}$ defined on X by the triple $(X,\tau,\mathscr{I})$ that will be called an ideal space.

Lemma 2.2. *[14] If $\mathscr{I} \neq \emptyset$ is an ideal on X and F is a subset over X, then $\mathscr{I}_F = \{F \cap I : I \in \mathscr{I}\}$ is an ideal on X.*

Definition 2.3. *[17] Let $(X,\tau,\mathscr{I})$ be an ideal space. A set operator $()^* : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$, called local function of F with respect to $\mathscr{I}$ and τ, is defined as follows*

$$F^*(\mathscr{I},\tau) = \{x \in X : (F \cap G) \notin \mathscr{I} \text{ for every } G \in \tau_{(x)}\}$$

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where $\tau_{(x)} = \{G \subseteq X : x \in G \text{ and } G \in \tau\}$. *We simply write* F^* *instead of* $F^*(\mathscr{I},\tau)$ *in case there is no chance for confusion.*

Definition 2.4. *[15] Let* $(X,\tau,\mathscr{I})$ *be an ideal space. A Kuratowski closure operator* $cl()^*$ *for a topology* $\tau^*(\mathscr{I},\tau)$ *(also denoted by* τ^* *), called* $*-$*topology, finer than* τ *is defined by* $cl^*(F) = F \cup F^*$. *A basis* $\beta(\mathscr{I},\tau)$ *for* τ^* *can be described as follows*

$$\beta(\mathscr{I},\tau) = \{V - I : V \in \tau \text{ and } I \in \mathscr{I}\}.$$

Definition 2.5. *[10] Let* (X,τ) *be a topological space and* $F \subseteq X$. *Then,* F *is said to be a* $\beta-$*open (semi-preopen [3]) set if* $F \subseteq cl(int(cl(F)))$. *The complement of* $\beta-$*open set is said to be a* $\beta-$*closed set. The collection of all* $\beta-$*open (* $\beta-$*closed) subsets of* X *is denoted by* $\beta O(X,\tau)$ *(* $\beta C(X,\tau)$*).*

Definition 2.6. *[20] Let* (X,τ) *be a topological space and* $F \subseteq X$. F *is said to be an* $\alpha-$*open set if* $F \subseteq int(cl(int(F)))$. *The collection of all* $\alpha-$*open subsets of* X *is denoted by* τ^α*, forms a topology on* X*, finer than* τ.

Definition 2.7. *[3, 12] Let* (X,τ) *be a topological space and* $F \subseteq X$. *The intersection of all* $\beta-$*closed sets over* X *containing* F *is called* $\beta-$*closure of* F*, and it is denoted by* $\beta cl(F)$.

Theorem 2.8. *[19] Let* (X,τ) *be a topological space,* $F \subset Y \subset X$ *and* Y *be a* $\beta-$*open set over* X. *Then* F *is a* $\beta-$*open set over* X *if and only if* F *is a* $\beta-$*open set over* (Y,τ_Y).

Lemma 2.9. *[3, 12] Let* (X,τ) *be a topological space and* $F \subseteq X$. *Then, the set* $\beta cl(F)$ *is a* $\beta-$*closed set over* X.

Definition 2.10. *[11] Let* (X,τ) *be a topological space. If for each* $\beta-$*open set* U *and each* $x \in U$*, there exists a* $\beta-$*open set* F *over* X *such that* $x \in F \subseteq \beta cl(F) \subseteq U$*, then it is called* $\beta-$*regular space.*

Definition 2.11. *[30] Let* (X,τ) *be a topological space. Then it is called extremally disconnected if the closure of every open set is an open set over* X.

Definition 2.12. *[5] Let* (X,τ) *be a topological space. Then it is called submaximal if each dense subset of* X *is an open set over* X.

Example 2.13. *Let* (X,τ) *be a topological space where* $X = \{x_1,x_2\}$ *and* $\tau = \{\emptyset, X, \{x_2\}\}$. *Clearly,* (X,τ) *is an extremally disconnected and submaximal topological space.*

Lemma 2.14. *[16] Let* (X,τ) *be an extremally disconnected and submaximal topological space. Then all semi-open sets over* X *are open.*

Definition 2.15. *[18] Let* (X,τ) *and* (Y,σ) *be topological spaces. A function* $f : (X,\tau) \to (Y,\sigma)$ *is said to be a* $\beta-$*irresolute if for every* $\beta-$*open set* F *over* Y*,* $f^{-1}(F)$ *is a* $\beta-$*open set over* X.

Definition 2.16. *[30] Let* (X,τ) *be a topological space. A collection* $\mathscr{V} = \{V_\lambda \subseteq X : \lambda \in \wedge\}$ *is said to be a locally finite if for each* $x \in X$*, there exists an open set* U *containing* x *such that* $V_\lambda \cap U \neq \emptyset$ *for all* $\lambda \in \{\lambda_1,...,\lambda_n\}$.

Lemma 2.17. *[4] The union of a family of locally finite collection of sets in a topological space is a locally finite family of sets.*

Theorem 2.18. *[5] Let* (X,τ) *be a topological space and* $\mathscr{U} = \{U_\lambda : \lambda \in \wedge\}$ *be a locally finite collection. If* $V_\lambda \subset U_\lambda$ *for each* $\lambda \in \wedge$*, then* $\{V_\lambda : \lambda \in \wedge\}$ *is a locally finite collection.*

Definition 2.19. *[14] Let* $(X,\tau,\mathscr{I})$ *be an ideal space. An ideal* $\mathscr{I}$ *is called weakly* $\tau-$*local on* X *if* $F^* = \emptyset$ *implies* $F \in \mathscr{I}$.

Example 2.20. *Let* (X,τ) *be as in Example 2.13 with the ideal* $\mathscr{I} = \{\emptyset, \{x_2\}\}$. *It is obvious that,* $\mathscr{I}$ *is a weakly* $\tau-$*local on* X.

Definition 2.21. *[14] Let* $(X,\tau,\mathscr{I})$ *be an ideal space. An ideal* $\mathscr{I}$ *is called* $\tau-$*locally finite on* X *if the union of each locally finite collection contained in* $\mathscr{I}$ *belongs to* $\mathscr{I}$.

Theorem 2.22. *[14] Let* $(X,\tau,\mathscr{I})$ *be an ideal space. If* $\mathscr{I}$ *is a weakly* $\tau-$*local on* X*, then* $\mathscr{I}$ *is a* $\tau-$*locally finite on* X.

Definition 2.23. *[9] Let* (X,τ) *be a topological space. Then it is said to be a paracompact space, if every open cover of* X *has a locally finite open refinement which covers to* X.

Definition 2.24. *[14] Let* $(X,\tau,\mathscr{I})$ *be an ideal space. Then it is said to be an* $\mathscr{I}-$*paracompact space if every open cover* $\mathscr{U}$ *of* X *has a locally finite open refinement* $\mathscr{V}$ *such that* $X - \bigcup\left\{V \subseteq X : V \in \mathscr{V}\right\} \in \mathscr{I}$.

Definition 2.25. *[31] Let* (X,τ) *be a topological space. The collection* $\mathscr{V}$ *satisfying* $X - \bigcup\left\{V : V \in \mathscr{V}\right\} \in \mathscr{I}$ *is called an* $\mathscr{I}-$*cover of* X.

Theorem 2.26. *[26] Let* (X,τ) *be a topological space. If* (X,τ) *is a paracompact space, then it is a* $\mathscr{I}-$*paracompact space.*

Definition 2.27. *[1] Let* (X,τ) *be a topological space. Then it is called* β_1- *paracompact if every* $\beta-$*open cover of* X *has a locally finite open refinement.*

Theorem 2.28. *[1] Let* (X,τ) *be a topological space. If* (X,τ) *is a* β_1-*paracompact space, then it is a paracompact space.*

Definition 2.29. *[6] Let* (X,τ) *be a topological space and* $F \subseteq X$. *Then* F *is said to be a* $N-$*closed relative to* X *(briefly,* $N-$*closed) [9] if for every cover* $\{U_\alpha : \alpha \in \wedge\}$ *of* F *by open sets over* X*, there exists a finite subfamily* $\wedge_0$ *of* $\wedge$ *such that* $F \subset \bigcup\{int(cl(U_\alpha)) : \alpha \in \wedge_0\}$.

Definition 2.30. *[27] Let* (X,τ) *and* (Y,σ) *be topological spaces and* $f : (X,\tau) \to (Y,\sigma)$ *be a function. Then* f *is said to be an almost closed mapping if* $f(F)$ *is closed over* Y *for each regular closed set* F *over* X.

Lemma 2.31. *[21] Let* (X,τ) *and* (Y,σ) *be topological spaces and* $f : (X,\tau) \to (Y,\sigma)$ *be an almost closed surjection with* $N-$*closed point inverse. If* $\{U_\alpha : \alpha \in \wedge\}$ *is a locally finite open cover of* X*, then* $\{f(U_\alpha) : \alpha \in \wedge\}$ *is a locally finite cover of* Y.

3. $\beta_1 - \mathscr{I}$ – *paracompact spaces*

Definition 3.1. *An ideal space* $(X,\tau,\mathscr{I})$ *is said to be a* $\beta_1 - \mathscr{I}-$*paracompact space if every* $\beta-$*open cover* $\mathscr{U}$ *of* X *has a locally finite open refinement* $\mathscr{V}$ *such that* $X - \bigcup \left\{ V \subseteq X : V \in \mathscr{V} \right\} \in \mathscr{I}$.

It is clear that every β_1-*paracompact space* (X,τ) *is a* $\beta_1 - \mathscr{I}-$*paracompact space for any ideal* $\mathscr{I}$ *on* X. *But the following example shows that the converse is not true in general.*

Example 3.2. *Let* $(X,\tau,\mathscr{I})$ *be an ideal space where* $X = \{a,b,c\}$ *with the topology* $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b,c\}\}$ *and the ideal* $\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. *Then, one can verify* $(X,\tau,\mathscr{I})$ *is a* $\beta_1 - \mathscr{I}-$*paracompact space, but* (X,τ) *is not a* β_1-*paracompact since the collection* $\mathscr{U} = \{\{a,b\},\{a,c\}\}$ *is a* $\beta-$*open cover of* X *which admits no locally finite open refinement of* $\mathscr{U}$ *which covers to* X.

Remark 3.3. *Definition 3.1 coincides with* β_1-*paracompactness when the ideal* $\mathscr{I}$ *just consists of empty set.*

Theorem 3.4. *If* $(X,\tau,\mathscr{I})$ *is a* $\beta_1 - \mathscr{I}-$*paracompact space, then it is an* $\mathscr{I}-$*paracompact space.*

Proof. The simple proof is omitted. □

The converse of Theorem 3.4 is not necessarily true as we can see in the following example.

Example 3.5. *Let* $X = \mathbb{Z}$ *be the set of integer numbers with the topology* $\tau = \{\emptyset, X, \{0\}\}$ *and the ideal* $\mathscr{I} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. *Observe that* $(X,\tau,\mathscr{I})$ *is an* $\mathscr{I}-$ *paracompact space, but is not a* $\beta_1 - \mathscr{I}-$*paracompact space. Since the collection* $\mathscr{U} = \{\{0,x\} : x \in \mathbb{Z}\}$ *is a* $\beta-$*open cover of* X *which admits no locally finite open refinement which is an* $\mathscr{I}-$*cover of* X.

Remark 3.6. *Let* (X,τ) *be a topological space and* $\mathscr{I}$ *be an ideal on* X. *Then, the following diagram obtains immediately from Theorem 2.26, Theorem 2.28, Definition 3.1 and Theorem 3.4.*

$$\begin{array}{ccc} (X,\tau)\ \ \beta_1-\text{paracompact} & \Longrightarrow & (X,\tau,\mathscr{I})\ \ \beta_1-\mathscr{I}-\text{paracompact} \\ \Downarrow & & \Downarrow \\ (X,\tau)\ \ \text{paracompact} & \Longrightarrow & (X,\tau,\mathscr{I})\ \ \mathscr{I}-\text{paracompact} \end{array}$$

Theorem 3.7. *Let* $(X,\tau,\mathscr{I})$ *be an ideal space. If* $(X,\tau,\mathscr{I})$ *is a* $\beta_1 - \mathscr{I}-$*paracompact space, then* $(X,\tau^{\alpha},\mathscr{I})$ *is a* $\beta_1 - \mathscr{I}-$ *paracompact space.*

Proof. The proof follows immediately from $\beta O(X,\tau) = \beta O(X,\tau^{\alpha})$ and $\tau \subseteq \tau^{\alpha}$. □

The following example shows that the converse of Theorem 3.7 may not be true, in general.

Example 3.8. *Let* $(X,\tau,\mathscr{I})$ *be an ideal space where* $X = \{0,1,2,3\}$ *with the topology* $\tau = \{\emptyset, X, \{0\}\}$ *and the ideal* $\mathscr{I} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. *Then, one can readily verify* $(X,\tau^{\alpha},\mathscr{I})$ *is a* $\beta_1 - \mathscr{I}-$*paracompact space, but* $(X,\tau,\mathscr{I})$ *is not a* $\beta_1 - \mathscr{I}-$*paracompact since the collection* $\{\{0,1\},\{0,2\},\{0,3\}\}$ *is a* $\beta-$*open cover of* X *which admits no locally finite open refinement which is an* $\mathscr{I}-$*cover of* X.

Theorem 3.9. *Let* $(X,\tau,\mathscr{I})$ *be an ideal space. If* $(X,\tau^{\alpha},\mathscr{I})$ *is a* $\beta_1 - \mathscr{I}-$*paracompact space, then* $(X,\tau,\mathscr{I})$ *is an* $\mathscr{I}-$*paracompact space.*

Proof. Let $\mathscr{U} = \{U_\gamma \subseteq X : \gamma \in \wedge\}$ be an open cover of X. Since $\beta O(X,\tau) = \beta O(X,\tau^{\alpha})$, then $\mathscr{U}$ is a β-open cover of (X,τ^{α}). By hypothesis, there exists a locally finite open collection $\mathscr{V} = \{V_\lambda \subseteq X : \lambda \in \vee\}$ of (X,τ^{α}) which refines $\mathscr{U}$ such that $X - \bigcup\{V_\lambda : \lambda \in \vee\} \in \mathscr{I}$.

Let $V_\lambda \in \mathscr{V}$. Since $\mathscr{V}$ refines $\mathscr{U}$, there is some $U_{\gamma_\lambda} \in \mathscr{U}$ such that $V_\lambda \subseteq U_{\gamma_\lambda}$ which implies that the collection $\mathscr{G} = \{int(cl(int(V_\lambda))) \cap U_{\gamma_\lambda} : V_\lambda \subseteq U_{\gamma_\lambda}$ and $U_{\gamma_\lambda} \in \mathscr{U}\}$ is an open refinement of $\mathscr{U}$ such that $X - \bigcup\{int(cl(int(V_\lambda))) \cap U_{\gamma_\lambda} : V_\lambda \subseteq U_{\gamma_\lambda}$ and $U_{\gamma_\lambda} \in \mathscr{U}\} \in \mathscr{I}$.

Now, we shall show that $\mathscr{G}$ is a locally finite collection of (X,τ). Let $x \in X$. Since $\mathscr{V}$ is a locally finite collection of (X,τ^{α}), there exists an $F_x \in \tau^{\alpha}$ containing x such that $V_\lambda \cap F_x \neq \emptyset$ for all $\lambda \in \{\lambda_1,...,\lambda_n\}$. Therefore, we get $F_x \subseteq int(cl(int(F_x)))$ and

$$V_\lambda \cap F_x \subseteq \left(int(cl(int(V_\lambda))) \cap U_{\gamma_\lambda}\right) \cap int(cl(int(F_x)))$$

and so that $\left(int(cl(int(V_\lambda))) \cap U_{\gamma_\lambda}\right) \cap int(cl(int(F_x))) \neq \emptyset$ for all $\lambda \in \{\lambda_1,...,\lambda_n\}$. Hence, $(X,\tau,\mathscr{I})$ is an $\mathscr{I}-$paracompact space. □

The converse of Theorem 3.9 need not be true as shown by the following example.

Example 3.10. *Let* $(X,\tau,\mathscr{I})$ *be an ideal space where* $X = \{a,b,c\}$ *with the topology* $\tau = \{\emptyset, X, \{c\}, \{a,b\}\}$ *and the ideal* $\mathscr{I} = \{\emptyset, \{c\}\}$. *Observe that* $(X,\tau,\mathscr{I})$ *is an* $\mathscr{I}-$*paracompact space, but* $(X,\tau^{\alpha},\mathscr{I})$ *is not a* $\beta_1 - \mathscr{I}-$*paracompact since the collection* $\{\{a\},\{b\},\{c\}\}$ *is a* $\beta-$*open cover of* (X,τ^{α}) *which admits no open locally finite refinement of* (X,τ^{α}) *which is an* $\mathscr{I}-$*cover of* X.

Remark 3.11. *Let* (X,τ) *be a topological space and* $\mathscr{I}$ *be an ideal on* X. *Then, the following diagram obtains immediately from Theorem 2.26, Theorem 2.28, Definition 3.1, Theorem 3.4, Theorem 3.7 and Theorem 3.9.*

$$\begin{array}{ccc} (X,\tau)\ \ \beta_1-paracompact & \Longrightarrow & (X,\tau,\mathscr{I})\ \ \beta_1-\mathscr{I}-paracompact \\ & & \Downarrow \\ \Downarrow & & (X,\tau^{\alpha},\mathscr{I})\ \ \beta_1-\mathscr{I}-paracompact \\ & & \Downarrow \\ (X,\tau)\ \ paracompact & \Longrightarrow & (X,\tau,\mathscr{I})\ \ \mathscr{I}-paracompact \end{array}$$

Theorem 3.12. *Let $(X,\tau,\mathscr{I})$ be an extremally disconnected and submaximal space. If $(X,\tau^{\alpha},\mathscr{I})$ is a $\beta_1-\mathscr{I}-$paracompact space, then $(X,\tau,\mathscr{I})$ is a $\beta_1-\mathscr{I}-$paracompact space.*

Proof. This follows directly from the fact that if $(X,\tau,\mathscr{I})$ is an extremally disconnected and submaximal space, then from Lemma 2.14 $\tau=\tau^{\alpha}$. □

Theorem 3.13. *Let $(X,\tau,\mathscr{I})$ be an ideal space. If it is a $\beta_1-\mathscr{I}-$paracompact space and the collection $\mathscr{F}$ is an ideal on X such that $\mathscr{I}\subseteq\mathscr{F}$, then $(X,\tau,\mathscr{F})$ is a $\beta_1-\mathscr{F}-$paracompact space.*

Proof. Let $\mathscr{U}=\{U_\lambda\subseteq X:\lambda\in\wedge\}$ be a $\beta-$open cover of X. By hypothesis, $\mathscr{U}$ has a locally finite open refinement $\mathscr{V}$ such that $X-\bigcup\{V:V\in\mathscr{V}\}\in\mathscr{I}$. Since $\mathscr{I}\subseteq\mathscr{F}$, $X-\bigcup\{V:V\in\mathscr{V}\}\in\mathscr{F}$. Thus, $(X,\tau,\mathscr{F})$ is a $\beta_1-\mathscr{F}-$paracompact space. □

Theorem 3.14. *Let $(X,\tau,\mathscr{I})$ be an ideal space. If $(X,\tau,\mathscr{I})$ is a $\beta_1-\mathscr{I}-$paracompact space and $\mathscr{I}$ is weakly $\tau-$local, then $(X,\tau^*,\mathscr{I})$ is a $\beta_1-\mathscr{I}-$paracompact space.*

Proof. Let $\mathscr{U}^*=\{U_\gamma-I_\gamma:U_\gamma\in\tau,I_\gamma\in\mathscr{I},\gamma\in\wedge\}$ be a $\beta-$open cover of (X,τ^*). Then $\mathscr{U}=\{U_\gamma\subseteq X:U_\gamma\in\tau,\gamma\in\wedge\}$ is a $\beta-$open cover of (X,τ). By hypothesis, there exists locally finite open collection $\mathscr{V}=\{V_\lambda\subseteq X:\lambda\in\vee\}$ which refines $\mathscr{U}$ such that $X-\bigcup\{V_\lambda:V_\lambda\in\mathscr{V}\}\in\mathscr{I}$.

It is clear that $\mathscr{V}^*=\{V_\lambda-I_\gamma:\lambda\in\vee,\gamma\in\wedge\}$ is an open collection of (X,τ^*) which refines $\mathscr{U}^*$. Also, since $\tau\subseteq\tau^*$, $\mathscr{V}^*$ is a locally finite collection of (X,τ^*). It remains only to show that $X-\bigcup\{V_\lambda-I_\gamma:\lambda\in\vee,\gamma\in\wedge\}\in\mathscr{I}$.

By Theorem 2.18, $\{V_\lambda\cap I_\gamma:\lambda\in\vee,\gamma\in\wedge\}$ is a locally finite collection. Since $\mathscr{I}$ is weakly $\tau-$local on X, by Theorem 2.22, we have $\mathscr{I}$ is $\tau-$locally finite on X. It follows that $\bigcup(V_\lambda\cap I_\gamma)\in\mathscr{I}$. Then,

$$X-\bigcup\{V_\lambda-I_\gamma:\lambda\in\vee,\gamma\in\wedge\}\subseteq(X-\bigcup\{V_\lambda:\lambda\in\vee\})\cup(\bigcup(V_\lambda\cap I_\gamma))\in\mathscr{I}.$$

Thus, we have $X-\bigcup\{V_\lambda-I_\gamma:\lambda\in\vee,\gamma\in\wedge\}\in\mathscr{I}$. Therefore, $(X,\tau^*,\mathscr{I})$ is a $\beta_1-\mathscr{I}-$paracompact space. □

Theorem 3.15. *Let (X,τ) be a $\beta-$regular space. If $(X,\tau,\mathscr{I})$ is $\beta_1-\mathscr{I}-$paracompact then every $\beta-$open cover of X has a locally finite $\beta-$closed $\mathscr{I}-$cover refinement.*

Proof. Let $\mathscr{U}$ be a $\beta-$open cover of X. By $\beta-$regularity of X, for each $x\in X$ and $x\in U_x\in\mathscr{U}$, there exists $F_x\in\beta O(X,\tau)$ such that $x\in F_x\subset\beta cl(F_x)\subset U_x$. Then the collection $\mathscr{F}=\{F_x:x\in X\}$ is a $\beta-$open cover of X. By hypothesis, there exists a locally finite open collection $\mathscr{V}=\{V_\lambda:\lambda\in\Lambda\}$ which refines $\mathscr{F}$ such that $X-\bigcup\{V_\lambda:\lambda\in\Lambda\}\in\mathscr{I}$. Then $X-\bigcup\{\beta cl(V_\lambda):\lambda\in\Lambda\}\in\mathscr{I}$.

Let $x\in X$. Since $\mathscr{V}$ is locally finite, there exists a $G\in\tau$ containing x such that $V_\lambda\cap G\neq\emptyset$ for all $\lambda\in\{\lambda_1,...,\lambda_n\}$. Since $V_\lambda\cap G\subset\beta cl(V_\lambda)\cap G$, we get $\beta cl(V_\lambda)\cap G\neq\emptyset$ for all $\lambda\in\{\lambda_1,...,\lambda_n\}$. So, the collection $\mathscr{H}=\{\beta cl(V_\lambda):\lambda\in\Lambda\}$ is locally finite. Let $\beta cl(V_\lambda)\in\mathscr{H}$. Then $V_\lambda\in\mathscr{V}$. Since, $\mathscr{V}$ refines $\mathscr{F}$, there exists $F_x\in\mathscr{F}$ such that $V_\lambda\subset F_x$ so that $\beta cl(V_\lambda)\subset\beta cl(F_x)\subset U_x$. Hence, $\mathscr{H}$ refines $\mathscr{U}$. Moreover by Lemma 2.9, $\mathscr{H}=\{\beta cl(V_\lambda):\lambda\in\Lambda\}$ is a $\beta-$closed collection. Therefore, $\mathscr{H}$ is a locally finite $\beta-$closed $\mathscr{I}-$cover refinement of $\mathscr{U}$. □

If $\mathscr{I}=\{\emptyset\}$ in the above theorem, we have the Remark 3.16.

Remark 3.16. *[1, Theorem 2.12] Let (X,τ) be a $\beta-$regular space. If each $\beta-$open cover of the space X has a locally finite refinement, then each $\beta-$open cover of X has a locally finite $\beta-$closed refinement.*

4. $\beta_1-\mathscr{I}-$ *paracompact subsets*

Definition 4.1. *A subset F of X is called a $\beta_1-\mathscr{I}-$paracompact relative to $(X,\tau,\mathscr{I})$ if for every $\beta-$open cover $\mathscr{U}$ of F, there exists an $I\in\mathscr{I}$ and a locally finite open refinement $\mathscr{V}$ such that $F\subset\bigcup\{V\subseteq X:V\in\mathscr{V}\}\cup I$.*

Theorem 4.2. *$(X,\tau,\mathscr{I})$ be an ideal space and A, $B\subseteq X$. If A and B are $\beta_1-\mathscr{I}-$paracompact relative to $(X,\tau,\mathscr{I})$, then $A\cup B$ is a $\beta_1-\mathscr{I}-$paracompact relative to $(X,\tau,\mathscr{I})$.*

Proof. Let $\mathscr{U}=\{U_\gamma:\gamma\in\wedge\}$ be a $\beta-$open cover of $A\cup B$. Then $\mathscr{U}=\{U_\gamma:\gamma\in\wedge\}$ is a $\beta-$open cover of A and B. By hypothesis, there exists $I_1,I_2\in\mathscr{I}$ and locally finite open refinements $\mathscr{V}=\{V_\lambda:\lambda\in\Delta\}$ of A and $\mathscr{G}=\{G_i:i\in\bigtriangledown\}$ of B such that

$$A\subseteq\bigcup\{V_\lambda:\lambda\in\Delta\}\cup I_1 \text{ and } B\subseteq\bigcup\{G_i:i\in\bigtriangledown\}\cup I_2.$$

Take $\mathscr{H}=\{V_\lambda:\lambda\in\Delta\}\cup\{G_i:i\in\bigtriangledown\}$. Since the families $\mathscr{V}$ and $\mathscr{G}$ are locally finite by Lemma 2.17, $\mathscr{H}$ is a locally finite collection. Therefore, $\mathscr{H}$ is a locally finite open refinement of $\mathscr{U}$.

Now we shall show that $A\cup B\subseteq\bigcup\{H:H\in\mathscr{H}\}\cup I$ for some $I\in\mathscr{I}$. Since $A\subseteq\bigcup\{V_\lambda:\lambda\in\Delta\}\cup I_1$ and $B\subseteq\bigcup\{G_i:i\in\bigtriangledown\}\cup I_2$, we get

$$A \cup B \subseteq \bigcup\{V_\lambda : \lambda \in \Delta\} \cup I_1 \cup \bigcup\{G_i : i \in \bigtriangledown\} \cup I_2 = \bigcup\{V_\lambda : \lambda \in \Delta\} \cup \bigcup\{G_i : i \in \bigtriangledown\} \cup I$$

where $I = I_1 \cup I_2$. Thus, $A \cup B$ is a $\beta_1 - \mathscr{I}-$paracompact relative to $(X, \tau, \mathscr{I})$. □

Theorem 4.3. *$(X, \tau, \mathscr{I})$ be an ideal space and A, $B \subseteq X$. If A is a $\beta_1 - \mathscr{I}-$paracompact relative to $(X, \tau, \mathscr{I})$ and B is a $\beta-$closed set over X, then $A \cap B$ is a $\beta_1 - \mathscr{I}-$paracompact relative to $(X, \tau, \mathscr{I})$.*

Proof. Let $\mathscr{U} = \{U_\delta : \delta \in \wedge\}$ be a cover of $A \cap B$ such that $U_\delta \in \beta O(X, \tau)$. Then $\mathscr{G} = \{U_\delta : \delta \in \wedge\} \cup (X - B)$ is a $\beta-$open cover of A. By hypothesis, there exists $I \in \mathscr{I}$ and locally finite open collection $\mathscr{V} = \{V_\lambda : \lambda \in \vee\} \cup V$ $(V \subset X - B)$ which refines $\mathscr{U}$ such that $A \subset \bigcup\{V_\lambda : \lambda \in \vee\} \cup V \cup I$. Then,

$$A \cap B \subset (\bigcup\{V_\lambda : \lambda \in \vee\} \cup V \cup I) \cap B \subseteq \bigcup\{V_\lambda \cap B : \lambda \in \vee\} \cup (I \cap B)$$

which $A \cap B \subseteq \bigcup\{V_\lambda : \lambda \in \vee\} \cup I_B$ where $I_B = I \cap B$.

Since $\mathscr{V}$ is a locally finite collection, by Teorem 2.18, $\mathscr{H} = \{V_\lambda : \lambda \in \vee\}$ is a locally finite refinement of $\mathscr{U}$. Thus $A \cap B$ is a $\beta_1 - \mathscr{I}-$paracompact relative to $(X, \tau, \mathscr{I})$. □

Remark 4.4. *If $(X, \tau, \mathscr{I})$ is a $\beta_1 - \mathscr{I}-$paracompact space and B is a $\beta-$closed set over X, then B is a $\beta_1 - \mathscr{I}-$paracompact relative to $(X, \tau, \mathscr{I})$.*

Proof. The proof is direct from Theorem 4.3. □

Theorem 4.5. *Let $(X, \tau, \mathscr{I})$ be an ideal space and $A \subset B \subset X$. If A is a $\beta_1 - \mathscr{I}-$paracompact relative to $(X, \tau, \mathscr{I})$ and B is a $\beta-$open set over X then A is a $\beta_1 - \mathscr{I}-$paracompact relative to $(B, \tau_B, \mathscr{I}_B)$.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A such that $U_\alpha \in \beta O(B, \tau_B)$. Since B is a $\beta-$open set over X, by Theorem 2.8, $\mathscr{U}$ is a $\beta-$open cover of A. By hypothesis, there exists $I \in \mathscr{I}$ and locally finite open collection $\mathscr{V} = \{V_\beta : \beta \in \Delta_1\}$ which refines $\mathscr{U}$ such that $A \subseteq \bigcup\{V_\beta : \beta \in \Delta_1\} \cup I$, which implies $A \subseteq \bigcup\{V_\beta \cap B : \beta \in \Delta_1\} \cup I_B$ where $I_B = I \cap B$.

Let $x \in B$. Since $\mathscr{V}$ is a locally finite collection of X, there exists an open set F containing x such that $F \cap V_\beta = \emptyset$ for $\beta \notin \{\beta_1, ..., \beta_n\}$ which implies $(F \cap B) \cap (V_\beta \cap B) = \emptyset$ for $\beta \notin \{\beta_1, ..., \beta_n\}$. Thus, the collection $\mathscr{V}_{\mathscr{B}} = \{V_\beta \cap B : \beta \in \Delta_1\}$ is a locally finite open collection of (B, τ_B) which refines $\mathscr{U}$. Therefore, A is a $\beta_1 - \mathscr{I}-$paracompact relative to $(B, \tau_B, \mathscr{I}_B)$. □

Theorem 4.6. *$f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{F})$ be an open, $\beta-$irresolute, almost closed bijective function with $N-$closed point inverse with $f(\mathscr{I}) \subseteq \mathscr{F}$. If $(X, \tau, \mathscr{I})$ is a $\beta_1 - \mathscr{I}-$paracompact space, then $(Y, \sigma, \mathscr{F})$ is a $\beta_1 - \mathscr{F}-$paracompact space.*

Proof. Let $\mathscr{U} = \{U_\lambda : \lambda \in \wedge\}$ be a $\beta-$open cover of Y. Since f is a $\beta-$irresolute function, $\mathscr{G} = \{f^{-1}(U_\lambda) : \lambda \in \wedge\}$ is a $\beta-$ open cover of X. Since $(X, \tau, \mathscr{I})$ is a $\beta_1 - \mathscr{I}-$paracompact space, the collection $\mathscr{G}$ has a locally finite open refinement $\mathscr{V} = \{V_\lambda : \lambda \in \vee\}$ such that $X - \bigcup\{V_\lambda : V_\lambda \in \mathscr{V}\} \in \mathscr{I}$. Since f is an open function, by Lemma 2.31, $f(\mathscr{V}) = \{f(V_\lambda) : \lambda \in \vee\}$ is a locally finite open collection which refines $\mathscr{U}$. Also, $f(\mathscr{V})$ is a $\mathscr{F}$ cover of Y, because

$$Y = f(X) = f(\bigcup\{V_\lambda : \lambda \in \wedge\} \cup I) = f(\bigcup\{V_\lambda : \lambda \in \wedge\}) \cup f(I) = \bigcup\{f(V_\lambda) : \lambda \in \wedge\} \cup f(I).$$

This implies that $Y - \bigcup\{f(V_\lambda) : \lambda \in \wedge\} \in f(\mathscr{I}) \subseteq \mathscr{F}$. So, $(Y, \sigma, \mathscr{F})$ is a $\beta_1 - \mathscr{F}-$paracompact space. □

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