On the New Wirtinger Type Inequalities

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Abstract

The aim of this paper to establish some generalized and refinement of Wirtinger type inequality.

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1. Introduction

The classical Wirtinger inequality is given by

\[ \int_a^b (f(x))^2 \, dx \leq \int_a^b (f'(x))^2 \, dx \] \hspace{2cm} (1.1)

for any \( f \in C^1([a,b]) \) satisfying \( f(a) = f(b) = 0 \) in [6]. Then, Beesack extended the inequality (1.1) and proved that

\[ \int_a^b (f(x))^4 \, dx \leq \frac{4}{3} \int_a^b (f'(x))^4 \, dx \]

for any \( f \in C^2([a,b]) \) satisfying \( f(a) = f(b) = 0 \) in [5].

One of the most impressive issues in inequality theory is integral inequalities involving a function and its derivative. Wirtinger inequality in this area of the theorems has been a particular attraction due to close coupling to linear differential equations and differential geometry. Wirtinger’s inequality compares the integral of a square of a function with that of a square of its first derivative. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Wirtinger’s inequality and its generalizations, see [1]-[11]. The purpose of this paper is to establish some generalized and refinement of Wirtinger type inequalities.

2. Main Results

Now, we present the main results:

**Theorem 2.1.** Let \( f \in C^4([a,b]) \) with \( f(a) = f(b) = 0 \) and \( f' \in L^2[a,b] \), then, we have the following inequality

\[ \int_a^b |f(x)|^2 \, dx \leq \frac{(b-a)^2}{6} \int_a^b |f'(x)|^2 \, dx. \]

**Proof.** From the hypotheses, we have

\[ |f(x)|^2 \leq \left( \int_a^x f'(t) \, dt \right)^2, \quad \text{with} \ f(a) = 0 \]
Then using Fubini’s theorem it follows that

\[ [f(x)]^2 \leq \left( \int_a^b f'(t) \, dt \right)^2, \quad \text{with } f(b) = 0. \]

By using the Cauchy-Shwartz inequality, we have

\[ [f(x)]^2 \leq (x-a) \int_a^b [f'(t)]^2 \, dt \quad \text{(2.1)} \]

and

\[ [f(x)]^2 \leq (b-x) \int_a^b [f'(t)]^2 \, dt. \quad \text{(2.2)} \]

By integrating both sides of the inequality (2.1) from \( a \) to \( a\lambda + (1-\lambda)b \) for \( \lambda \in [0,1] \), we get

\[
\int_a^{a\lambda + (1-\lambda)b} [f(x)]^2 \, dx \leq \int_a^{a\lambda + (1-\lambda)b} \int_a^b (x-a) [f'(t)]^2 \, dt \, dx.
\]

Then using Fubini’s theorem it follows that

\[
\int_a^{a\lambda + (1-\lambda)b} [f(x)]^2 \, dx \leq \frac{1}{2} \int_a^{a\lambda + (1-\lambda)b} \left( (1-\lambda)^2 (b-a)^2 - (t-a)^2 \right) \int_a^b [f'(t)]^2 \, dt.
\]

By the change of variable \( t = au + (1-u)b \), on the right hand sides integrals, we get

\[
\int_a^{a\lambda + (1-\lambda)b} [f(x)]^2 \, dx \leq \frac{(b-a)^3}{2} \int_a^{a\lambda + (1-\lambda)b} \left[ (1-\lambda)^2 (b-a)^2 - (u-a)^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, du.
\]

Similarly, by integrating both sides of the inequality (2.2) from \( a\lambda + (1-\lambda)b \) to \( b \) for \( \lambda \in [0,1] \), we get

\[
\int_{a\lambda + (1-\lambda)b}^b [f(x)]^2 \, dx \leq \int_{a\lambda + (1-\lambda)b}^b \int_a^b (b-x) [f'(t)]^2 \, dt \, dx.
\]

Then Fubini’s theorem it follows that

\[
\int_{a\lambda + (1-\lambda)b}^b [f(x)]^2 \, dx \leq \frac{1}{2} \int_{a\lambda + (1-\lambda)b}^b \left[ \lambda^2 (b-a)^2 - (t-b)^2 \right] \int_a^b [f'(t)]^2 \, dt.
\]

By the change of variable \( t = au + (1-u)b \), on the right hand sides integrals, we get

\[
\int_{a\lambda + (1-\lambda)b}^b [f(x)]^2 \, dx \leq \frac{(b-a)^3}{2} \int_{a\lambda + (1-\lambda)b}^b \left[ \lambda^2 - u^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, du.
\]

Adding (2.3) and (2.4), it follows that

\[
\int_a^b [f(x)]^2 \, dx \leq \frac{(b-a)^3}{2} \left\{ \int_0^1 \left[ (1-\lambda)^2 - (1-u)^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, du \right\} dx
\]

\[
+ \frac{(b-a)^3}{2} \int_0^1 \left[ \lambda^2 - u^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, du.
\]

By integrating both sides of the inequality from 0 to 1 with respect to \( \lambda \), we get

\[
\int_a^b [f(x)]^2 \, dx \leq \frac{(b-a)^3}{2} \left\{ \int_0^1 \left[ (1-\lambda)^2 - (1-u)^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, dud\lambda \right\}
\]

\[
+ \frac{(b-a)^3}{2} \int_0^1 \left[ \lambda^2 - u^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, dud\lambda.
\]
By using change of order of the integrals, we have
\[
\int_{a}^{b} |f(x)|^2 \, dx \leq \frac{(b-a)^3}{2} \int_{0}^{1} \left[ (1-\lambda)^2 - (1-u)^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, d\lambda \, du
\]
\[
+ \frac{(b-a)^3}{2} \int_{0}^{1} \left[ \lambda^2 - u^2 \right] \left[ f'(au + (1-u)b) \right]^2 \, d\lambda \, du
\]
\[
= \frac{(b-a)^3}{6} \int_{0}^{1} \left[ f'(au + (1-u)b) \right]^2 \, du,
\]
which is the desired inequality.

Theorem 2.2. Let \( f \in C^1([a,b]) \) with \( f(a) = f(b) = 0 \), \( p > 1 \), and \( f' \in L_p[a,b] \), then, we have the following inequality
\[
\int_{a}^{b} |f(x)|^p \, dx \leq \frac{(b-a)^{p-1}}{2^{p-1}p} \int_{a}^{b} |f'(x)|^p \, dx.
\]
\[(2.5)\]

Proof. From the hypotheses, we have
\[
|f(x)|^p \leq \left( \int_{a}^{b} |f'(t)| \, dt \right)^p, \quad \text{with } f(a) = 0
\]
\[
|f(x)|^p \leq \left( \int_{a}^{b} |f'(t)| \, dt \right)^p, \quad \text{with } f(b) = 0,
\]
and hence from Hölder’s inequality with indices \( p \) and \( \frac{p}{p-1} \), it follows that
\[
|f(x)|^p \leq (x-a)^{p-1} \int_{a}^{x} |f'(t)|^p \, dt
\]
\[(2.6)\]
and
\[
|f(x)|^p \leq (b-x)^{p-1} \int_{x}^{b} |f'(t)|^p \, dt.
\]
\[(2.7)\]

By integrating both sides of the inequality (2.6) from \( a \) to \( a\lambda + (1-\lambda) b \) for \( \lambda \in [0,1] \), we get
\[
\int_{a}^{a\lambda + (1-\lambda) b} |f(x)|^p \, dx
\]
\[
\leq \int_{a}^{a\lambda + (1-\lambda) b} (x-a)^{p-1} \int_{a}^{x} |f'(t)|^p \, dt \, dx
\]
\[
= \frac{(x-a)^p}{p} |f'(x)|_{a\lambda + (1-\lambda) b}^{a\lambda + (1-\lambda) b} - \frac{1}{p} \int_{a}^{a\lambda + (1-\lambda) b} (x-a)^p |f'(x)|^p \, dx
\]
\[
\leq \frac{(1-\lambda)^p (b-a)^p}{p} |f'(a\lambda + (1-\lambda) b)|^p
\]
Similarly, by integrating both sides of the inequality (2.7) from \( a\lambda + (1-\lambda) b \) to \( b \) for \( \lambda \in [0,1] \), we get
\[
\int_{a\lambda + (1-\lambda) b}^{b} |f(x)|^p \, dx
\]
\[
\leq \int_{a\lambda + (1-\lambda) b}^{b} (b-x)^{p-1} \int_{x}^{b} |f'(t)|^p \, dt \, dx
\]
\[
= \frac{(b-x)^p}{p} |f'(x)|_{a\lambda + (1-\lambda) b}^{b} - \frac{1}{p} \int_{a\lambda + (1-\lambda) b}^{b} (b-x)^p |f'(x)|^p \, dx
\]
\[
\leq \frac{\lambda^p (b-a)^p}{p} |f'(a\lambda + (1-\lambda) b)|^p.
\]
Adding (2.8) and (2.9), it follows that
\[ \int_a^b |f(x)|^p \, dx \leq \frac{(b-a)^p}{p} |(1-\lambda)^p + \lambda^p| \left| f'(a\lambda + (1-\lambda)b) \right|^p. \]

By integrating both sides of the inequality from 0 to 1 with respect to \( \lambda \), we get
\[ \int_a^b |f(x)|^p \, dx \leq \frac{(b-a)^p}{p} \int_0^1 |(1-\lambda)^p + \lambda^p| \left| f'(a\lambda + (1-\lambda)b) \right|^p \, d\lambda. \]

It is not difficult to reveal that the function
\[ h(u) = (1-\lambda)^p + \lambda^p \]
for all \( \lambda \in [0, 1] \) attains its maximum \( \frac{1}{2} \) at the point \( \lambda = \frac{1}{2} \in [0, 1] \) and using the change of variable \( u = a\lambda + (1-\lambda)b \), which is the same as (2.5). This completes the proof. \( \square \)

**Theorem 2.3.** Let \( f, g \in C^1([a,b]) \) with \( f(a) = f(b) = 0 \), \( g(a) = g(b) = 0 \), and \( f', g' \in L^2[a,b] \), then, we have the following inequality
\[ \int_a^b |f(x)||g(x)| \, dx \leq \frac{(b-a)^2}{8} \int_a^b \left[ |f'(x)|^2 + |g'(x)|^2 \right] \, dx. \] (2.10)

**Proof.** From the hypotheses, we have
\[ f(x) = \frac{1}{2} \left( \int_a^x f'(t) \, dt - \int_x^b f'(t) \, dt \right) \] (2.11)
and
\[ g(x) = \frac{1}{2} \left( \int_a^x g'(t) \, dt - \int_x^b g'(t) \, dt \right). \] (2.12)

Using the properties of modulus we have
\[ |f(x)| \leq \frac{1}{2} \left( \int_a^b |f'(t)| \, dt \right), \] (2.13)
\[ |g(x)| \leq \frac{1}{2} \left( \int_a^b |g'(t)| \, dt \right). \] (2.14)

Multiplying the left sides and right sides of (2.13) and (2.14) and then integrating both sides of the inequality from \( a \) to \( b \) with respect to \( x \), we get
\[ \int_a^b |f(x)||g(x)| \, dx \leq \frac{(b-a)^2}{4} \left( \int_a^b |f'(t)|^2 \, dt \right) \left( \int_a^b |g'(t)|^2 \, dt \right). \]

By using the Cauchy-Schwartz inequality, and then using elementary inequality \( \sqrt{mn} \leq \frac{1}{2}(m+n) \), \( m,n \geq 0 \), we have
\[ \int_a^b |f(x)||g(x)| \, dx \leq \frac{(b-a)^2}{4} \left( \int_a^b |f'(t)|^2 \, dt \right)^{\frac{1}{2}} \left( \int_a^b |g'(t)|^2 \, dt \right)^{\frac{1}{2}} \leq \frac{(b-a)^2}{8} \int_a^b \left[ |f'(x)|^2 + |g'(x)|^2 \right] \, dx \]
which is the desired inequality. \( \square \)

**Remark 2.4.** By taking \( f = g \) and \( f' = g' \) in Theorem 2.3, we have
\[ \int_a^b |f(x)|^2 \, dx \leq \frac{(b-a)^2}{4} \int_a^b |f'(x)|^2 \, dx. \]
References