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Eigenfunction Expansion in the Singular Case for Dirac Systems on Time Scales

Bilender P. Allahverdiev[1] and Hüseyin Tuna[2*]

[1]*Department of Mathematics, Faculty of Science and Arts, Süleyman Demirel University, , 32260 Isparta, TURKEY*
[2]*Department of Mathematics, Faculty of Science and Arts, Mehmet Akif Ersoy University, , 15030 Burdur, TURKEY*
[*]*Corresponding author E-mail: hustuna@gmail.com*

Abstract

In this work, we prove the existence of a spectral function for one dimensional singular Dirac system on time scales. Further, we establish a Parseval equality and expansion formula in eigenfunctions by terms of the spectral function.

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1. Introduction

The theory of time scales attempts to unify continuous and discrete mathematics. It was introduced at first by Stefan Hilger in [12]. This theory represents an effective tool for applications to insect population models, quantum physics, maximization problems in economics, epidemic models among others. Hence, it has recently received a lot of attention (see [1], [4]- [7], [9]-[10], [15]).
Eigenfunction expansions theorems are important for solving varies problems in mathematics. We lead to the problem of expanding an arbitrary function as a series of eigenfunctions whenever we seek a solution of a partial differential equation by the Fourier method. There are a lot of studies about eigenfunction expanding problems (for instance, see [2]-[3], [10]-[11], [16], [19]).
In this paper, we consider the one dimensional singular Dirac system

$$\begin{aligned} -\Delta y_2^{\rho}+p(t)\,y_1 &= \lambda y_1, \\ \Delta y_1+r(t)\,y_2 &= \lambda y_2, \end{aligned} \tag{1.1}$$

where $p(.)$ and $r(.)$ are real-valued functions defined on $[a,\infty)_{\mathbb{T}}$ and $p,r\in L^1_{\Delta,loc}([a,\infty)_{\mathbb{T}})$, where
$L^1_{\Delta,loc}([a,\infty)_{\mathbb{T}}) := \{f:[a,\infty)_{\mathbb{T}}\to\mathbb{R}:=(-\infty,\infty),\ \int_I |f(t)|\,\Delta t<\infty, \forall I \text{ finite subinterval of } [a,\infty)_{\mathbb{T}}\}$.
If $\mathbb{T}=\mathbb{R}$, the system (1.1) describe a relativistic electron in the electrostatic field (see [20]). For these systems, we prove the existence of a spectral function. A Parseval equality and an expansion formula in eigenfunctions are established.
On the other hand, there is a few research about Dirac system on time scales ([8], [13]). Hence, our study can fill the important gap in this subject.
Now, we recall some necessary fundamental concepts of time scale calculus. These definitions and properties can be found in [6]-[7].
Let $\mathbb{T}$ be a time scale, i.e, a non-empty closed subset of real numbers $\mathbb{R}$. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},\ \text{where } t\in\mathbb{T}$$

and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\},\ \text{where } t\in\mathbb{T}.$$

It is convenient to have graininess operators $\mu_\sigma:\mathbb{T}\to[0,\infty)$ and $\mu_\rho:\mathbb{T}\to(-\infty,0]$ defined by

$$\mu_\sigma(t)=\sigma(t)-t$$

and

$$\mu_\rho(t)=\rho(t)-t,$$

respectively.

Email addresses: **bilenderpasaoglu@sdu.edu.tr (Bilender P. Allahverdiev), hustuna@gmail.com (Hüseyin Tuna)**

Definition 1.1. *A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$.*

Now, we introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$ which are derived form the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 1.2. *A function f on $\mathbb{T}$ is said to be Δ- differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \textit{where } s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show (see [9])

$$f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions.

If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f'(t).$$

If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, then

$$f^\Delta(t) = \frac{f(t+h) - f(t)}{h}.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, then

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

The product and quotient rules on time scales have the following form: If $f, g : \mathbb{T} \to \mathbb{R}$, then

$$\begin{aligned}
(fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t), \\
(fg)^\nabla(t) &= f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t), \\
\left(\frac{f}{g}\right)^\Delta(t) &= \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}, \\
\left(\frac{f}{g}\right)^\nabla(t) &= \frac{f^\nabla(t)g(t) - f(t))g^\nabla(t)}{g(t)g(\rho(t))}.
\end{aligned}$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Analogously one may define the notion of ∇-antiderivative of some function.
If $\mathbb{T} = \mathbb{R}$ and f is continuous, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$ and $a = hx,\ b = hy,\ x < y$, then

$$\int_a^b f(t)\Delta t = h\sum_{k=x}^{y-1} f(hk).$$

If $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$ and $a = q^x, b = q^y,\ x < y$, then

$$\int_a^b f(t)\Delta t = (q-1)\sum_{k=x}^{y-1} q^k f\left(q^k\right).$$

Let $\mathbb{T}$ be a time scale which is bounded from below and unbounded from above such that $\inf\mathbb{T} = a > -\infty$ and $\sup\mathbb{T} = \infty$. We will denote $\mathbb{T}$ also as $[a,\infty)_\mathbb{T}$.
Let $L^2_\Delta[a,\infty)_\mathbb{T}$ be the space of all functions defined on $\mathbb{T}$ such that

$$\|f\| := \left(\int_a^\infty |f(t)|^2 \Delta t\right)^{1/2} < \infty.$$

The space $L^2_\Delta[a,\infty)_\mathbb{T}$ is a Hilbert space with the inner product (see [17])

$$\langle f, g\rangle := \int_a^\infty f(t)\overline{g(t)}\Delta t, \quad f, g \in L^2_\Delta[a,\infty)_\mathbb{T}.$$

Now, we introduce convenient Hilbert space $\mathscr{H} := L^2_\Delta\left([a,\infty)_{\mathbb{T}};E\right)$ $(E := \mathbb{R}^2)$ of vector-valued functions using the inner product

$$(f,g) := \int_a^\infty (f(t),g(t))_E \Delta t,\ \ f,g \in \mathscr{H}\ .$$

Now let $y(.) = \begin{pmatrix} y_1(.) \\ y_2(.) \end{pmatrix}$, $z(.) = \begin{pmatrix} z_1(.) \\ z_2(.) \end{pmatrix} \in \mathscr{H}$. Then, we define the Wronskian of $y(t)$ and $z(t)$ by

$$W(y,z)(t) = y_1(t)z_2^\rho(t) - z_1(t)y_2^\rho(t), \tag{1.2}$$

where $f^\rho(t) := f(\rho(t))$.

2. Main Results

Let us consider

$$\tau(y) := \begin{cases} -\Delta y_2^\rho + p(t)y_1 \\ \Delta y_1 + r(t)y_2 \end{cases},$$

$$\tau(y) = \lambda y,\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ t \in [a,\infty)_{\mathbb{T}}, \tag{2.1}$$

with the boundary condition

$$y_1(a,\lambda)\sin\beta + y_2^\rho(a,\lambda)\cos\beta = 0,\ \beta \in \mathbb{R}, \tag{2.2}$$

where $\Delta f(t) := f^\Delta(t)$, λ is a complex eigenvalue parameter, $p(.)$ and $r(.)$ are real-valued functions defined on $[a,\infty)_{\mathbb{T}}$ and $p,r \in L^1_{\Delta,loc}([a,\infty)_{\mathbb{T}})$.

Denote by $\phi(t,\lambda) = \begin{pmatrix} \phi_1(t,\lambda) \\ \phi_2(t,\lambda) \end{pmatrix}$, the solution of the system (2.1) subject to the initial conditions

$$\phi_1(a,\lambda) = \cos\beta,\ \phi_2^\rho(a,\lambda) = -\sin\beta. \tag{2.3}$$

Further, we adjoin to problem (2.1)-(2.2) the boundary condition

$$y_2^\rho(b,\lambda)\cos\alpha + y_1(b,\lambda)\sin\alpha = 0,\ b \in (a,\infty)_{\mathbb{T}}\ ,\alpha \in \mathbb{R}. \tag{2.4}$$

It is clear that the problem (2.1), (2.2), (2.4) is a regular problem for a Dirac system.
Let $\lambda_{m,b}$ $(m \in \mathbb{Z} := \{0,\pm1,\pm2,...\})$ denote the eigenvalues of this problem and by

$$\phi_{m,b}(t) = \begin{pmatrix} \phi_{m,b}^{(1)}(t) \\ \phi_{m,b}^{(2)}(t) \end{pmatrix} = \phi\left(t,\lambda_{m,b}\right) = \begin{pmatrix} \phi_1\left(t,\lambda_{m,b}\right) \\ \phi_2\left(t,\lambda_{m,b}\right) \end{pmatrix}$$

the corresponding eigenfunction which satisfy the conditions (2.2). If $f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$,

$$\int_a^b \left(f_1^2(t) + f_2^2(t)\right)\Delta t < +\infty,$$

and

$$\alpha_{m,b}^2 = \int_a^b \left(\left(\phi_{m,b}^{(1)}(t)\right)^2 + \left(\phi_{m,b}^{(2)}(t)\right)^2\right)\Delta t,$$

then we have

$$\int_a^b \left(f_1^2(t) + f_2^2(t)\right)\Delta t$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,b}^2}\left\{\int_a^b \left(f_1(t)\phi_{m,b}^{(1)}(t) + f_2(t)\phi_{m,b}^{(2)}(t)\right)\Delta t\right\}^2. \tag{2.5}$$

which is called the Parseval equality.
Now, let us define the nondecreasing step function ω_b on $(-\infty,\infty)$ by

$$\omega_b(\lambda) = \begin{cases} -\sum_{\lambda<\lambda_{m,b}<0} \frac{1}{\alpha_{m,b}^2}, & \text{for } \lambda \le 0 \\ \sum_{0\le\lambda_{m,b}<\lambda} \frac{1}{\alpha_{m,b}^2} & \text{for } \lambda \ge 0. \end{cases}$$

Then equalities (2.5) can be written as

$$\int_a^b \left(f_1^2(t) + f_2^2(t)\right)\Delta t = \int_{-\infty}^{\infty} F^2(\lambda)\,d\omega_b(\lambda), \tag{2.6}$$

where

$$F(\lambda)=\int_a^b (f_1(t)\phi_1(t,\lambda)+f_2(t)\phi_2(t,\lambda))\Delta t.$$

We will show that the Parseval equality for the problem (2.1), (2.2) can be obtained from (2.6) by letting $b\to\infty$. For this purpose, we shall prove a lemma.

Lemma 2.1. *For any positive κ, there is a positive constant $\Upsilon=\Upsilon(\kappa)$ not depending on b such that*

$$\bigvee_{-\kappa}^{\kappa}\{\omega_b(\lambda)\}=\sum_{-\kappa\le\lambda_{m,b}<\kappa}\frac{1}{\alpha_{m,b}^2}=\omega_b(\kappa)-\omega_b(-\kappa)<\Upsilon. \tag{2.7}$$

Proof. Let $\sin\beta\neq 0$. Since $\phi_2(t,\lambda)$ is continuous on the region

$$\{(t,\lambda):-\kappa\le\lambda\le\kappa, a\le t\le b\},$$

by condition $\phi_2^{\rho}(a,\lambda)=-\sin\beta$, there is a positive number k and near by a such that

$$\left(\frac{1}{k}\int_a^k\phi_2(t,\lambda)\Delta t\right)^2>\frac{1}{2}\sin^2\beta. \tag{2.8}$$

Let us define $f_k(t)=\begin{pmatrix} f_{1k}(t)\\ f_{2k}(t)\end{pmatrix}$ by

$$f_{1k}(t)=0,\ f_{2k}(t)=\begin{cases}\frac{1}{k}, & a\le t<k\\ 0, & t\ge k.\end{cases}$$

From (2.6), (2.7) and (2.8), we get

$$\begin{aligned}\int_a^k\left(f_{1k}^2(t)+f_{2k}^2(t)\right)\Delta t=\frac{k-a}{k^2}&=\int_{-\infty}^{\infty}\left(\frac{1}{k}\int_a^k\phi_2(t,\lambda)\Delta t\right)^2 d\omega_b(\lambda)\\ &\ge\int_{-\kappa}^{\kappa}\left(\frac{1}{k}\int_a^k\phi_2(t,\lambda)\Delta t\right)^2 d\omega_b(\lambda)\\ &>\frac{1}{2}\sin^2\beta\{\omega_b(\kappa)-\omega_b(-\kappa)\},\end{aligned}$$

which proves the inequality (2.7).

If $\sin\beta=0$, then we define the function $f_k(t)=\begin{pmatrix} f_{1k}(t)\\ f_{2k}(t)\end{pmatrix}$ by the formula

$$f_{1k}(t)=\begin{cases}\frac{1}{k^2}, & a\le t<k\\ 0, & t\ge k\end{cases},\ f_{2k}(t)=0.$$

So, we obtain the inequality (2.7) by applying the Parseval equality. □

Now, we recall that the following well-known theorems of Helly's.

Theorem 2.2 ([14]). *Let $(u_n)_{n\in\mathbb{N}}$ ($\mathbb{N}:=\{1,2,3,...\}$) be a uniformly bounded sequence of real nondecreasing function on a finite interval $c\le\lambda\le d$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function u such that*

$$\lim_{k\to\infty}u_{n_k}(\lambda)=u(\lambda),\ c\le\lambda\le d.$$

Theorem 2.3 ([14]). *Assume $(u_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded, sequence of nondecreasing function on a finite interval $c\le\lambda\le d$, and suppose*

$$\lim_{n\to\infty}u_n(\lambda)=u(\lambda),\ c\le\lambda\le d.$$

If f is any continuous function on $c\le\lambda\le d$, then

$$\lim_{n\to\infty}\int_c^d f(\lambda)\,du_n(\lambda)=\int_c^d f(\lambda)\,du(\lambda).$$

Let ω be any nondecreasing function on $-\infty<\lambda<\infty$. Denote by $L_\omega^2(-\infty,\infty)$ the Hilbert space of all functions $f:(-\infty,\infty)\to(-\infty,\infty)$ which are measurable with respect to the Lebesque-Stieltjes measure defined by ω and such that

$$\int_{-\infty}^{\infty}f^2(\lambda)\,d\omega(\lambda)<\infty,$$

with the inner product

$$(f,g)_\omega:=\int_{-\infty}^{\infty}f(\lambda)g(\lambda)\,d\omega(\lambda).$$

The main result of this paper is the following theorem.

Theorem 2.4. *For the Dirac system (2.1)-(2.2), there exists a nondecreasing function $\omega(\lambda)$ on $-\infty<\lambda<\infty$ with the following properties.*
(i) If $f(.)=\begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix}\in\mathscr{H}$, there exist a function $F\in L^2_\omega(-\infty,\infty)$ such that

$$\lim_{b\to\infty}\int_{-\infty}^{\infty}\left\{F(\lambda)-\int_a^b (f_1(t)\phi_1(t,\lambda)+f_2(t)\phi_2(t,\lambda))\Delta t\right\}d\omega(\lambda)=0, \tag{2.9}$$

and the Parseval equality

$$\int_a^{\infty}\left(f_1^2(t)+f_2^2(t)\right)\Delta t=\int_{-\infty}^{\infty}F^2(\lambda)\,d\omega(\lambda) \tag{2.10}$$

holds.
(ii) The integrals

$$\int_{-\infty}^{\infty}F(\lambda)\phi_1(t,\lambda)\,d\omega(\lambda) \text{ and } \int_{-\infty}^{\infty}F(\lambda)\phi_2(t,\lambda)\,d\omega(\lambda)$$

converge to f_1 and f_2 in $L^2_\Delta[a,\infty)_{\mathbb{T}}$, respectively. That is,

$$\lim_{b\to\infty}\int_a^b\left\{f_1(t)-\int_{-\infty}^{\infty}F(\lambda)\phi_1(t,\lambda)\,d\omega(\lambda)\right\}^2\Delta t=0,$$
$$\lim_{b\to\infty}\int_a^b\left\{f_2(t)-\int_{-\infty}^{\infty}F(\lambda)\phi_2(t,\lambda)\,d\omega(\lambda)\right\}^2\Delta t=0.$$

We note that the function ω is called a spectral function for the system (2.1)-(2.2).

Proof. Assume that the function $f_\xi(x)=\begin{pmatrix} f_{1\xi}(x) \\ f_{2\xi}(x) \end{pmatrix}$ satisfies the following conditions.
1) $f_\xi(t)$ vanishes outside the interval $[a,\xi]_{\mathbb{T}},\ \xi\in\mathbb{T},\xi<b$.
2) The function $f_\xi(t)$ is $\Delta-$differentiable.
3) $f_\xi(t)$ satisfies the boundary condition (2.2).
If we apply to $f_\xi(t)$ the Parseval equality (2.6), we obtain

$$\int_a^{\xi}\left(f_{1\xi}^2(t)+f_{2\xi}^2(t)\right)\Delta t=\int_{-\infty}^{\infty}F_\xi^2(\lambda)\,d\omega(\lambda), \tag{2.11}$$

where

$$F_\xi(\lambda)=\int_a^{\xi}\left(f_{1\xi}(x)\phi_1(t,\lambda)+f_{2\xi}(t)\phi_2(t,\lambda)\right)\Delta t. \tag{2.12}$$

Since $\phi(t,\lambda)$ satisfies the system (2.1), we see that

$$\phi_1(t,\lambda)=\frac{1}{\lambda}\left[-\Delta\phi_2^{\rho}(t,\lambda)+p(t)\phi_1(t,\lambda)\right],$$
$$\phi_2(t,\lambda)=\frac{1}{\lambda}\left[\Delta\phi_1(t,\lambda)+r(t)\phi_2(t,\lambda)\right].$$

By (2.12), we get

$$F_\xi(\lambda)=\frac{1}{\lambda}\int_a^b f_{1\xi}(t)\left[-\Delta\phi_2^{\rho}(t,\lambda)+p(t)\phi_1(t,\lambda)\right]\Delta t$$
$$+\frac{1}{\lambda}\int_a^b f_{2\xi}(t)\left[\Delta\phi_1(t,\lambda)+r(t)\phi_2(t,\lambda)\right]\Delta t.$$

Since $f_\xi(t)$ vanishes in a neighborhood of the point b and $f_\xi(t)$ and $\phi(t,\lambda)$ satisfy the boundary condition (2.3), we obtain

$$F_\xi(\lambda)=\frac{1}{\lambda}\int_a^b \phi_1(t,\lambda)\left[-\Delta f_{2\xi}^{\rho}(t)+p(t)f_{1\xi}(t)\right]\Delta t$$
$$+\frac{1}{\lambda}\int_a^b \phi_2(t,\lambda)\left[\Delta f_{1\xi}(t)+r(t)f_{2\xi}(t)\right]\Delta t,$$

by integration by parts.
For any finite $\kappa>0$, using (2.6), we have
$\int_{|\lambda|>\kappa}F_\xi^2(\lambda)\,d\omega_b(\lambda)$
$\leq\frac{1}{\kappa^2}\int_{|\lambda|>\kappa}\left\{\int_a^b\left[\begin{array}{c}\phi_1(t,\lambda)\left[-\Delta f_{2\xi}^{\rho}(t)+p(t)f_{1\xi}(t)\right]\\ +\phi_2(t,\lambda)\left[\Delta f_{1\xi}(t)+r(t)f_{2\xi}(t)\right]\end{array}\right]\Delta t\right\}^2 d\omega_b(\lambda)$
$\leq\frac{1}{\kappa^2}\int_{-\infty}^{\infty}\left\{\int_a^b\left[\begin{array}{c}\phi_1(t,\lambda)\left[-\Delta f_{2\xi}^{\rho}(t)+p(t)f_{1\xi}(t)\right]\\ +\phi_2(t,\lambda)\left[\Delta f_{1\xi}(t)+r(t)f_{2\xi}(t)\right]\end{array}\right]\Delta t\right\}^2 d\omega_b(\lambda)$

$= \frac{1}{\kappa^2}\int_a^\xi \left\{ \left[-\Delta f_{2\xi}^{\rho}(t) + p(t) f_{1\xi}(t) \right]^2 + \left[\Delta f_{1\xi}(t) + r(t) f_{2\xi}(t) \right]^2 \right\} \Delta t.$

From (2.11), we see that

$$\left| \int_a^\xi \left(f_{1\xi}^2(t) + f_{2\xi}^2(t) \right) \Delta t - \int_{-\kappa}^{\kappa} F_\xi^2(\lambda)\, d\omega_b(\lambda) \right| \leq$$

$$\frac{1}{\kappa^2}\int_a^\xi \{ \left[-\Delta f_{2\xi}^{\rho}(t) + p(t) f_{1\xi}(t) \right]^2 + \left[\Delta f_{1\xi}(t) + r(t) f_{2\xi}(t) \right]^2 \} \Delta t. \tag{2.13}$$

By Lemma 2.1, the set $\{\omega_b(\lambda)\}$ is bounded. Using Theorems 2.2 and 2.3, we can find a sequence $\{b_k\}$ such that the function $\omega_{b_k}(\lambda)$ $(\kappa \to \infty)$ converge to a monotone function $\omega(\lambda)$. Passing to the limit with respect to $\{b_k\}$ in (2.13), we get

$$\left| \int_a^\xi \left(f_{1\xi}^2(t) + f_{2\xi}^2(t) \right) \Delta t - \int_{-\kappa}^{\kappa} F_\xi^2(\lambda)\, d\omega(\lambda) \right|$$

$$\leq \frac{1}{\kappa^2}\int_a^\xi \left\{ \left[-\Delta f_{2\xi}^{\rho}(t) + p(t) f_{1\xi}(t) \right]^2 + \left[\Delta f_{1\xi}(t) + r(t) f_{2\xi}(t) \right]^2 \right\} \Delta t.$$

Hence, letting $\kappa \to \infty$, we obtain

$$\int_a^\xi \left(f_{1\xi}^2(t) + f_{2\xi}^2(t) \right) \Delta t = \int_{-\infty}^{\infty} F_\xi^2(\lambda)\, d\omega(\lambda).$$

Now, let f be an arbitrary function on $\mathscr{H}$. It is known that there exists a sequence of function $\left\{ f_\xi(t) \right\}$ satisfying the condition 1-3 and such that

$$\lim_{\xi \to \infty} \int_a^\infty \left\| f(t) - f_\xi(t) \right\|^2 \Delta t = 0.$$

Let

$$F_\xi(\lambda) = \int_a^\infty \left\| f_\xi^T(t)\, \phi(t,\lambda) \right\| \Delta t,$$

where the norm $\|.\|$ is the convenient norm in E. Then, we have

$$\int_a^\infty \left(f_{1\xi}^2(t) + f_{2\xi}^2(t) \right) \Delta t = \int_{-\infty}^{\infty} F_\xi^2(\lambda)\, d\omega(\lambda).$$

Since

$$\int_a^\infty \left\| f_{\xi_1}(t) - f_{\xi_2}(t) \right\|^2 \Delta t \to 0 \text{ as } \xi_1, \xi_2 \to \infty,$$

we have

$$\int_{-\infty}^{\infty} \left(F_{\xi_1}(\lambda) - F_{\xi_2}(\lambda) \right)^2 d\omega(\lambda) = \int_a^\infty \left\| f_{\xi_1}(t) - f_{\xi_2}(t) \right\|^2 \Delta t \to 0$$

as $\xi_1, \xi_2 \to \infty$. Consequently, there is a limit function F which satisfies

$$\int_a^\infty \left(f_1^2(t) + f_2^2(t) \right) \Delta t = \int_{-\infty}^{\infty} F^2(\lambda)\, d\omega(\lambda),$$

by the completeness of the space $L_\omega^2(-\infty,\infty)$.
Our next goal is to show that the function

$$K_\xi(\lambda) = \int_a^\xi f_1(t)\,\phi_1(t,\lambda) + f_2(t)\,\phi_2(t,\lambda)\,\Delta t$$

converges as $\xi \to \infty$ to F in the metric of space $L_\omega^2(-\infty,\infty)$. Let g be another function in $\mathscr{H}$. By a similar arguments, $G(\lambda)$ be defined by g. It is clear that

$$\int_a^\infty \|f(t) - g(t)\|^2 \Delta t = \int_{-\infty}^{\infty} \{F(\lambda) - G(\lambda)\}^2 d\omega(\lambda).$$

Set

$$g(t) = \begin{cases} f(t), & t \in [a,\xi] \\ 0, & t \in (\xi,\infty). \end{cases}$$

Then we have

$$\int_{-\infty}^{\infty} \left\{ F(\lambda) - K_\xi(\lambda) \right\}^2 d\omega(\lambda) = \int_\xi^\infty \left(f_1^2(t) + f_2^2(t) \right) \Delta t \to 0 \ (\xi \to \infty),$$

which proves that K_ξ converges to F in $L_\omega^2(-\infty,\infty)$ as $\xi \to \infty$. This proves (i).

Now, we will prove (ii). Suppose that the functions $f(.)=\begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix}$, $g(.)=\begin{pmatrix} g_1(.) \\ g_2(.) \end{pmatrix} \in \mathscr{H}$, and $F(\lambda)$ and $G(\lambda)$ are their Fourier transforms. Then $F \mp G$ are transforms of $f \mp g$. Consequently, by (2.10), we have

$$\int_a^{\infty} \left([f_1(t)+g_1(t)]^2 + [f_2(t)+g_2(t)]^2 \right) \Delta t = \int_{-\infty}^{\infty} (F(\lambda)+G(\lambda))^2 \, d\omega(\lambda),$$
$$\int_a^{\infty} \left([f_1(t)-g_1(t)]^2 + [f_2(t)-g_2(t)]^2 \right) \Delta t = \int_{-\infty}^{\infty} (F(\lambda)-G(\lambda))^2 \, d\omega(\lambda).$$

Subtracting the second relation from the first, we get

$$\int_a^{\infty} [f_1(t) g_1(t) + f_2(t) g_2(t)] \Delta t = \int_{-\infty}^{\infty} F(\lambda) G(\lambda) \, d\omega(\lambda) \tag{2.14}$$

which is called the generalized Parseval equality.
Set

$$f_\tau(t) = \begin{pmatrix} \int_{-\tau}^{\tau} F(\lambda)\phi_1(t,\lambda)\, d\omega(\lambda) \\ \int_{-\tau}^{\tau} F(\lambda)\phi_2(t,\lambda)\, d\omega(\lambda) \end{pmatrix}, \tau > 0,$$

where F is the function defined in (2.9). Let $g(.)=\begin{pmatrix} g_1(.) \\ g_2(.) \end{pmatrix}$ be a vector-function which equals zero outside the finite interval $[a,\mu]_{\mathbb{T}}$, $\mu > a$. Thus, we obtain

$$\begin{aligned}
(f_\tau, g) &= \int_a^{\mu} \left\{ \int_{-\tau}^{\tau} F(\lambda)\phi_1(t,\lambda)\, d\omega(\lambda) \right\} g_1(t)\Delta t \\
&\quad + \int_0^{\mu} \left\{ \int_{-\tau}^{\tau} F(\lambda)\phi_2(t,\lambda)\, d\omega(\lambda) \right\} g_2(t)\Delta t \\
&= \int_{-\tau}^{\tau} F(\lambda) \left\{ \int_0^{\mu} \phi_1(t,\lambda) g_1(t)\Delta t \right\} d\omega(\lambda) \\
&\quad + \int_{-\tau}^{\tau} F(\lambda) \left\{ \int_0^{\mu} \phi_2(t,\lambda) g_2(t)\Delta t \right\} d\omega(\lambda) \\
&= \int_{-\tau}^{\tau} F(\lambda) G(\lambda)\, d\omega(\lambda).
\end{aligned} \tag{2.15}$$

From (2.14), we get

$$(f,g) = \int_{-\infty}^{\infty} F(\lambda) G(\lambda)\, d\omega(\lambda). \tag{2.16}$$

Subtracting (2.15) and (2.16), we have

$$(f_\tau - f, g) = \int_{|\lambda|>\tau} F(\lambda) G(\lambda)\, d\omega(\lambda).$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|(f_\tau - f, g)|^2 &\leq \int_{|\lambda|>\tau} F^2(\lambda)\, d\omega(\lambda) \int_{|\lambda|>\tau} G^2(\lambda)\, d\omega(\lambda) \\
&\leq \int_{|\lambda|>\tau} F^2(\lambda)\, d\omega(\lambda) \int_{-\infty}^{\infty} G^2(\lambda)\, d\omega(\lambda).
\end{aligned}$$

Apply this inequality to the function

$$g(t) = \begin{cases} f_\tau(t) - f(t), & t \in [0,\mu]_{\mathbb{T}} \\ 0, & t \in (\mu,\infty)_{\mathbb{T}}, \end{cases}$$

we get

$$\|f_\tau - f\|^2 \leq \int_{|\lambda|>\tau} F^2(\lambda)\, d\omega(\lambda).$$

Letting $\tau \to \infty$ yields the desired result. □

3. Conclusion

In this paper, we have considered one dimensional singular Dirac system on time scales. In this context, we prove the existence of a spectral function for one dimensional singular Dirac system on time scales. Finally, we establish a Parseval equality and expansion formula in eigenfunctions by terms of the spectral function.

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