Konuralp Journal of Mathematics, 7 (1) (2019) 199-202

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

An Application of an Operator Associated with Generalized Mittag-Leffler Function

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Abstract

The main object of this paper is to give an application of an operator associated with generalized Mittag-Leffler function in the unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$ to the differential inequalities.

Keywords: *Analytic functions; Mittag-Leffler function.*
2010 Mathematics Subject Classification: *30C45, 30C80, 33E12.*

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $E_\alpha(z)$ be the Mittag-Leffler [10] defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \mathrm{Re}(\alpha) > 0\,). \tag{1.2}$$

A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [13] and defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0\,). \tag{1.3}$$

Moreover, Srivastava and Tomovski [12] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z)$ as

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(\alpha n + \beta) n!},$$

$$(\alpha, \beta, \gamma \in \mathbb{C}; \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k) - 1\}; \mathrm{Re}(k) > 0),$$

where $(\gamma)_n$ is Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) is given in term of the Gamma functions can be written as

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1, & \text{if } n = 0; \\ \gamma(\gamma+1)\ldots(\gamma+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \tag{1.4}$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2, 3, 4, 5, 6, 7, 8, 10, 11, 12].

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In [1], Attiya defined the operator $\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f):\mathscr{A}\to\mathscr{A}$ by

$$\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z)=Q^{\gamma}_{\alpha,\beta,k}(z)*f(z), \qquad (z\in\mathscr{U}),$$

where

$$Q^{\gamma}_{\alpha,\beta,k}(z)=\frac{\Gamma(\alpha+\beta)}{(\gamma)_k}\left(E^{\gamma,k}_{\alpha,\beta}(z)-\frac{1}{\Gamma(\beta)}\right), \qquad (z\in\mathscr{U}),$$

$$(\alpha,\beta,\gamma\in\mathbb{C};\mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\};\mathrm{Re}(k)>0$$
$$\text{and } \mathrm{Re}(\alpha)=0 \text{ when } \mathrm{Re}(k)=1 \text{ with } \beta\neq 0)$$

and the symbol $(*)$ denotes the Hadamard product (or convolution).
We note that,

$$\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!}a_nz^n. \tag{1.5}$$

It can be easily verified from (1.5) that

$$z\left(\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)'=\left(\frac{\gamma+k}{k}\right)(\mathscr{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z))-\frac{\gamma}{k}(\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z)). \tag{1.6}$$

We note that

1. $\mathscr{H}^{1}_{0,\beta,1}(f)(z)=f(z)$.
2. $\mathscr{H}^{2}_{0,\beta,1}(f)(z)=\frac{1}{2}\left(f(z)+zf'(z)\right)$.
3. $\mathscr{H}^{0}_{0,\beta,1}(f)(z)=\int_0^z\frac{1}{t}f(t)dt$.
4. $\mathscr{H}^{1}_{1,0,1}(\frac{z}{1-z})=ze^z$.
5. $\mathscr{H}^{1}_{1,1,1}(\frac{z}{1-z})=e^z-1$.
6. $\mathscr{H}^{1}_{2,1,1}(\frac{z}{1-z})=\cosh(\sqrt{z})-2$.

The object of the present paper is to give an application of the above differential operator $\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z)$ to Miller and Mocanu's result (Lemma 2.1 below).
For our purpose, we introduce

Definition 1.1. *Let H be the set of complex valued functions $h(r,s,t):\mathbb{C}^3\to\mathbb{C}$ such that*
(i) $h(r,s,t)$ is continuous in a domain $\mathbb{D}\subset\mathbb{C}^3$;
(ii) $(0,0,0)\in\mathbb{D}$ and $|h(0,0,0)|<1$;
(iii) $\left|h\left(e^{i\theta},\left(\frac{\gamma+k\delta}{\gamma+k}\right)e^{i\theta},\frac{k^2\beta+(k^2+2k\gamma+k)\delta+\gamma^2+\gamma)e^{i\theta}}{(\gamma+k)(\gamma+k+1)}\right)\right|\geq 1$ whenever

$$\left(e^{i\theta},\left(\frac{\gamma+k\delta}{\gamma+k}\right)e^{i\theta},\frac{1}{(\gamma+k)^2}[k^2\beta+(k(k+2\gamma)+\gamma^2)e^{i\theta}]\right)\in\mathbb{D}$$

with $Re\{\beta e^{-i\theta}\}$ for real θ, $\delta\geq 1$.

2. Main Result

In order to prove our main result, we recall the following lemma due to Miller and Mocanu [9].

Lemma 2.1. *Let a function $w(z)\in\mathscr{A}$ with $w(z)\neq 0$ in $\mathscr{U}$. If $z_0=r_0e^{i\theta}$ $(0<r_0<1)$ and $|w(z_0)|=\max_{|z|\leq r_0}|w(z)|$. Then*

$$zw'(z_0)=\delta w(z_0) \tag{2.1}$$

and

$$Re\left\{1+\frac{z_0w''(z_0)}{w'(z_0)}\right\}\geq\delta, \tag{2.2}$$

where δ is a real number and $\delta\geq 1$.

Applying Lemma 2.1, we derive the following result.

Theorem 2.2. *Let $h(r,s,t)\in H$ and let $f(z)\in\mathscr{A}$ satisfy*

$$\left(\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z),\mathscr{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z),\mathscr{H}^{\gamma+2}_{\alpha,\beta,k}(f)(z)\right)\in\mathbb{D}\subset\mathbb{C}^3 \tag{2.3}$$

and

$$\left|h\left(\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z),\mathscr{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z),\mathscr{H}^{\gamma+2}_{\alpha,\beta,k}(f)(z)\right)\right|<1 \tag{2.4}$$

for all $z\in\mathscr{U}$. Then we have

$$\left|\mathscr{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right|<1 \qquad (z\in\mathscr{U}).$$

Proof. Define the function $w(z)$ by

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z) = w(z). \tag{2.5}$$

Then it follows that $w(z) \in \mathcal{A}$ and $w(z) \neq 0$ $(z \in \mathcal{U})$. Using the identity (1.6), we have

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma+1}(f)(z) = \frac{1}{\gamma+k}[kzw'(z) + \gamma w(z)]$$

and

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma+2}(f)(z) = \frac{k^2z^2w''(z) + (k^2+2k\gamma+k)zw'(z) + \gamma(\gamma+1)w(z)}{(\gamma+k)(\gamma+k+1)}.$$

Suppose that $z_0 = r_0e^{i\theta}$ $(0 < r_0 < 1)$ and

$$\max_{|z|\leq r_0} |w(z)| = |w(z_0)| = 1.$$

Letting $w(z_0) = e^{i\theta}$ and using (2.1) of Lemma 2.1, we have

$$\begin{aligned}
\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z_0) &= w(z_0) = e^{i\theta}, \\
\mathcal{H}_{\alpha,\beta,k}^{\gamma+2}(f)(z_0) &= \frac{1}{\gamma+k}[kz_0w'(z_0) + \gamma w(z_0)] \\
&= \left(\frac{\gamma+k\delta}{\gamma+k}\right)e^{i\theta},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}_{\alpha,\beta,k}^{\gamma+2}(f)(z_0) &= \frac{k^2z_0^2w''(z_0) + (k^2+2k\gamma+k)z_0w'(z_0) + \gamma(\gamma+1)w(z_0)}{(\gamma+k)(\gamma+k+1)} \\
&= \frac{1}{(\gamma+k)(\gamma+k+1)}[k^2\beta + (k^2+2k\gamma+k)\delta + \gamma^2 + \gamma)e^{i\theta}].
\end{aligned}$$

where

$$\beta = z_0^2w''(z_0) \quad \text{and} \quad \delta \geq 1\,.$$

Further, an application of (2.2) in Lemma 2.1 gives

$$\text{Re}\left\{\frac{z_0w''(z_0)}{w'(z_0)}\right\} = \text{Re}\left\{\frac{z_0^2w''(z_0)}{\delta e^{i\theta}}\right\} \geq \delta - 1,$$

or

$$\text{Re}\{\beta e^{-i\theta}\} \geq \delta(\delta-1).$$

Since $h(r,s,t) \in H$, we have

$$\begin{aligned}
&\left|h\left(\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z), \mathcal{H}_{\alpha,\beta,k}^{\gamma+2}(f)(z), \mathcal{H}_{\alpha,\beta,k}^{\gamma+2}(f)(z)\right)\right| \\
= &\left|h\left(e^{i\theta}, \left(\frac{\gamma+k\delta}{\gamma+k}\right)e^{i\theta}, \frac{k^2\beta + (k^2+2k\gamma+k)\delta + \gamma^2 + \gamma)e^{i\theta}}{(\gamma+k)(\gamma+k+1)}\right)\right| > 1
\end{aligned}$$

which contradicts the condition (2.4) of Theorem 2.2. Therefore, we conclude that

$$\left|D_{m,\lambda}^{\zeta}f(z)\right| < 1 \qquad (z \in \mathcal{U}).$$

The proof is complete. □

Corollary 2.3. *Let $h(r,s,t) \in H$ and let $f(z) \in \mathcal{A}$ satisfy*

$$\left(\int_0^z \frac{1}{t}f(t)dt, f(z), \frac{1}{2}\left(f(z) + zf'(z)\right)\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.6}$$

and

$$\left|h\left(\int_0^z \frac{1}{t}f(t)dt, f(z), \frac{1}{2}\left(f(z) + zf'(z)\right)\right)\right| < 1 \tag{2.7}$$

for all $z \in \mathcal{U}$. Then we have

$$\left|\int_0^z \frac{1}{t}f(t)dt\right| < 1 \qquad (z \in \mathcal{U}).$$

Acknowledgement

The author would like to thank the referee for his helpful comments and suggestions.

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