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Close-to-Convex Functions By Means Of Bounded Boundary Rotation

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Abstract

In the present paper, we study on the class $\mathscr{CC}_k(A,B)$ of the close-to-convex functions with bounded boundary rotation. We investigate distortion theorem, growth theorem and coefficient inequality for the class $\mathscr{CC}_k(A,B)$.

Keywords: *Bounded boundary rotation, Coefficient inequality, Distortion theorem, Growth theorem*
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1. Introduction

Let $\mathscr{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and let g be an element of $\mathscr{A}$. If g satisfies the condition

$$Re\left(1 + z\frac{g''(z)}{g'(z)}\right) > 0,$$

then g is called convex function. The class of such functions is denoted by $\mathscr{C}$. For more details of convex functions one may refer to [1]. Let Ω be the family of functions ϕ regular in $\mathbb{D}$ and satisfying the condition $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. For arbitrary fixed number A, B, $-1 \le B < A \le 1$ denote by $\mathscr{P}(A,B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 \ldots$, analytic in $\mathbb{D}$ such that p in $\mathscr{P}(A,B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

for some function $\phi \in \Omega$ and every $z \in \mathbb{D}$. A function p of the form $p(z) = 1 + p_1 z + p_2 z^2 \ldots$, analytic in $\mathbb{D}$ with $p(0) = 1$ is said to be in the class of $\mathscr{P}_k(A,B)$, $k \ge 2, -1 \le B < A \le 1$ if and only if there exists $p_1^{(1)}, p_2^{(2)} \in \mathscr{P}(A,B)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z).$$

We now give the following definition.

Definition 1.1. Let f of the form (1.1) be an element of $\mathscr{A}$. Then $f \in \mathscr{CC}_k(A,B)$, if there exists a function $g \in \mathscr{C}$ such that

$$\frac{f'(z)}{g'(z)} = p(z), \quad p \in \mathscr{P}_k(A,B), \tag{1.2}$$

with $k \ge 2, -1 \le B < A \le 1$, then f is called close-to-convex function with bounded boundary rotation denoted by $\mathscr{CC}_k(A,B)$.

We investigate distortion theorem, growth theorem and coefficient inequality for the class $\mathscr{CC}_k(A,B)$. Details of bounded boundary rotation can be found in [3],[4].

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2. Main Results

Let p be an element of $\mathscr{P}(A,B)$ and $|z|=r<1$, then we have

$$\begin{aligned} &\frac{1-Ar}{1-Br} \le Rep(z) \le |p(z)| \le \frac{1+Ar}{1+Br}, \quad B \neq 0, \\ &1-Ar \le Rep(z) \le |p(z)| \le 1+Ar, \qquad B=0. \end{aligned} \tag{2.1}$$

After simple calculations in (2.1), we get

$$\begin{aligned} &\frac{1-\frac{k}{2}(A-B)r-ABr^2}{1-B^2r^2} \le Rep(z) \le \frac{1+\frac{k}{2}(A-B)r-ABr^2}{1-B^2r^2}, \quad B \neq 0, \\ &1-\frac{k}{2}Ar \le Rep(z) \le 1+\frac{k}{2}Ar, \qquad B=0. \end{aligned} \tag{2.2}$$

(2.2) shows that the set of variability of $p \in \mathscr{P}_k(A,B)$ is the closed disc

$$\begin{aligned} &\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{\frac{k}{2}(A-B)r}{1-B^2r^2}, \quad B \neq 0, \\ &|p(z)-1| \le \frac{k}{2}Ar, \qquad B=0. \end{aligned} \tag{2.3}$$

On the other hand from definition of $\mathscr{CC}_k(A,B)$, we can write

$$\begin{aligned} &\left| \frac{f'(z)}{g'(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{\frac{k}{2}(A-B)r}{1-B^2r^2}, \quad B \neq 0, \\ &\left| \frac{f'(z)}{g'(z)} - 1 \right| \le \frac{k}{2}Ar, \qquad B=0. \end{aligned} \tag{2.4}$$

Since g is convex, using distortion bound of convex functions and above inequalities, we obtain

$$\begin{aligned} &\frac{1-\frac{k}{2}(A-B)r-ABr^2}{(1+r)^2(1-B^2r^2)} \le |f'(z)| \le \frac{1+\frac{k}{2}(A-B)r-ABr^2}{(1-r)^2(1-B^2r^2)}, \quad B \neq 0, \\ &\frac{1-\frac{k}{2}Ar}{(1+r)^2} \le |f'(z)| \le \frac{1+\frac{k}{2}Ar}{(1-r)^2}, \qquad B=0. \end{aligned} \tag{2.5}$$

Sharpness follows from the function f and g (convex) where

$$\begin{aligned} &\frac{f'(z)}{g'(z)} = (\frac{k}{4}+\frac{1}{2})\frac{1-Az}{1-Bz} - (\frac{k}{4}-\frac{1}{2})\frac{1+Az}{1+Bz}, \qquad B \neq 0, \\ &\frac{f'(z)}{g'(z)} = (\frac{k}{4}+\frac{1}{2})(1-Az) - (\frac{k}{4}-\frac{1}{2})(1+Az), \quad B=0. \end{aligned} \tag{2.6}$$

Since $p(z) = \frac{f'(z)}{g'(z)}, g \in \mathscr{C}$,

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{\frac{k}{2}(A-B)r}{1-B^2r^2}$$

then the set of variability of $\frac{f'}{g'}$ is the closed disc

$$\left| \frac{f'(z)}{g'(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{\frac{k}{2}(A-B)r}{1-B^2r^2}.$$

Therefore this set could be written in the following form

$$w(\mathbb{D}_r) = \left\{ \frac{f'(z)}{g'(z)} : \left| \frac{f'(z)}{g'(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{\frac{k}{2}(A-B)r}{1-B^2r^2}, \quad 0<r<1 \right\}. \tag{2.7}$$

On the other hand, we define the function ϕ by

$$\begin{aligned} &\frac{f(z)}{g(z)} = (\frac{k}{4}+\frac{1}{2})\frac{1-A\phi(z)}{1-B\phi(z)} - (\frac{k}{4}-\frac{1}{2})\frac{1+A\phi(z)}{1+B\phi(z)} = \frac{1-\frac{k}{2}(A-B)\phi(z)-AB(\phi(z))^2}{1-B^2(\phi(z))^2}, \quad B \neq 0, \\ &\frac{f(z)}{g(z)} = (\frac{k}{4}+\frac{1}{2})(1-A\phi(z)) - (\frac{k}{4}-\frac{1}{2})(1+A\phi(z)) = 1-\frac{k}{2}A\phi(z), \qquad B=0. \end{aligned} \tag{2.8}$$

Note that ϕ is well-defined analytic function and $\phi(0)=0$. Now, we want to show that $|\phi(z)|<1$ for every $z\in\mathbb{D}$. Taking derivative and simplifying both sides of (2.8), we get

$$\frac{f'(z)}{g'(z)}=\frac{1-\frac{k}{2}(A-B)\phi(z)-AB(\phi(z))^2}{1-B^2(\phi(z))^2}$$
$$+\frac{z\phi'(z)\Big(-\frac{k}{2}(A-B)(1+B^2(\phi(z))^2)-2B(A-B)\phi(z)\Big)}{\big(1-B^2(\phi(z))^2\big)^2}\frac{g(z)}{zg'(z)},\quad B\neq 0,$$

$$\frac{f'(z)}{g'(z)}=1-\frac{k}{2}A\phi(z)-\frac{k}{2}Az\phi'(z)\frac{g(z)}{zg'(z)},\quad B=0.$$

Suppose that there exists a $z_0\in\mathbb{D}_r$ such that $|\phi(z_0)|=1$, then by Jack's Lemma [2], we obtain

$$\frac{f'(z_0)}{g'(z_0)}=\frac{1-\frac{k}{2}(A-B)\phi(z_0)-AB(\phi(z_0))^2}{1-B^2(\phi(z_0))^2}$$
$$+\frac{m\phi(z_0)\Big(-\frac{k}{2}(A-B)(1+B^2(\phi(z_0))^2)-2B(A-B)\phi(z_0)\Big)}{\big(1-B^2(\phi(z_0))^2\big)^2}\frac{g(z_0)}{z_0g'(z_0)}\notin w(\mathbb{D}_r), B\neq 0,$$

$$\frac{f'(z_0)}{g'(z_0)}=1-\frac{k}{2}A\phi(z_0)-\frac{k}{2}Am\phi(z_0)\frac{g(z_0)}{z_0g'(z_0)}\notin w(\mathbb{D}_r),\quad B=0,$$

where $m\geq 1$ is a real number and $\frac{g(z_0)}{z_0g'(z_0)}=(1+\phi(z_0))$. But this contradicts with (2.7), therefore we have $|\phi(z)|<1$ for all $z\in\mathbb{D}$, and obtain the following subordination

$$\frac{f(z)}{g(z)}\prec(\frac{k}{4}+\frac{1}{2})\frac{1-Az}{1-Bz}-(\frac{k}{4}-\frac{1}{2})\frac{1+Az}{1+Bz},\quad B\neq 0,$$
$$\frac{f(z)}{g(z)}\prec(\frac{k}{4}+\frac{1}{2})(1-Az)-(\frac{k}{4}-\frac{1}{2})(1+Az),\quad B=0.$$

Using the subordination principle, we obtain

$$\begin{aligned}&|g(z)|\frac{1-\frac{k}{2}(A-B)r-ABr^2}{1-B^2r^2}\leq|f(z)|\leq|g(z)|\frac{1+\frac{k}{2}(A-B)r-ABr^2}{1-B^2r^2},\quad B\neq 0,\\ &|g(z)|(1-\tfrac{k}{2}Ar)\leq|f(z)|\leq|g(z)|(1+\tfrac{k}{2}Ar),\quad B=0.\end{aligned}\tag{2.9}$$

Using the growth theorem for convex functions, the inequalities in (2.9) can be written in the following form

$$\begin{aligned}&rF(A,B,k,-r)\leq|f(z)|\leq rF(A,B,k,r),\\ &rF(A,k,-r)\leq|f(z)|\leq rF(A,k,r),\end{aligned}\tag{2.10}$$

where

$$F(A,B,k,r)=\frac{1+\frac{k}{2}(A-B)r-ABr^2}{(1-r)(1-B^2r^2)},$$
$$F(A,k,r)=\frac{1+\frac{k}{2}Ar}{(1-r)}.$$

These inequalities are sharp because the extremal function is given in (2.6).

The following lemma helps us to prove the next theorem.

Lemma 2.1. *Let $p(z)=1+p_1z+p_2z^2+...$ be an element of $\mathscr{P}_k(A,B)$, then*

$$|p_n|\leq\frac{k}{2}(A-B).$$

for all $n\geq 1,k\geq 2,-1\leq B<A\leq 1$.This result is sharp for the functions

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1^{(1)}(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2^{(1)}(z),$$

where $p_1^{(1)},p_2^{(2)}\in\mathscr{P}(A,B)$.

Proof. Let $p_1^{(1)} = 1 + a_1z + a_2z^2 + ...$ and $p_2^{(2)} = 1 + b_1z + b_2z^2 + ...$. Since $p \in \mathscr{P}_k(A,B)$, then we have

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z)$$
$$= \left(\frac{k}{4} + \frac{1}{2}\right) (1 + a_1z + a_2z^2 + ...) - \left(\frac{k}{4} - \frac{1}{2}\right) (1 + b_1z + b_2z^2 + ...).$$

Then, for n th term, we have

$$p_n = \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n.$$

Since $p_1^{(1)}, p_2^{(2)} \in \mathscr{P}(A,B)$, then $|a_n| \leq (A-B), |b_n| \leq (A-B)$ for all $n \geq 1$, and

$$|p_n| = \left| \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n \right|$$
$$\leq \left(\frac{k}{4} + \frac{1}{2}\right) |a_n| + \left(\frac{k}{4} - \frac{1}{2}\right) |b_n|$$
$$\leq \left(\frac{k}{4} + \frac{1}{2}\right) (A-B) + \left(\frac{k}{4} - \frac{1}{2}\right) (A-B).$$

This shows that, $|p_n| \leq \frac{k}{2}(A-B)$. □

Theorem 2.2. *If f of the form (1.1) is an element of $\mathscr{CC}_k(A,B)$, then*

$$|a_n| \leq 1 + \frac{k(A-B)(n-1)}{4}. \tag{2.11}$$

This result is sharp for each $n \geq 2$.

Proof. Using Definition 1.1 and subordination principle, we write

$$\frac{f'(z)}{g'(z)} = p(z)$$

for some $g \in \mathscr{C}$, where $g(z) = z + \sum_{n=2}^{\infty} b_nz^n$, $z \in \mathbb{D}$. Since $p(0) = 1$, it shows that $p \in \mathscr{P}_k(A,B)$, where $p(z) = 1 + \sum_{n=1}^{\infty} p_nz^n$. Therefore, above equality is equivalent to

$$\left(1 + \sum_{n=2}^{\infty} na_nz^{n-1}\right) = \left(1 + \sum_{n=2}^{\infty} nb_nz^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_nz^n\right).$$

This equation yields,

$$1 + 2a_2z + 3a_3z^2 + ... = 1 + (2b_2 + p_1)z + (3b_3 + 2b_2p_1 + p_2)z^2 + ... \tag{2.12}$$

Comparing the coefficients of z^{n-1} on both sides, we obtain

$$na_n = nb_n + (n-1)b_{n-1}p_1 + (n-2)b_{n-2}p_2 + ... + 2b_2p_{n-2} + p_{n-1}.$$

Using Lemma 2.1, we get

$$n|a_n| \leq n|b_n| + \frac{k}{2}(A-B)\Big[(n-1)|b_{n-1}| + ... + 2|b_2| + 1\Big].$$

Since g is convex, then $|b_n| \leq 1$, it follows from that

$$|a_n| \leq 1 + \frac{k(A-B)(n-1)}{4}.$$

This completes our proof. □

Remark 2.3. Letting $k = 2, A = 1, B = -1$, Theorem 2.2 gives the well-known coefficient inequality for close-to-convex functions.

We can conclude that an analytic functions $p \in \mathscr{P}_k(A,B)$, $k \geq 2$ if and only if there exists $p_1, p_2 \in \mathscr{P}(A,B)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

then new results are obtained for the class $\mathscr{CC}_k(A,B)$ by means of classes of bounded boundary rotation and bounded radius rotation. Giving the special values to A, B and k we obtain the growth theorem, distortion theorem and coefficient inequality for the other subclasses.

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