Konuralp Journal of Mathematics, 7 (1) (2019) 91-96

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

A Note on Quasi-Statistical Convergence of Order α in Rectangular Cone Metric Space

Nihal Turan[1] and Metin Başarır[1*]

[1] *Department of Mathematics, Sakarya University, 54050, Sakarya, Turkey*
[*] *Corresponding author E-mail: basarir@sakarya.edu.tr*

Abstract

The main purpose of this paper is to describe the quasi-statistical convergence of order α in the rectangular cone metric space and investigate some relations of these sequences.

Keywords: *Rectangular cone metric space, statistical convergence, quasi-statistical convergence.*
2010 Mathematics Subject Classification: *40G15,40A30,41A25,41A36*

1. Introduction

The concept of statistical convergence has been studied by many researchers from past to now. The idea of statistical convergence which is a generalization of convergence, was first introduced by Zygmund in 1935 and later introduced independently by Steinhaus [1] and Fast [2] in 1951. In 1959, Schoenberg [3] gave some basic properties of statistical convergence and also examined the concept of a summability method. The relationship between statistical convergence, Cesaro summability and strong p-Cesaro summability concepts was studied by some authors, [4], [5]. This concept has been studied under different names in different spaces such as real space, topological space, cone metric space [6], [7], [8], [9]. Recently, İlkhan and Kara [10] have defined a new type statistical Cauchy sequence in metric spaces. On the other hand, the authors in [11] defined the concept of quasi-statistical filter. Sakaoğlu and Yurdakadim [12] have defined the notion of quasi-statistical convergence by motivating from [11]. In 2007, Guang and Xian [13] introduced the idea of a cone metric space by replacing the set of real numbers with an ordering Banach space and generalized the concept of metric space. Later Azam et all [14] introduced the notion of a rectangular cone metric space by replacing the triangular inequality of a cone metric space by a rectangular inequality. In this paper, we introduce the concepts of statistical convergence, Cesaro summability, strongly p-Cesaro summability and quasi-statistical convergence of order α in rectangular cone metric space. Later, we give some theorems related to statistical convergence and quasi-statistical convergence of order α in the rectangular cone metric space.

2. Preliminaries and lemmas

Definition 2.1. *[13] Let $(E,\|.\|)$ be a real Banach space. A subset P of E is called a cone if it satisfies the following conditions:*
(1) $P\neq\oslash$, $P\neq\{0\}$ and P is closed.
(2) $ax+by\in P$ for all $x,y\in P$ and $a,b\in\mathbb{R}$ with $a,b\geq 0$.
(3) If $x\in P$ and $-x\in P$, then $x=0$ for all $x,y\in P$.

A partial ordering " $\preceq$ " with respect to P is defined by $x\preceq y\Leftrightarrow y-x\in P$. Also, we mean $x\prec y\Leftrightarrow x\preceq y$, $x\neq y$ and $x\prec\prec y\Leftrightarrow y-x\in E^{+}$, where E^{+} denotes the interior of P; that is $E^{+}=\{c\in E:0\prec\prec c\}$. The cone P is called normal if there is a number $K>0$ such that for all $x,y\in E$, $0\preceq x\preceq y$ implies $\|x\|\leq K\|y\|$. The least positive number satisfying this inequality is called the normal constant of P.
In this paper, we always assume that E is a Banach space, P is a cone in E with $E^{+}\neq\oslash$ and " $\preceq$ " is a partial ordering with respect to P.

Definition 2.2. *[13] Let X be a non-empty set. Suppose the mapping $\rho:X\times X\to E$ satisfies following*
(1) $0\preceq\rho(x,y)$ for all $x,y\in X$ and $\rho(x,y)=0$ if and only if $x=y$,
(2) $\rho(x,y)=\rho(y,x)$ for $x,y\in X$,
(3) $\rho(x,y)\preceq\rho(x,z)+\rho(z,y)$ for all $x,y\in X$.

Then ρ is called a cone metric on X, (X,ρ) is called a cone metric space.

Email addresses: **nihan.turan2@ogr.sakarya.edu.tr (Nihan Turan), basarir@sakarya.edu.tr (Metin Başarır)**

Definition 2.3. *[14] Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies following*
(1) $0 \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$ for $x,y \in X$,
(3) $d(x,y) \preceq d(x,w) + d(w,z) + d(z,y)$ for all $x,y \in X$ and for all distinct points $w,z \in X - \{x,y\}$ [rectangular property].

Then ρ is called a rectangular cone metric on X, (X,d) is called a rectangular cone metric space. Note that any cone metric space is a rectangular cone metric space but the converse is not true in general , see [15].

Definition 2.4. *[15] Let (X,d) be a rectangular cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$, $c \succ\succ 0$ there is N such that for all $n > N$, $d(x_n,x) \prec\prec c$, then (x_n) is said to be convergent to x and x is the limit of (x_n). We denote this by $x_n \to x$ as $n \to +\infty$.*

Definition 2.5. *[8] A sequence (x_n) in cone metric space (X,d) is said to be statistically convergent to $x \in X$ if for every $c \in E^+$*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : d(x_k,x) \prec\prec c\}| = 1$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : c \prec\prec d(x_k,x)\}| = 0.$$

where $|E|$ denotes the cardinality of E. Then it is denoted by $st - \lim_{n\to\infty} x_n = x$.

Definition 2.6. *[12] Let $s = (s_n)$ be a sequence of positive real numbers such that*

$$\lim_{n\to\infty} s_n = \infty \text{ and } \limsup_n \frac{s_n}{n} < \infty. \tag{2.1}$$

The quasi density of a subset $M \subset \mathbb{N}$ with respect to the sequence $s = (s_n)$ is defined by

$$\delta_s(M) = \lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : k \in M\}|.$$

A sequence (x_n) in $\mathbb{R}$ is called quasi-statistical convergent to x provided that for every $\varepsilon > 0$ the set $M_\varepsilon = \{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}$ has quasi-density zero. It is denoted by $st_q - \lim_{n\to\infty} x_n = x$.
Throughout the paper, we assume that $s = (s_n)$ is a sequence of positive real numbers satisfying the conditions (2.1).

Definition 2.7. *[12] A sequence (x_n) in $\mathbb{R}$ is said to be strongly quasi-summable to $x \in \mathbb{R}$ if*

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} |x_k - x| = 0.$$

Lemma 2.8. *[13] Let (X,d) be a cone metric space. Given $c \in E^+$. Then, for $c \in E^+$ with $0 \prec\prec c$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in E^+$.*

3. Main results

Definition 3.1. *A sequence (x_n) in the rectangular cone metric space (X,d) is said to be statistical convergent to a point $x \in X$ if for every $c \in E^+$ we have*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : d(x_k,x) \prec\prec c\}| = 1$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : c \prec\prec d(x_k,x)\}| = 0.$$

We denote it by $st - \lim_{n\to\infty} x_n = x$.

Definition 3.2. *Let (x_n) be a sequence in the rectangular cone metric space (X,d). The sequence (x_n) is said to be Cesaro summable if there is a $L \in X$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} d(x_k,L) \right\| = 0.$$

Definition 3.3. *Let (X,d) be a rectangular cone metric space, (x_n) be a sequence in X and let p be a positive real number. Then, the sequence (x_n) is said to be strongly p-Cesaro summable to L if there is a $L \in X$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|d(x_k,L)\|^p = 0.$$

Lemma 3.4. *Let P be a normal cone with normal constant K. The following statements hold for sequences (x_n) and (y_n) in rectangular cone metric space (X,d).*
(1) $st-\lim_{n\to\infty} x_n = x \Leftrightarrow st-\lim_{n\to\infty} d(x_n,x) = 0$
(2) If $st-\lim_{n\to\infty} x_n = x$ and $st-\lim_{n\to\infty} y_n = y$ then $st-\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$.

Proof. (1) Suppose that $st-\lim_{n\to\infty} x_n = x$. Then, for every $c \in E^+$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : d(x_k,x) \prec\prec c\}| = 1.$$

□

Given any for $\varepsilon > 0$, choose $c \in E^+$ such that $K\|c\| < \varepsilon$. Suppose that $k \in \mathbb{N}$ satisfies $d(x_k,x) \prec\prec c$. Since P is a normal cone with normal constant K, we can write

$$\|d(x_k,x)\| \le K\|c\| < \varepsilon.$$

Consequently, we obtain

$$\frac{1}{n} |\{k \le n : d(x_k,x) \prec\prec c\}| \le \frac{1}{n} |\{k \le n : \|d(x_k,x)\| < \varepsilon\}|.$$

Hence, we conclude that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \|d(x_k,x)\| < \varepsilon\}| = 1$$

which means that $st-\lim_{n\to\infty} d(x_n,x) = 0$.
Conversely, suppose that $st-\lim_{n\to\infty} d(x_n,x) = 0$. Then for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \|d(x_k,x)\| < \varepsilon\}| = 1.$$

Given any for $c \in E^+$, we can find an $\varepsilon > 0$ such that $c - a \in E^+$ for all $a \in E$ with $\|a\| < \varepsilon$. Hence if we choose $k \in \mathbb{N}$ such that $\|d(x_k,x)\| < \varepsilon$ then we obtain $d(x_k,x) \prec\prec c$ which implies that the inclusion $\{k : \|d(x_k,x)\| < \varepsilon\} \subset \{k : d(x_k,x) \prec\prec c\}$ holds. It follows that

$$\frac{1}{n} |\{k \le n : \|d(x_k,x)\| < \varepsilon\}| \le \frac{1}{n} |\{k \le n : d(x_k,x) \prec\prec c\}|.$$

Thus, we conclude that $\lim_{n\to\infty} \frac{1}{n} |\{k \le n : d(x_k,x) \prec\prec c\}| = 1$ and so $st-\lim_{n\to\infty} x_n = x$.
(2) Suppose $st-\lim_{n\to\infty} x_n = x$ and $st-\lim_{n\to\infty} y_n = y$. Given any $\varepsilon > 0$, choose $c \in E^+$ such that $\|c\| < \frac{\varepsilon}{4K+2}$. For $k \in \mathbb{N}$ with $d(x_k,x) \prec\prec c$ and $d(y_k,y) \prec\prec c$, we have $\|d(x_k,y_k) - d(x,y)\| < \varepsilon$ from the proof of Lemma in [13]. Hence, the inclusion

$$\{k : \varepsilon \le \|d(x_k,y_k) - d(x,y)\|\} \subset \{k : c \prec\prec d(x_k,x)\} \cup \{k : c \prec\prec d(y_k,y)\}$$

holds. It follows that

$$\lim_{n\to\infty} \frac{1}{n} \{k : \varepsilon \le \|d(x_k,y_k) - d(x,y)\|\} = 0$$

which means that $st-\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$.

Remark 3.5. *Note that P does not need to be a normal cone to prove the sufficiency in (1) of Lemma. That is; if $st-\lim_{n\to\infty} d(x_n,x) = 0$ in a rectangular cone metric space (X,d) then we have $st-\lim_{n\to\infty} x_n = x$.*

Theorem 3.6. *Let (x_n) be a sequence in rectangular cone metric space (X,d).*
(i) If (x_n) is Cesaro summable to L then it is statistically convergent to L.
(ii) Let P be a normal cone with normal constant K. If a bounded sequence is statistically convergent to L then it is Cesaro summable to L.

Proof. (i) Let $\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{k=1}^{n} d(x_k,L)\right\| = 0$. From the fact that

$$\begin{aligned}\left\|\sum_{k=1}^{n} d(x_k,L)\right\| &= \left\|\sum_{\substack{k=1\\ d(x_k,L)\prec\prec c}}^{n} d(x_k,L)+\sum_{\substack{k=1\\ d(x_k,L)\succeq c}}^{n} d(x_k,L)\right\| \\ &\geq \sum_{\substack{k=1\\ d(x_k,L)\succeq c}}^{n} d(x_k,L) \\ &\geq \|\ c\ \ |\{k\leq n : d(x_k,L)\succeq c\}|\| \\ &= \|c\|\ \ |\{k\leq n : d(x_k,L)\succeq c\}|,\end{aligned}$$

□

we have the following inequality

$$\frac{1}{\|c\|}\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{k=1}^{n} d(x_k,L)\right\| \geq \lim_{n\to\infty}\frac{1}{n}|\{k\leq n : d(x_k,L)\succ\succ c\}|.$$

Consequently, we find that $st-\lim_{n\to\infty}x_n = L$.
(ii) Let $A_n=\left\{k\leq n : \|d(x_k,L)\|\geq\frac{\varepsilon}{2K}\right\}$. Suppose that (x_n) is bounded and statistically convergent to L. Let M be a set created by the term of the sequence (x_n) .Then, M is bounded and

$$\delta(M)=\sup_{x,y\in M}\|d(x,y)\|<\infty.$$

Now, we suppose that $\sup\|d(x,y)\| = K$ and $st-\lim_{n\to\infty}x_n = L$. From (ii) of Lemma (1), it is $st-\lim_{n\to\infty}d(x_n,x)=0$ since P is a normal cone with normal constant K . Then, we can write down the inequality below for every $\|c\|=\frac{\varepsilon}{2K}>0$

$$\frac{1}{n}\left|\left\{k\leq n : \|d(x_k,L)\|\geq\frac{\varepsilon}{2}\right\}\right| < \frac{\varepsilon}{2K}.$$

Thus, we obtain that

$$\begin{aligned}\left\|\sum_{k=1}^{n} d(x_k,L)\right\| &\leq \sum_{k=1}^{n}\|d(x_k,L)\| \\ &= \sum_{\substack{k\in A_n\\ k\leq n}}\|d(x_k,L)\|+\sum_{\substack{k\notin A_n\\ k\leq n}}\|d(x_k,L)\| \\ &\leq \sum_{\substack{k\in A_n\\ k\leq n}}^{n}\sup\|d(x_k,L)\|+(n-|A_n|)\frac{\varepsilon}{2} \\ &\leq |A_n|.K+(n-|A_n|)\frac{\varepsilon}{2}\end{aligned}$$

and

$$\frac{1}{n}\left\|\sum_{k=1}^{n} d(x_k,L)\right\| \leq \frac{1}{n}\left(|A_n|.K+(n-|A_n|)\frac{\varepsilon}{2}\right). \tag{3.1}$$

If we take the limit in (3.1) as $n\to\infty$, we find that (x_n) is Cesaro summable to L.

Theorem 3.7. *Let (x_n) be a sequence in rectangular cone metric space (X,d) and $1\leq p<\infty$.*
(i) If (x_n) is strongly p-Cesaro summable to L then it is statistically convergent to L.
(ii) Let P be a normal cone with normal constant K. If a bounded sequence is statistically convergent to L then it is strongly p-Cesaro summable to L.

Proof. The proof can be done in the same way as in the proof of the previous theorem. □

Definition 3.8. *Let (x_n) be a sequence in a rectangular cone metric space (X,d) and $0<\alpha\leq 1$ and let $s=(s_n)$ be the sequence of positive real numbers satisfying the conditions in (2.1). Then (x_n) is said to be quasi statistically convergent to L of order α if for every $c\in E^+$*

$$\lim_{n\to\infty}\frac{1}{s_n^{\alpha}}\left|\{k\leq n : c \prec\prec d(x_k,L)\}\right| = 0$$

or equivalently

$$\lim_{n\to\infty}\frac{1}{s_n^{\alpha}}\left|\{k\leq n : d(x_k,L) \prec\prec c\}\right| = 1.$$

We denote it by $st_q^{\alpha}-\lim_{n\to\infty} x_n = L$.
If we take $\alpha = 1$ then (x_n) is said to be quasi statistically convergent to L and it is denoted by $st_q - \lim_{n\to\infty} x_n = L$.

Theorem 3.9. *Let (x_n) be a sequence in a cone rectangular metric space (X,d). If (x_n) is convergent to $L\in X$ then it is quasi statistically convergent to L of order α.*

Proof. Let $\lim_{n\to\infty} x_n = L$. Then, for every $c\in E^{+}$ there exists $n_0\in\mathbb{N}$ such that $d(x_n,L)\prec\prec c$ for every $n>n_0$. It follows that □

$$\frac{1}{s_n^{\alpha}}\left|\{k\leq n : c \prec\prec d(x_k,L)\}\right| \leq \frac{n_0}{s_n^{\alpha}}$$

which means that $\lim_{n\to\infty}\frac{1}{s_n^{\alpha}}\left|\{k\leq n : c \prec\prec d(x_k,L)\}\right| = 0$.
Hence, (x_n) is quasi statistically convergent to L of order α.

Theorem 3.10. *Let (x_n) be a sequence in a rectangular cone metric space (X,d). If (x_n) is quasi statistically convergent to L of order α then it is statistically convergent to L.*

Proof. Suppose that $st_q^{\alpha}-\lim_{n\to\infty} x_n = L$ and let $M=\sup_n \frac{s_n^{\alpha}}{n}$. Then, for every $c\in E^{+}$ we have $\lim_{n\to\infty}\frac{1}{s_n^{\alpha}}\left|\{k\leq n : c \preceq d(x_k,L)\}\right| = 0$. The statistical convergence of the sequence (x_n) follows from the following inequality □

$$\frac{1}{n}\left|\{k\leq n : c \prec\prec d(x_k,L)\}\right| \leq \frac{M}{s_n^{\alpha}}\left|\{k\leq n : c \prec\prec d(x_k,L)\}\right|.$$

Consequently, we have the following diagram:

convergent $\Longrightarrow$ quasi statistical convergent of order α $\Longrightarrow$ statistical convergent.

Theorem 3.11. *Let (x_n) be a sequence in a rectangular cone metric space (X,d). Assume that*

$$h = \inf_n \frac{s_n}{n} > 0. \tag{3.2}$$

If a sequence (x_n) in a rectangular cone metric space (X,d) is statistical convergent to $L\in X$ then it is quasi statistical convergent to L of order α.

Proof. Let $st-\lim_{n\to\infty} x_n = L$. The proof follows from the inequality

$$\frac{1}{n}\left|\{k\leq n : c \preceq d(x_k,L)\}\right| \geq \frac{h}{s_n^{\alpha}}\left|\{k\leq n : c \preceq d(x_k,L)\}\right|.$$

□

We can give the following.

Corollary 3.12. *Suppose that the sequence (s_n) satisfies the condition in (3.2). Then (x_n) is statistical convergent to L if and only if (x_n) is quasi statistical convergent to L of order α.*

Theorem 3.13. *Let (x_n) be a sequence in a rectangular cone metric space (X,d). If $st_q^{\alpha}-\lim_{n\to\infty} x_n = L$ then $st_q-\lim_{n\to\infty} x_n = L$.*

Proof. Suppose that $st_q^{\alpha}-\lim_{n\to\infty} x_n = L$. Then, the result follows from the following inequality

$$\frac{1}{s_n}\left|\{k\leq n : c \preceq d(x_k,L)\}\right| \leq \frac{1}{s_n^{\alpha}}\left|\{k\leq n : c \preceq d(x_k,L)\}\right|.$$

□

References

[1] Steinhaus, H.: Sur la convergence ordinarie et la convergence asimptotique. Colloq. Math. (2), 73-74 (1951)
[2] Fast, H.: Sur la convergence statistique. Colloq. Math. 2, 241-244 (1951)
[3] Schoenberg, I., J.: The integrability of certain functions and related summability methods. Amer. Math. Montly 86, 361-375 (1959)
[4] Connor, J.: The statistical and strong p-Cesaro convergence of sequences. Analysis 8, 47-63 (1988)
[5] Connor, J.: On strong matrix summability with respect to a modulus and statistical convergence. Canad. Math. Bull. 32, 194-198 (1989)
[6] Fridy, J. A.: On statistical convergence. Analysis 5, 301-313, (1985)
[7] Maio, G. D., Kočinac, L. D. R.: Statistical convergence in topology. Topol. Appl. 156, 28-45 (2008)
[8] Li, K., Lin, S., Ge, Y.: On statistical convergence in cone metric space. Topol. Appl. 196, 641-651 (2015)
[9] Turan, N., Kara, E. E., İlkhan, M.: Quasi statistical convergence in cone metric spaces. Facta Univ. Ser. Math. Inform. 33(4), 613-626 (2018)
[10] İlkhan, M., Kara, E. E.: A new type of statistical Cauchy sequence and its relation to Bourbaki completeness. Cogent Math. Stat. 5(1), 1-9 (2018)
[11] Ganichev, M., Kadets, V.: “Filter convergence in Banach spaces and generalized bases,” in General Topology in Banach Spaces. USA: Nova Science 61-69 (2001)
[12] Sakaoğlu Özgüç, İ., Yurdakadim, T.: On quasi-statistcal convergence. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 61(1), 11-17 (2012)
[13] Long-Guang, H., Xian, Z.: Cone metric space and fixed point theorems of contractive mappings. J. Math. Anal. Appl. 332(2), 1467-1475 (2007)
[14] Azam, A., Arshad, M., Beg, I.: Banach contraction principle on cone rectangular metric space. Appl. Anal. Discrete Math. 3, 236-241 (2009)
[15] Jleli, M., Samet, B.: The Kannas’s fixed point theorem in a cone rectangular metric space. J. Nonlinear Sci. Appl. 2(3), 161-167 (2009)