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Oscillation Criteria for Higher Order Fractional Differential Equations with Mixed Nonlinearities

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Abstract

In this article, we consider higher order fractional nonlinear differential equation of type

$$
\begin{aligned}
&{}_aD_t^q x(t) - p(t)x(t) + \sum_{i=1}^{m} q_i(t)|x(t)|^{\lambda_i - 1}x(t) = v(t)\\
&\lim_{t\to a^+} J_a^{n-q}x(t) = a_n\\
&{}_aD_t^{q-k}x(a) = a_k, \qquad k = 1,\dots,n-1
\end{aligned}
$$

where ${}_aD_t^q$is Riemann-Liouville fractional differential operator of order q, $m-1 < q \leq m, m \geq 1$ is an integer. We obtain some oscillation criteria for this equation.

Keywords: *Fractional differential, Oscillation*
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1. Introduction

In this paper, we consider the oscillation theory for a fractional differential equation with mixed nonlinearities of the type

$$
\begin{aligned}
&{}_aD_t^q x(t) - p(t)x(t) + \sum_{i=1}^{m} q_i(t)|x(t)|^{\lambda_i - 1}x(t) = v(t)\\
&\lim_{t\to a^+} J_a^{n-q}x(t) = a_n\\
&{}_aD_t^{q-k}x(a) = a_k, \qquad k = 1,\dots,n-1
\end{aligned}
\tag{1.1}
$$

where $\{p(t)\}$,$\{v(t)\}$ and $\{q_i(t)\}$ $(1 \leqslant i \leqslant m)$ are continuous functions on $[a,+\infty)$ and λ_i $(1 \leqslant i \leqslant m)$ are ratios of odd positive integers with $\lambda_1 > \cdots > \lambda_l > 1 > \lambda_{l+1} > \cdots > \lambda_m$.
By a solution of equation (1.1) we mean a function $x(t)$ which is defined for $t \geqslant a$ and satisfies equation (1.1). Such a solution is said to be oscillatory if it has arbitrarily large zeros on $[a,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.
By ${}_aD_t^q$ we denote the Riemann-Liouville differential operator of order q with $0 < q \leqslant 1$. For $p \geqslant 0$, the operator J_a^q defined by

$$
J_a^q x(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{p-1}x(s)ds, \qquad\qquad J_a^0 x = x
$$

is called the Riemann-Liouville fractional integral operator. The Riemann-Liouville differential operator ${}_aD_t^q$ of order q for $0 < q \leqslant 1$ is defined by ${}_aD_t^q x(t) = \frac{d}{dt}J_a^{1-q}x(t)$ and, more generally, if $n \geqslant 1$ is an integer and $n-1 < q \leqslant n$, then

$$
{}_aD_t^q x(t) = \frac{d^n}{dt^n}J_a^{n-q}x(t)
$$

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In [4,Lemma 5.3], under much weaker assumptions on $p(t)$,$v(t)$ and $q_i(t)$, the initial value problem (1.1) is equivalent to be Volterra fractional integral equation

$$x(t)=\sum_{k=1}^{n}\frac{a_k(t-a)^{q-k}}{\Gamma(q-k+1)}+\frac{1}{\Gamma(q)}\int_a^t(t-s)^{q-1}\left[v(s)+p(s)x(s)-\sum_{i=1}^{m}q_i(s)|x(s)|^{\lambda_i-1}x(s)\right]ds \tag{1.2}$$

Therefore, a function $x(t)$ is a solution of (1.2) if and only if it is a solution of fractional differential equation (1.1).

2. Preliminaries

Definition 2.1. *The Riemann-Liouville fractional derivative of order $q>0$ of a function $x:[a,\infty)\to\mathbb{R}$ is defined by*

$$(I_a^q x)(t):=\frac{1}{\Gamma(q)}\int_a^t(t-s)^{q-1}x(s)ds \tag{2.1}$$

provided the right-hand side is pointwise defined on $[a,\infty)$, where Γ is the gamma function. Furthermore, $I_a^0 x:=x$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $q>0$ of a function $x:[a,\infty)\to\mathbb{R}$ is defined by*

$$({}_aD_t^q x)(t):=\frac{d^m}{dt^m}(I_a^{m-q}x)(t) \tag{2.2}$$

provided the right-hand side is pointwise defined on $[a,\infty)$,where $n-1<q\leqslant n$ and $n\geqslant 1$ is an integer. Furthermore, we set $D_a^0 x:=x$.

Lemma 2.3. *Suppose that X,Y and U,V are nonnegative, then*

$$\lambda XY^{\lambda-1}-X^{\lambda}\leqslant(\lambda-1)Y^{\lambda},\quad \lambda>1 \tag{2.3}$$

$$\mu UV^{\mu-1}-U^{\mu}\geqslant(\mu-1)V^{\mu},\ \ 0<\mu<1 \tag{2.4}$$

where each equality holds if and only if $X=Y$ or $U=V$

Lemma 2.4. *Let $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ be an m-tuple satisfying $\alpha_1>\alpha_2>\ldots>\alpha_l>1>\alpha_{l+1}>\ldots>\alpha_m>0$. Then there exists an m-tuple $(\eta_1,\eta_2,\ldots,\eta_m)$ satisfying*

$$\sum_{i=1}^{l}\alpha_i\eta_i=\sum_{i=l+1}^{m}\alpha_i\eta_i$$

with $\sum_{i=1}^{m}\eta_i=1$ and $0<\eta_i<1$ for $i=1,2,\ldots,m$.

3. Main Results

Theorem 3.1. *Assume*

$$p(t)>0,\ \ q_i(t)\begin{cases}\geqslant 0 & for\ \ 1\leqslant i\leqslant l;\\ \leqslant 0 & for\ \ l+1\leqslant i\leqslant m.\end{cases} \tag{3.1}$$

If for some constant $K>0$,

$$\liminf_{t\to\infty} t^{1-q}\int_a^t(t-s)^{q-1}\left(v(s)+K\sum_{i=1}^{m}p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}\right)ds=-\infty \tag{3.2}$$

and

$$\limsup_{t\to\infty} t^{1-q}\int_a^t(t-s)^{q-1}\left(v(s)+K\sum_{i=1}^{m}p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}\right)ds=\infty \tag{3.3}$$

then every solution of (1.1) is oscillatory.

Proof. Suppose to the contrary that there exist a nonoscillatory solution $x(t)$ of equation (1.1). Without loss of generality, we may suppose that $x(t)>0$ for $t\geqslant T$. It follows from equation (1.2) that

$$\begin{aligned}
x(t) &\leqslant \sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}+\frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}|F(s)|ds\\
&=\sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}+\frac{1}{\Gamma(q)}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+\frac{1}{\Gamma(q)}\int_{T_1}^t (t-s)^{q-1}|F(s)|ds\\
&=\sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}+\frac{1}{\Gamma(q)}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+\frac{1}{\Gamma(q)}\int_{T_1}^t (t-s)^{q-1}\nu(s)ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{T_1}^t (t-s)^{q-1}\left(p(s)x(s)-\sum_{i=1}^{m}q_i(s)x^{\lambda_i}(s)\right)ds\\
&=\sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}+\frac{1}{\Gamma(q)}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+\frac{1}{\Gamma(q)}\int_{T_1}^t (t-s)^{q-1}\nu(s)ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{T_1}^t (t-s)^{q-1}\left[\sum_{i=1}^{l}(\lambda_i p(s)x(s)-q_i(s)x^{\lambda_i}(s))\right]ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{T_1}^t (t-s)^{q-1}\left[\sum_{i=l+1}^{m}(-Ap(s)x(s)+|q_i(s)|x^{\lambda_i}(s))\right]ds \qquad (3.4)\\
&\qquad (3.5)
\end{aligned}$$

where $F(s)=\nu(s)+p(s)x(s)-\sum_{i=1}^{m}q_i(s)x^{\lambda_i}(s)$ and $A=(\sum_{i=1}^{l}\lambda_i-1)/(m-l)>0$. For $t\geqslant T$, set

$$X_i=q_i^{\frac{1}{\lambda_i}}(s)x(s) \quad\text{and}\quad Y_i=\left(p(s)q_i^{-\frac{1}{\lambda_i}}(s)\right)^{\frac{1}{\lambda_i-1}}(s),\quad 1\leqslant i\leqslant l,$$

$$U_i=|q_i(s)|^{\frac{1}{\lambda_i}}x(s) \quad\text{and}\quad V_i=\left(\frac{A}{\lambda_i}p(s)|q_i(s)|^{-\frac{1}{\lambda_i}}\right)^{\frac{1}{\lambda_i-1}}(s),\quad l+1\leqslant i\leqslant m.$$

For $t\geqslant T$, multiplying the inequality (3.4) by $\Gamma(q)t^{1-q}$ and using (2.3) and (2.4) we find that

$$\begin{aligned}
\Gamma(q)t^{1-q}x(t)&\leqslant \sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}\Gamma(q)t^{1-q}+t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+t^{1-q}\int_{T_1}^t (t-s)^{q-1}\nu(s)ds\\
&\quad+t^{1-q}\int_{T_1}^t (t-s)^{q-1}\sum_{i=1}^{l}(\lambda_i-1)p^{\frac{\lambda_i}{\lambda_i-1}}(s)q_i^{\frac{1}{1-\lambda_i}}(s)ds\\
&\quad+t^{1-q}\int_{T_1}^t (t-s)^{q-1}\sum_{i=l+1}^{m}(1-\lambda_i)\left(\frac{\lambda_i}{A}\right)^{\frac{\lambda_i}{1-\lambda_i}}p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}ds\\
&\leqslant \sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}\Gamma(q)t^{1-q}+t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+t^{1-q}\int_{T_1}^t (t-s)^{q-1}\nu(s)ds\\
&\quad+t^{1-q}\int_{T_1}^t (t-s)^{q-1}K\sum_{i=1}^{m}p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}ds,\quad t\geqslant T_1, \qquad (3.6)
\end{aligned}$$

where $K=\max\{\lambda_1-1,\max_{l+1\leqslant i\leqslant m}(1-\lambda_i)(\frac{\lambda_i}{A})^{\frac{\lambda_i}{1-\lambda_i}}\}$. Take $T_2>T_1$. Next, we consider the cases $0<q\leqslant 1$ and $q>1$.
Case 1. Let $0<q\leqslant 1$.Then we get $n=1$,

$$|a_1|t^{1-q}(t-a)^{q-1}\leqslant |a_1|\left(\frac{T_2}{T_2-a}\right)^{1-q}:=c(T_2)\qquad \text{for } t\geqslant T_2 \tag{3.7}$$

and

$$t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds\leqslant \int_a^{T_1}\left(\frac{T_2}{T_2-s}\right)^{1-q}|F(s)|ds:=c_2(T_1,T_2)\qquad \text{for } t\geqslant T_2 \tag{3.8}$$

it follows from (3.7)-(3.8) that

$$\Gamma(q)t^{1-q}x(t)\leqslant c_1(T_2)+c_2(T_1,T_2)+t^{1-q}\int_{T_1}^t (t-s)^{q-1}\left(\nu(s)+K\sum_{i=1}^{m}p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}\right)ds\qquad \text{for } t\geqslant T_2 \tag{3.9}$$

Taking the limit inferior of both sides of inequality (3.9) as $t\to\infty$, we get a contradiction to (3.2). In the case $x(t)$ is eventually negative, a similar argument leads to contradiction to (3.3).
Case 2. Let $q>1$. Then we have $n\geqslant 2$,

$$\Gamma(q)t^{1-q}\sum_{k=1}^{n}\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}\leqslant \sum_{k=1}^{n}\frac{\Gamma(q)|a_k|(T_2-a)^{1-k}}{\Gamma(q-k+1)}:=c_3(T_2)\qquad \text{for } t\geqslant T_2 \tag{3.10}$$

and

$$\Gamma(q)t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds=\Gamma(q)\int_a^{T_1}\left(\frac{t-s}{t}\right)^{q-1}|F(s)|ds\leqslant\Gamma(q)\int_a^{T_1}|F(s)|ds:=c_4(T_1)\qquad\text{for }\ t\geqslant T_2 \tag{3.11}$$

From (3.10) and (3.11), we conclude

$$\Gamma(q)t^{1-q}x(t)\leqslant c_3(T_2)+c_4(T_1)+t^{1-q}\int_{T_1}^t(t-s)^{q-1}\left(v(s)+K\sum_{i=1}^m p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}\right)ds\qquad\text{for }\ t\geqslant T_2 \tag{3.12}$$

Taking the limit inferior of both sides of inequality (3.12) as $t\to\infty$, we get a contradiction to condition (3.2).This completes the proof. □

Corollary 3.2. *Suppose $p(t)>0$, $q_i(t)\geq 0$, $1\leq i\leq m$.If (3.2),(3.3) hold for some constant $K_1>0$, then equation (1.1)is a oscillatory.*

Proof. Suppose to the contrary that there exists a nonoscillatory solution $x(t)$ of equation (1.1). Without loss of generality, we may suppose that $x(t)$ is an ultimately positive solution of equation (1.1). So, there exists $T>a$ such that $x(t)>0$ for $t\geq T_1$. It follows from equation (1.1) that

$$\begin{aligned}\Gamma(q)t^{1-q}x(t)\leqslant&\sum_{k=1}^n\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}\Gamma(q)t^{1-q}+t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+t^{1-q}\int_{T_1}^t(t-s)^{q-1}v(s)ds\\&+t^{1-q}\int_{T_1}^t(t-s)^{q-1}\left[\sum_{i=1}^m(\frac{1}{m}p(s)x(s)-q_i(s)x^{\lambda_i}(s))\right]ds,\end{aligned} \tag{3.13}$$

For $t\geqslant T_1$, set

$$X_i=q_i^{\frac{1}{\lambda_i}}(s)x(s)\quad\text{and}\quad Y_i=\left(\frac{1}{m\lambda_i}p(s)q_i^{-\frac{1}{\lambda_i}}(s)\right)^{\frac{1}{\lambda_i-1}}(s),\quad 1\leqslant i\leqslant m,$$

and, using (2.3), we obtain

$$\begin{aligned}\Gamma(q)t^{1-q}x(t)\leqslant&\sum_{k=1}^n\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}\Gamma(q)t^{1-q}+t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+t^{1-q}\int_{T_1}^t(t-s)^{q-1}v(s)ds\\&+t^{1-q}\int_{T_1}^t(t-s)^{q-1}K_1\sum_{i=1}^m p^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}ds,\quad t\geqslant T_1\end{aligned} \tag{3.14}$$

where $K_1\geq\dfrac{\lambda_1-1}{m}$. The remaining part is similar to that of Theorem 1, so we omit the details. The proof Corollary 1 is finished. □

If $l=0$ in equation (1.1), then $1>\lambda_1>\lambda_2>\cdots>\lambda_m$.Similarly, we obtain the following corollary.

Corollary 3.3. *Suppose $p(t)<0$, $q_i(t)\leq 0$, $1\leq i\leq m$.If (3.2),(3.3) hold for some constant $K_2>0$, then equation (1.1)is a oscillatory.*
If $p(s)\equiv 0$ and $1<l<m$ in equation (1.1), we obtain the following corollary.

Corollary 3.4. *Assume*

$$q_i(t)\begin{cases}\geqslant 0 & \text{for }\ 1\leqslant i\leqslant l;\\ \leqslant 0 & \text{for }\ l+1\leqslant i\leqslant m.\end{cases} \tag{3.15}$$

If there exists a positive function $r(t)$ on $[a,\infty)$ such that for some constant $K_3>0$,

$$\liminf_{t\to\infty}t^{1-q}\int_a^t(t-s)^{q-1}\left(v(s)+K_3\sum_{i=1}^m r^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}\right)ds=-\infty \tag{3.16}$$

and

$$\limsup_{t\to\infty}t^{1-q}\int_a^t(t-s)^{q-1}\left(v(s)+K_3\sum_{i=1}^m r^{\frac{\lambda_i}{\lambda_i-1}}(s)|q_i(s)|^{\frac{1}{1-\lambda_i}}\right)ds=\infty \tag{3.17}$$

then every solution of equation (1.1) is oscillatory.

Proof. For $\lambda_1>\cdots>\lambda_l>1>\lambda_{l+1}>\ldots>\lambda_m$, by Lemma 2, there exist an m-tuple $(\eta_1,\ldots,\eta_m)$ satisfying

$$\sum_{i=1}^l\lambda_i\eta_i=\sum_{i=l+1}^m\lambda_i\eta_i.$$

Suppose to the contrary that there exists a nonoscillatory positive solution $x(t)$ for $t\geq T$.It follows from equation (1.1) that

$$\begin{aligned}\Gamma(q)t^{1-q}x(t)\leqslant&\sum_{k=1}^n\frac{|a_k|(t-a)^{q-k}}{\Gamma(q-k+1)}+t^{1-q}\int_a^{T_1}(t-s)^{q-1}|F(s)|ds+t^{1-q}\int_{T_1}^t(t-s)^{q-1}v(s)ds\\&+t^{1-q}\int_{T_1}^t(t-s)^{q-1}\left[\sum_{i=1}^l(\lambda_i\eta_i r(s)x(s)-q_i(s)x^{\lambda_i}(s))\right]ds\\&+t^{1-q}\int_{T_1}^t(t-s)^{q-1}\left[\sum_{i=l+1}^m(-\lambda_i\eta_i r(s)x(s)+|q_i(s)|x^{\lambda_i}(s))\right]ds\end{aligned}$$

The remainder of the proof is similar, so we omit the details. □

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